Pearson Edexcel

Mark Scheme (Results)

Summer 2019

Pearson Edexcel Advanced Extension Award
In Mathematics (9811) Paper 01

Edexcel and BTEC Qualifications

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General Marking Guidance

- All candidates must receive the same treatment. Examiners must mark the first candidate in exactly the same way as they mark the last.
- Mark schemes should be applied positively. Candidates must be rewarded for what they have shown they can do rather than penalised for omissions.
- Examiners should mark according to the mark scheme not according to their perception of where the grade boundaries may lie.
- There is no ceiling on achievement. All marks on the mark scheme should be used appropriately.
- All the marks on the mark scheme are designed to be awarded. Examiners should always award full marks if deserved, i.e. if the answer matches the mark scheme. Examiners should also be prepared to award zero marks if the candidate's response is not worthy of credit according to the mark scheme.
- Where some judgement is required, mark schemes will provide the principles by which marks will be awarded and exemplification may be limited.
- When examiners are in doubt regarding the application of the mark scheme to a candidate's response, the team leader must be consulted.
- Crossed out work should be marked UNLESS the candidate has replaced it with an alternative response.

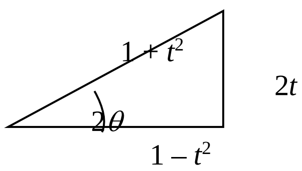
Question	Scheme	Marks	Notes
1. (a)	$u = \log_4 r \Rightarrow 4^u = r$ and taking logs to base 2 Accept $u = \log_4 r = \dfrac{\log_2 r}{\log_2 4} = \ldots$	M1	Accept use of the "Change of base" formula, but **not** $\log_4 r = \log_2 r^{\frac{1}{2}}$ without further working
	$\Rightarrow u \log_2 4 = \log_2 r \Rightarrow u = \log_4 r = \dfrac{\log_2 r}{\log_2 4} = \dfrac{\log_2 r}{2}$	A1	Correct work leading to given answer
		(2)	
(b)	$\log_4\left(5x^2 - 11\right) = \frac{1}{2}\log_2\left(5x^2 - 11\right)$ or $\log_2(3x-5) = 2\log_4(3x-5)$	M1	For getting both terms to the same base
	$\frac{1}{2}\log_2\left(5x^2-11\right) = \log_2(3x-5)$ $\Rightarrow 5x^2 - 11 = (3x-5)^2$	M1	Removing the logs (FT but must involve a power)
	$4x^2 - 30x + 36 = 0$ or $2x^2 - 15x + 18 = 0 \Rightarrow x = \ldots$	M1	Forms and solves a three term quadratic (need not be the correct one)
	$x = \frac{3}{2}$ or 6	A1	Both answers (or $x = 6$, with $x = \frac{3}{2}$ seen in a factorisation.)
	Noting that $3x - 5 < 0$ when $x = \frac{3}{2}$ (so log. term is undefined) and hence $x = 6$	B1	Single final answer with reason
		(5)	
			Total 7 marks

Question	Scheme	Marks	Notes
2. (a)	$P(X = m) = \binom{n}{m} p^m (1-p)^{n-m}$ Accept ${}^nC_m$ or factorial form etc for the binomial coefficient.	B1 **(1)**	Correct expression
(b)	Want $P(X = m-1) \le P(X = m)$ **or** $P(X = m+1) \le P(X = m)$	M1	Allow ≤ for < at endpoints throughout (may be implied by final answer)
	i.e. $\binom{n}{m-1} p^{m-1}(1-p)^{n-m+1} \le \binom{n}{m} p^m (1-p)^{n-m}$ **or** $\binom{n}{m+1} p^{m+1}(1-p)^{n-m-1} \le \binom{n}{m} p^m (1-p)^{n-m}$	M1	Attempted with at least one correct inequality. Allow with = used for this and the next two Ms.
	i.e. $\frac{(n!)\, p^{m-1}(1-p)^{n-m+1}}{(m-1)!\ (n-m+1)!} \le \frac{(n!)\, p^m (1-p)^{n-m}}{m!\ (n-m)!}$ **or** $\frac{(n!)\, p^{m+1}(1-p)^{n-m-1}}{(m+1)!\ (n-m-1)!} \le \frac{(n!)\, p^m (1-p)^{n-m}}{m!\ (n-m)!}$	M1	Use of factorial form for the binomial coefficients in at least two probability expressions.
	i.e. $\frac{1-p}{n-m+1} \le \frac{p}{m}$ **or** $\frac{p}{m+1} \le \frac{1-p}{n-m}$	M1	Appropriate cancelling of terms attempted
	leading to $(n+1)p - 1 \le m \le (n+1)p$	A1 **(5)**	Correct inequality
(c)	e.g. Taking $p = 0.1$, we require both ends of the above interval to be attained … … that is, $\frac{1}{10}(n+1)$ to be an integer Try $n = 109$ (e.g.), giving modes 10 and 11	M1 A1 **(2)**	Sensible choice for p and a strategy to calculate a suitable n ie sets the end points of their interval **of length 1** to be integers. Valid answers given
			Total 8 marks

