AEA summer exam 2002

1. Solve the following equation, for $0 \le x \le \pi$, giving your answers in terms of π.

$$\sin 5x - \cos 5x = \cos x - \sin x.$$ **(8)**

Hints Short answer Full working

2. In the binomial expansion of

$$(1-4x)^p, \quad |x| < \tfrac{1}{4},$$

The coefficient of x^2 is equal to the coefficient of x^4 and the coefficient of x^3 is positive. Find the value of p. **(9)**

Hints Short answer Full working

3. The curve C has parametric equations

$$x = 15t - t^3, \quad y = 3 - 2t^2.$$

Find the values of t at the points where the normal to C at (14, 1) cuts C again. **(11)**

Hints Short answer Full working

4. Find the coordinates of the stationary points of the curve with equation

$$x^3 + y^3 - 3xy = 48$$

and determine their nature. **(11)**

Hints Short answer Full working

5. **Figure 1**

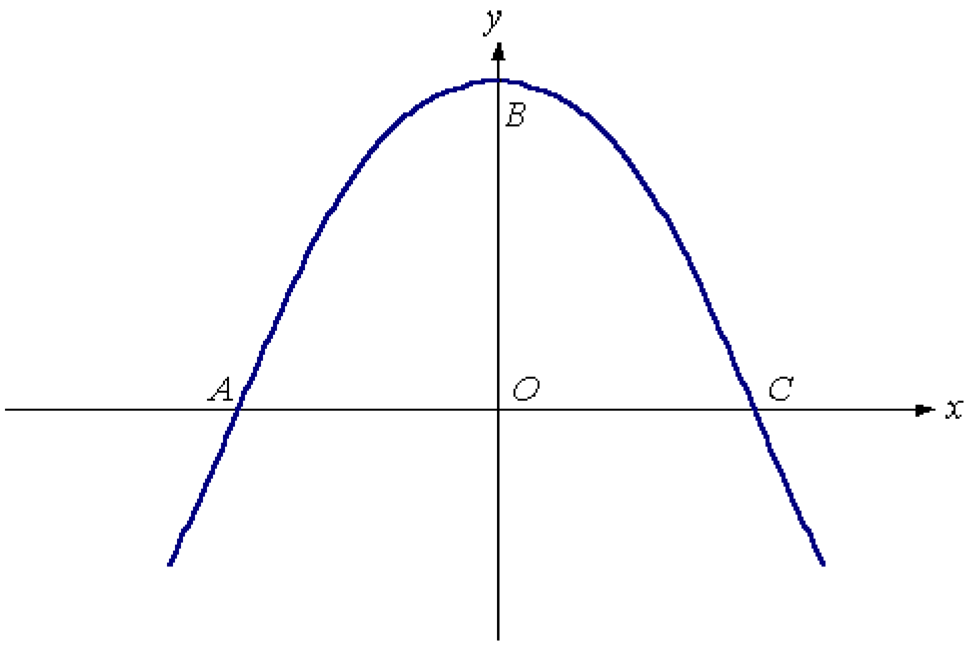

Figure 1 shows a sketch of part of the curve with equation

$$y = \sin(\cos x).$$

The curve cuts the x-axis at the points A and C and the y-axis at the point B.

(*a*) Find the coordinates of the points A, B and C. **(3)**

(*b*) Prove that B is a stationary point. **(4)**

(*c*) Given that the region OCB is convex, show that, for $0 \le x \le \frac{\pi}{2}$,

$$\sin(\cos x) \le \cos x$$

and

$$(1 - \frac{2\pi}{x})\sin 1 \le \sin(\cos x)$$

and state in each case the value or values of x for which equality is achieved. **(6)**

(*d*) Hence show that

$$\frac{\pi}{4}\sin 1 \le \int_0^{\frac{\pi}{2}} \sin(\cos x)\,dx < 1.$$

(4)

Hints Short answer Full working

6. **Figure 2**

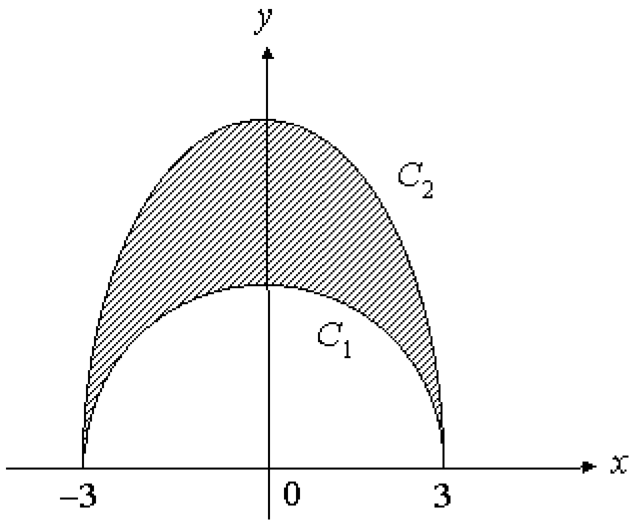

Figure 2 shows a sketch of part of two curves C_1 and C_2 for $y \geq 0$.

The equation of C_1 is $y = m_1 - x^{n_1}$ and the equation of C_2 is $y = m_2 - x^{n_2}$, where m_1, m_2, n_1 and n_2 are positive integers with $m_2 > m_1$.

Both C_1 and C_2 are symmetric about the line $x = 0$ and they both pass through the points (3, 0) and (−3, 0).

Given that $n_1 + n_2 = 12$, find

(a) the possible values of n_1 and n_2, **(4)**

(b) the exact value of the smallest possible area between C_1 and C_2 , simplifying your answer, **(8)**

(c) the largest value of x for which the gradients of the two curves can be the same. Leave your answer in surd form. **(5)**

Hints Short answer Full working

7. A student was attempting to prove that $x = \frac{1}{2}$ is the only real root of

.

$$x^3 + \tfrac{3}{4}x - \tfrac{1}{2} = 0.$$

The attempted solution was as follows.

$$x^3 + \tfrac{3}{4}x = \tfrac{1}{2}$$

$$\therefore \quad x(x^2 + \tfrac{3}{4}) = \tfrac{1}{2}$$

$$\therefore \quad x = \tfrac{1}{2}$$

$$\text{or} \quad x^2 + \tfrac{3}{4} = \tfrac{1}{2}$$

$$\text{i.e.} \quad x^2 = -\tfrac{1}{4} \text{ no solution}$$

$$\therefore \quad x = \tfrac{1}{2} \text{ is only real root.}$$

(*a*) Explain clearly the error in the above attempt. **(1)**

(*b*) Give a correct version of the proof. **(5)**

The equation

$$x^3 + \beta x - \alpha = 0 \qquad \text{(I)}.$$

where α, β are real, $\alpha \neq 0$, has a real root at $x = \alpha$

(*c*) Find and simplify an expression for β in terms of α and prove that α is the only real root provided $|\alpha| < 2$. **(6)**

An examiner chooses a positive number α so that α is the only real root of the equation (I) but the incorrect method used by the student produces 3 distinct real 'roots'.

(*d*) Find the range of possible values for α. **(7)**

Hints Short answer Full working