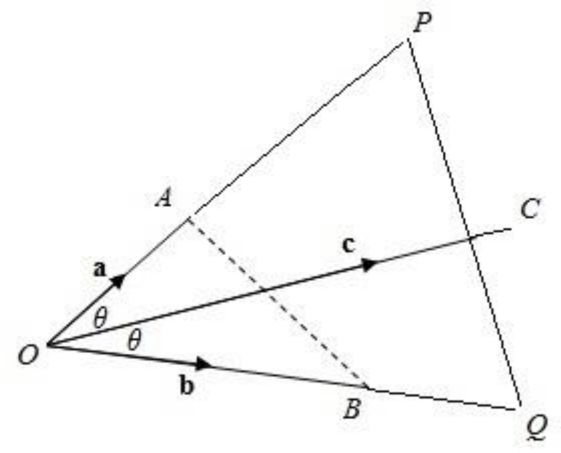
Question	Scheme	Marks	Notes
3. (a)	**(i)** $\phi^2 = \frac{1}{4}\left(5+2\sqrt{5}+1\right)$	M1	Good attempt at squaring
	$= \frac{1}{4}\left(6+2\sqrt{5}\right) = \frac{1}{2}\left(3+\sqrt{5}\right) = \frac{1}{2}\left(1+\sqrt{5}\right)+1 = \phi+1$	A1	All legitimately shown
	(ii) $\frac{1}{\phi} = \frac{2}{\sqrt{5}+1} \times \left(\frac{\sqrt{5}-1}{\sqrt{5}-1}\right)$	M1	Relevant surd-work attempted
	$= \frac{1}{2}\left(\sqrt{5}-1\right) = \frac{1}{2}\left(\sqrt{5}+1\right) - 1 = \phi - 1$	A1	All legitimately shown
	Alt. Divide $\phi^2 = \phi+1$ by ϕ (M1) and rearrange $\Rightarrow$ result (A1)	(S+)	
		(4)	
(b)	$\frac{dy_1}{dx} = \frac{1}{x} - 1$ and $\frac{dy_2}{dx} = -\frac{1}{x^2}$	M1	Both gradts. attempted
	$\Rightarrow\ x^2 = x+1$	M1 A1	Gradients equated and correct eqn. deduced
	(Since $x > 0$,) $x = \phi$, $y = \frac{1}{\phi}$ or $\phi - 1$	A1 A1 (S+)	x coordinate, y coordinate Allow SC A1A0 here if both correct but not in terms of ϕ
	and $k = \frac{1}{\phi} + \phi - \ln(\phi) = \left(\sqrt{5} - \ln\phi\right)$ or $2\phi - 1 - \ln\phi$	A1	Accept any sensible equivalent in terms of ϕ .
		(6)	
S1	S1 mark: Award S1 for a clear solution that is EITHER - fully correct and concise OR - that scores 9+ and includes an S+ but may be slightly laboured	**(1)**	
	Notes **(a) S+** for those who realise the result can be achieved without the need for calculation. **(b) S+** for those who justify the choice of the correct root of the quadratic equation $x^2 = x+1$; the second root is $-\frac{1}{\phi} < 0$.		
			Total 10 + 1 marks

Question	Scheme	Marks	Notes
4. (a)	LHS $\equiv S_xC_xC_y + S_x{}^2S_y + C_xC_y{}^2 + S_xS_yC_y$	M1	Applies cos(*x* – *y*) formula and expands the brackets.
	$\equiv S_xC_xC_y + (1 - C_x{}^2)\,S_y + C_x(1 - S_y{}^2) + S_xS_yC_y$	M1	Replaces $\cos^2 x$ by $1 - \sin^2 x$ and $\sin^2 x$ by $1 - \cos^2 x$ respectively
	$\equiv S_y + C_x + C_x(S_xC_y - C_xS_y) + S_y(S_xC_y - C_xS_y)$	M1	Expands, rearranges and factorises appropriately
	$\equiv S_y + C_x + (S_y + C_x)(S_xC_y - C_xS_y)$	M1	Factors out the $(S_y + C_x)$
	$\equiv (S_y + C_x)(1 + \sin(x - y)) \equiv$ RHS	A1	Applies sin(*x* – *y*) formula and completes to RHS (no conclusion needed this way)
		(5)	(S+ for a direct approach)
	Alt 1 $\cos(x-y) = \cos x\cos y + \sin x\sin y$ and $\sin(x-y) = \sin x\cos y - \cos x\sin y$	M1	Use of both expansions (NB may be awarded for use of one and complete expansion of one side as per main)
	$S_xC_xC_y + S_x{}^2S_y + C_xC_y{}^2 + S_xS_yC_y$ $\equiv C_x + S_y + S_xC_xC_y - C_x{}^2S_y + S_xS_yC_y - C_xS_y{}^2$	M1	Full expansion both sides (may be seen separately)
	$\Leftrightarrow S_x{}^2S_y + C_xC_y{}^2 \equiv C_x + S_y - C_x{}^2S_y - C_xS_y{}^2$	M1	Cancelling terms
	$\Leftrightarrow S_x{}^2S_y + C_xC_y{}^2 \equiv C_x\left(1 - S_y{}^2\right) + S_y\left(1 - C_x{}^2\right)$	M1	Use of relevant trig. identities;
	$\equiv C_xC_y{}^2 + S_x{}^2S_y$ Hence the result is true	A1 (S+)	all correct, with concluding statement.
		(5)	
	Alt 2 $\sin x\cos(x-y) + \cos y\cos(x-y) \equiv$ $\cos x + \sin y + \cos x\sin(x-y) + \sin y\sin(x-y)$	M1	Multiplying out both sides. May be done as separate statements
	$\Leftrightarrow [\sin x\cos(x-y) - \cos x\sin(x-y)]$ $+ [\cos y\cos(x-y) - \sin y\sin(x-y)] \equiv$ $\cos x + \sin y$	M1 M1	(Equates sides and) attempts to rearrange to useful form Collecting into useable groupings
	Then L $\equiv \sin\left(x - (x-y)\right) + \cos\left(y + (x-y)\right)$	M1	Use of sin(*A* – *B*) and cos(*A* + *B*) formulae
	$\equiv \sin(y) + \cos(x) \equiv$ R Hence the identity is true	A1 (S+)	All correctly shown and conclusion
		(5)	
(b)	Re-arranging and setting $x = 5\theta$, $y = 3\theta$ $\Rightarrow \dfrac{\sin 5\theta + \cos 3\theta}{\cos 5\theta + \sin 3\theta} \equiv \dfrac{1 + \sin 2\theta}{\cos 2\theta}$	M1	Use of **(a)**'s result
	$= \dfrac{1 + 2sc}{c^2 - s^2}$	M1 A1	Use of double-angle formulae
	$= \dfrac{(c+s)^2}{(c-s)(c+s)} = \dfrac{c+s}{c-s}$ *	M1	Factorisation & cancelling
	$= \dfrac{c/c + s/c}{c/c - s/c} = \dfrac{1+t}{1-t}$ **	M1 A1	Converting to tans **Given Answer** all correctly obtained
		(6)	

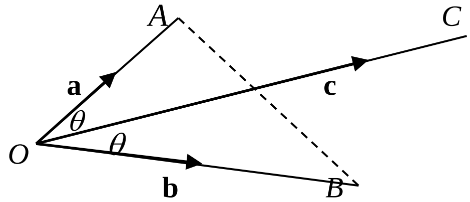
(c)	$k = \dfrac{1+t}{1-t} \Rightarrow k - kt = 1 + t \Rightarrow k - 1 = t(k+1)$ $\Rightarrow \tan\theta = \dfrac{k-1}{k+1}$ or $\theta = \arctan\dfrac{k-1}{k+1}$	M1 A1	Rearranging to tan θ = …
	Explain 1-1-ness of mapping $k \to \theta$	B1	e.g by graph of $y = \dfrac{x-1}{x+1}$ or its gradient > 0 always
	$0 < k - 1 < k + 1$ so $0 < \dfrac{k-1}{k+1} < 1$ $\Rightarrow 0 < \tan\theta < 1 \Rightarrow 0 < \theta < \frac{1}{4}\pi$	B1	For convincing reasoning that each k gives a θ in the required interval
		(4)	
S2	**S1**: Award S1 for a clear solution that EITHER Scores 9+ in (a) and (b) with one part fully correct and concise OR that scores 12+ in total and includes an S+ point but may be laboured in places. **S2**: Award S2 for a clear **and concise** solution throughout that scores at least 12 marks and includes an S+ point.	**(2)**	
	S+ opportunities: for a direct proof or correct use of $\Leftrightarrow$ notation through proof in (a) for noting that we require $c + s \neq 0$ (i.e. $\tan\theta \neq -1$ or $k \neq 0$) at * and/or that $t = \tan\theta \neq 1$ (i.e. $k \neq 1$) at ** for completely clear handling of the trig. identities throughout for any innovative ways used throughout the question.		
			Total 15 + 2 marks

Question	Scheme	Marks	Notes
4. (b)	**Alt. I** Re-arranging and setting $x = 5\theta, y = 3\theta$ $\Rightarrow \dfrac{\sin 5\theta + \cos 3\theta}{\cos 5\theta + \sin 3\theta} \equiv \sec 2\theta + \tan 2\theta$	M1	Use of **(a)**'s result
		M1	Trig. work for 2θ or other appropriate method e.g. half angle formulae to achieve an expression in tan
	$= \dfrac{1+t^2}{1-t^2} + \dfrac{2t}{1-t^2}$	A1	Correct expression in tan
	$= \dfrac{(1+t)^2}{(1+t)(1-t)}$	M1	Common denominator used
	$= \dfrac{1+t}{1-t}$	M1	Use of difference-of-two-squares factorisation
		A1 **(6)**	**Given Answer** all correct

	Alt. II $\dfrac{\sin(4\theta+\theta)+\cos(4\theta-\theta)}{\cos(4\theta+\theta)+\sin(4\theta-\theta)}$	M1	Correct use of θ and 4θ
	$=\dfrac{s_4c+c_4s+c_4c+s_4s}{c_4c-s_4s+s_4c-c_4s}$	M1 A1	Use of sin/cos($A \pm B$) formulae
	$=\dfrac{(s_4+c_4)(c+s)}{(c_4+s_4)(c-s)}$	M1	Factorising both numerator & denominator.
	$=\dfrac{c+s}{c-s}=\dfrac{\frac{c}{c}+\frac{s}{c}}{\frac{c}{c}-\frac{s}{c}}=\dfrac{1+t}{1-t}$	M1 A1 **(6)**	Converting to tans **Given Answer** all correct
	May work from right to left $\dfrac{1+\tan\theta}{1-\tan\theta}\equiv\dfrac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}$	M1	Multiplies through num. and denom. by cos θ
	$\equiv\dfrac{(\cos\theta+\sin\theta)^2}{(\cos\theta-\sin\theta)(\cos\theta+\sin\theta)}$	M1	Multiplies through by (cosθ +sinθ) and expands…
	$\equiv\dfrac{1+2\sin\theta\cos\theta}{\cos^2\theta-\sin^2\theta}$	A1	to this expression
	$\equiv\dfrac{1+\sin 2\theta}{\cos 2\theta}$	M1	Applies the double angle formulae
	$\equiv\dfrac{1+\sin(5\theta-3\theta)}{\cos(5\theta-3\theta)}\equiv\dfrac{\sin 5\theta+\cos 3\theta}{\cos 5\theta+\sin 3\theta}$	M1A1 **(6)**	Writes 2θ as $5\theta-3\theta$ and completes to the LHS
	NB If working from both sides to a common expression, then a conclusion must be made for the final A mark.		

Question	Scheme	Marks	Notes
5. (a)			(S+ for good diagram sketched)
	Let P and Q be points such that $\overrightarrow{OP} = b\mathbf{a}$ and $\overrightarrow{OQ} = a\mathbf{b}$.	M1	Extends OA and OB (may use unit vectors instead)
	Then $\left\|\overrightarrow{OP}\right\| = b\|\mathbf{a}\| = ba = ab = a\|\mathbf{b}\| = \left\|\overrightarrow{OQ}\right\|$ hence OPQ is isosceles.	A1	Deduce isosceles or equivalent.
	Hence the angle bisector from O is perpendicular to PQ.	M1	Use isosceles to deduce perpendicular
	But $\overrightarrow{QP} = \overrightarrow{OP} - \overrightarrow{OQ} = b\mathbf{a} - a\mathbf{b}$ and hence as C is on the angle bisector, so $b\mathbf{a} - a\mathbf{b}$ is perpendicular to $\mathbf{c}$.	A1	Draw correct conclusion.
		(4)	
(b)	$\overrightarrow{OD} = \overrightarrow{OA} + \lambda\overrightarrow{AB} \Rightarrow \mathbf{d} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$	M1	Sets up appropriate equation, either form.
	$\Rightarrow \mathbf{d} = (1 - \lambda)\mathbf{a} + \lambda\mathbf{b}$	A1	Correctly shown
	($0 < \lambda < 1$ since D is between A and B)	(S+)	(Reasoning for λ)
		(2)	
(c)	($\overrightarrow{OD} = k\mathbf{c}$ and from (a) $\mathbf{c} = K \times \frac{1}{2}\left(\overrightarrow{OP} + \overrightarrow{OQ}\right)$ hence) $\overrightarrow{OD} = k'\left(\overrightarrow{OP} + \overrightarrow{OQ}\right)$	M1	Makes deduction that **d** is a multiple of **p+q**
	Hence $\mathbf{d} = k'\left(\overrightarrow{OP} + \overrightarrow{OQ}\right) = (1-\lambda)\mathbf{a} + \lambda\mathbf{b}$	M1	Equates their **d** to **d** from (b)
	So $k'\left(b\mathbf{a} + a\mathbf{b}\right) = (1-\lambda)\mathbf{a} + \lambda\mathbf{b}$	M1	Forms equation in **a** and **b**
	Therefore (since **a** and **b** are not parallel) $k'b = 1-\lambda$ and $k'a = \lambda$	M1	Extracts simultaneous equations and solves for λ.
	$\Rightarrow \frac{\lambda}{a}b = 1-\lambda \Rightarrow \lambda = \frac{a}{a+b}$	A1	(S+ for non-parallel reasoning)

Question	Scheme	Marks	Notes
	$\overrightarrow{AD} = (1-\lambda)\mathbf{a} + \lambda\mathbf{b} - \mathbf{a} = \lambda(\mathbf{b}-\mathbf{a}) \Rightarrow AD = \lambda\lvert\mathbf{b}-\mathbf{a}\rvert$ $\overrightarrow{BD} = (1-\lambda)\mathbf{a} + \lambda\mathbf{b} - \mathbf{b} = (1-\lambda)(\mathbf{a}-\mathbf{b}) \Rightarrow BD = (1-\lambda)\lvert\mathbf{b}-\mathbf{a}\rvert$ So $\dfrac{AD}{BD} = \dfrac{\lambda}{1-\lambda}$ $= \dfrac{a/(a+b)}{b/(a+b)} = \dfrac{a}{b} = \dfrac{OA}{OB}$	M1 dM1 A1	Correct work to establish ratio (may just be quoted) Give M0 if division of vectors is used. Substitutes in for λ **Given result** established
		(8)	
S2	S2 mark: Award S2 for a clear and concise solution that is EITHER - fully correct with no majorly incorrect vector notation used OR - that scores 12+ and includes at least 2 S+ points but may have some poor notation and be slightly laboured Award S1 for - a clear solution that scores 10+ marks with at least one S+ point.	**(2)**	
	Notes **(a)** S+ for a clearly labelled diagram drawn showing at least **a, b** and **c** **(b)** for the explanation of why $0 < \lambda < 1$ **(c)** S+ for reason given for being able to equate coefficients, e.g. vectors cannot be parallel since *OAB* is a triangle. S+ for any innovative ways used throughout the question.		
		Total 14 + 2 marks	

Question	Scheme	Marks	Notes
5. (a) Alt with dot product	 $\cos\theta = \dfrac{\mathbf{a}\bullet\mathbf{c}}{ac} = \dfrac{\mathbf{b}\bullet\mathbf{c}}{bc} \Rightarrow b\mathbf{a}\bullet\mathbf{c} = a\mathbf{b}\bullet\mathbf{c}$ $\Rightarrow (b\mathbf{a} - a\mathbf{b})\bullet\mathbf{c} = 0$ and the scalar product = 0 ⇒ perpendicularity *	(S+) M1 A1 M1 A1 **(4)**	(Appropriate diagram) Use of the scalar product for both **a** and **b** with **c** Factorising
(b)	As main scheme	M1A1 (S+) **(2)**	As main scheme (with S+)

(c)	$(b\mathbf{a} - a\mathbf{b}) \bullet \mathbf{d} = 0$	M1	Use **(a)**'s result with **c** = **d**
	$b\mathbf{a} \bullet \left((1-\lambda)\mathbf{a} + \lambda\mathbf{b}\right) - a\mathbf{b} \bullet \left((1-\lambda)\mathbf{a} + \lambda\mathbf{b}\right) = 0$	M1	Substg. for **d** from (b)
	$b(1-\lambda)a^2 + b\lambda\mathbf{a} \bullet \mathbf{b} - a(1-\lambda)\mathbf{a} \bullet \mathbf{b} - ab^2\lambda = 0$	M1	Multiplying out and using **a•a** = a^2 etc.
	$(a\mathbf{a} \bullet \mathbf{b} - a^2 b + b\mathbf{a} \bullet \mathbf{b} - ab^2)\lambda = a\mathbf{a} \bullet \mathbf{b} - a^2 b$	M1 (S+) A1	Collecting up λ's and factorising Correct λ
	$(a+b)(\mathbf{a} \bullet \mathbf{b} - ab)\lambda = a(\mathbf{a} \bullet \mathbf{b} - ab)$ **		
	$\Rightarrow \lambda = \dfrac{a}{a+b}$	M1 dM1 A1 (8)	As main scheme **Given result** established
	$\dfrac{AD}{BD} = \dfrac{\lambda}{1-\lambda} = \dfrac{a/(a+b)}{b/(a+b)} = \dfrac{a}{b} = \dfrac{OA}{OB}$		
	NB candidates may use $\mathbf{a} \bullet \mathbf{b} = ab\cos\theta$ when rearranging : $(a.ab\cos\theta - a^2 b + b.ab\cos\theta - ab^2)\lambda = a.ab\cos\theta - a^2 b$	4th M1	
	$(a+b)ab(\cos\theta - 1)\lambda = a.ab(\cos\theta - 1)$	A1	
	$\Rightarrow \lambda = \dfrac{a}{a+b}$ Explaining that $\cos\theta \neq 1$ since this would mean θ = 0º **	(S+)	
	S+ opportunities: At * for noting that (**c** ≠ **0**) $b\mathbf{a} - a\mathbf{b} \neq \mathbf{0}$ since otherwise (e.g.) $\mathbf{a} = \dfrac{a}{b}\mathbf{b}$ and **a** and **b** would be parallel At ** for explaining that $\mathbf{a} \bullet \mathbf{b} - ab \neq 0$ since $\mathbf{a} \bullet \mathbf{b} = ab\cos\theta$ and this would mean θ = 0º (or 180º) for any innovative ways used throughout the question.		

Question	Scheme	Marks	Notes
6. (a)	Want $\sin(\ln x) = 0 \Rightarrow \ln x = (0,)\ \pi, (2\pi, \ldots)$ $\Rightarrow x = (1,)\ e^{\pi}, (\ldots)$	M1 M1 A1 **(3)**	Sets equal to 0 and extracts second solution for sin to $\ln x$ Sets e to their "π" Must give only the one answer
(b)	$y = x\sin(\ln x) \Rightarrow \frac{dy}{dx} = x.\cos(\ln x).\frac{1}{x} + 1.\sin(\ln x)$ $= \cos(\ln x) + \sin(\ln x)$ $y = x\cos(\ln x) \Rightarrow \frac{dy}{dx} = x.-\sin(\ln x).\frac{1}{x} + 1.\cos(\ln x)$ $= \cos(\ln x) - \sin(\ln x)$ Combining the two: $\int \cos(\ln x)\,dx = \frac{1}{2}x(\sin(\ln x) + \cos(\ln x))\ (+ k_1)$ $\int \sin(\ln x)\,dx = \frac{1}{2}x(\sin(\ln x) - \cos(\ln x))\ (+ k_2)$	M1 A1 M1 A1 M1 A1 A1 **(7)**	Use of the Product Rule accept $x\cos(\ln x)\times\ldots + 1.\sin(\ln x)$ All correct, simplified With Product Rule All correct, simplified Must be a genuine attempt (Ignore missing constants of integration)
	Allow alternative methods such as substitution $u = \ln x$ or integration by parts for the final method mark.		
(c) (i)	Let $S = \int x\sin(\ln x)\,dx$ and $C = \int x\cos(\ln x)\,dx$. **Method I** $S = \int x\sin(\ln x).1\,dx$ $= x^2\sin(\ln x) - \int x.[\cos(\ln x) + \sin(\ln x)]\,dx$ i.e. $2S = x^2\sin(\ln x) - C$ $C = \int x\cos(\ln x).1\,dx$ $= x^2\cos(\ln x) - \int x.[\cos(\ln x) - \sin(\ln x)]\,dx$ i.e. $2C = x^2\cos(\ln x) + S$ Eliminating C then gives $S = \frac{1}{2}x^2\sin(\ln x) - \frac{1}{4}x^2\cos(\ln x) - \frac{1}{4}S$ Recognising and removing the "loop" $\Rightarrow S = \frac{1}{5}x^2(2\sin(\ln x) - \cos(\ln x))\ (+ k)$	 M1 A1 M1 A1 M1 M1 A1 **(7)**	 Use of $\int$ n. by parts on S Correct (unsimplified) Use of $\int$ n. by parts on C Correct (unsimplified) Solves simultaneously for S and C Rearranges to find $S = \ldots$ Correct answer (Ignore missing "+ k")

	Method II $S = \int \sin(\ln\ x).x\,dx$ $= \tfrac{1}{2}x^2\sin(\ln x) - \int \tfrac{1}{2}x^2.\cos(\ln x).\frac{1}{x}\,dx$ i.e. $S = \tfrac{1}{2}x^2\sin(\ln x) - \tfrac{1}{2}C$ $C = \int \cos(\ln\ x).x\,dx$ $= \tfrac{1}{2}x^2\cos(\ln x) - \int \tfrac{1}{2}x^2.-\sin(\ln\ x).\frac{1}{x}\,dx$ Eliminating C then gives $S = \tfrac{1}{2}x^2\sin(\ln x) - \tfrac{1}{4}x^2\cos(\ln x) - \tfrac{1}{4}S$ Recognising and removing the "loop" $\Rightarrow S = \tfrac{1}{5}x^2\left(2\sin(\ln\ x) - \cos(\ln x)\right)$ $(+\ k)$ **Method III** Differentiate. $y_1 = x^2\sin(\ln x)$ $\frac{dy_1}{dx} = x^2\cos(\ln x).\frac{1}{x} + 2x\sin(\ln x)$ $= x\cos(\ln x) + 2x\sin(\ln x)$ Differentiate. $y_2 = x^2\cos(\ln x)$ $\frac{dy_2}{dx} = x^2.-\sin(\ln x).\frac{1}{x} + 2x\cos(\ln x)$ $= -x\sin(\ln x) + 2x\cos(\ln x)$ Then $2y_1 - y_2 = 5x\sin(\ln x)$ Turning it round from differentiation to integration $\int x\sin(\ln x)\,dx = \tfrac{1}{5}\left(2x^2\sin(\ln\ x) - x^2\cos(\ln\ x)\right)$ $(+\ k)$	M1 A1 M1 A1 M1 M1 A1 (7) M1 A1 M1 A1 M1 M1 A1 **(7)**	Use of $\int$n. by parts on S Correct (unsimplifed) Use of $\int$n. by parts on C Correct, unsimplified Solves simultaneously for S Rearranges to S =… Ignore missing "+ k" Must include use of the Product Rule Correct, unsimplified Must include use of the Product Rule Correct, unsimplified Attempt to eliminate the unwanted terms Rearranges to find S Correct answer (Ignore missing "+ k")
	Method IV $I = \int x.\sin(\ln x)\,dx =$ $\frac{x}{2}\left(\sin(\ln x) - \cos(\ln x)\right)x - \frac{1}{2}\int x\sin(\ln x) - x\cos(\ln x)\,dx$ $J = \int x.\cos(\ln x)\,dx =$ $\frac{x}{2}\left(\sin(\ln x) + \cos(\ln x)\right)x - \frac{1}{2}\int x\sin(\ln x) + x\cos(\ln x)\,dx$ $3I = x^2\left(\sin(\ln x) - \cos(\ln x)\right) + \int x\cos(\ln x)\,dx$ $= x^2\left(\sin(\ln x) - \cos(\ln x)\right) + J$ $3J = x^2\left(\sin(\ln x) + \cos(\ln x)\right) - \int x\sin(\ln x)\,dx$ $= x^2\left(\sin(\ln x) + \cos(\ln x)\right) - I$	M1 A1 M1 A1 M1 M1 A1	Uses the result of (b) in application of parts on I Correct unsimplifed Uses the result of (b) in application of parts on J Correct unsimplified Solves simultaneously to achieve and equation in I Rearranges to correct integral for I and achieves correct answer (ignore k)

	$\Rightarrow 9I = 3x^2\left(\sin(\ln x) - \cos(\ln x)\right) + x^2\left(\sin(\ln x) + \cos(\ln x)\right) - I$ $\Rightarrow I = \frac{1}{5}x^2\left(2\sin(\ln x) - \cos(\ln x)\right)(+k)$		
	Method V $u = \ln x \Rightarrow x\mathrm{d}u = \mathrm{d}x$ (oe) and substitutes to get $I = \int x\sin(\ln x)\mathrm{d}x = \int \mathrm{e}^u \sin(u).\mathrm{e}^u \mathrm{d}u$ $= -\mathrm{e}^{2u}.\cos u - 2\int \mathrm{e}^{2u}(-\cos u)\mathrm{d}u$ $= -\mathrm{e}^{2u}.\cos u + 2\left(\mathrm{e}^{2u}.\sin u - 2\int \mathrm{e}^{2u}.\sin u\mathrm{d}u\right)$ $\Rightarrow 5I = 2\mathrm{e}^{2u}.\sin u - \mathrm{e}^{2u}.\cos u$ $\Rightarrow I = \frac{1}{5}\left(2x^2\sin(\ln x) - x^2\cos(\ln x)\right)(+k)$	M1 A1 M1A1 M1 M1 A1 **(7)**	Selects and makes an appropriate substitution. Correct in terms of *u* Applies integration by parts either way round (mark appropriately) Applies parts a second time in same direction. Gathers *I* terms together Undoes substitution.
(c) (ii)	Use of limits (1, their ans. to (a)) in their *S* Area of *R* = $\frac{1}{5}\mathrm{e}^{2\pi}(2.0 - (-1)) - \frac{1}{5}(2.0 - 1)$ $= \frac{1}{5}\left(\mathrm{e}^{2\pi} + 1\right)$	M1 A1 **(2)**	Attempted Correct **Given Answer** from fully correct working
		Total 19 marks	

Question	Scheme	Marks	Notes
7. (a)	$DB = \sqrt{(a-x)^2+b^2}$	B1	May be implied by use in formula
	Time taken is thus $t = \frac{x}{\lambda} + \sqrt{(a-x)^2+b^2}$	B1	Correct expression
	Differentiating w.r.t. x $\frac{dt}{dx} = \frac{1}{\lambda} + \frac{1}{2}\left((a-x)^2+b^2\right)^{-\frac{1}{2}} \times 2(a-x)\times(-1)$	M1 A1	Attempts differentiation of square root using chain rule. Correct differentiation of square root term
	$\Rightarrow \frac{dt}{dx} = \frac{1}{\lambda} - \frac{a-x}{\sqrt{(a-x)^2+b^2}}$	A1 **(5)**	**Given Answer** legitimately obtained
(b) (i)	$\frac{dt}{dx} = 0 \Rightarrow \sqrt{(a-x)^2+b^2} = \lambda(a-x)$ $\Rightarrow (a-x)^2+b^2 = \lambda^2(a-x)^2$ $\Rightarrow b^2 = (\lambda^2-1)(a-x)^2$	M1	Setting $\frac{dt}{dx} = 0$ & squaring
	$\Rightarrow \frac{b^2}{\lambda^2-1} = (a-x)^2 \Rightarrow \frac{b}{\sqrt{\lambda^2-1}} = a-x$ (as $x < a$)	M1(S+)	Rearranging for x May expand and solve quadratic in x
	and $x = a - \frac{b}{\sqrt{\lambda^2-1}}$ or $x = a - \frac{b\sqrt{\lambda^2-1}}{\lambda^2-1}$ oe	A1 **(3)**	Correct solution selected and simplified (if using formula)
(b) (ii)	For $x \geq 0$, $\sqrt{\lambda^2-1} \geq \frac{b}{a} \Rightarrow \lambda \geq \sqrt{1+\frac{b^2}{a^2}}$	M1	Attempt to find x in required interval. Accept showing the reverse implication for the M.
	(S+ if the $x \leq a$ case is commented/shown)	A1 **(2)**	**Given Answer** legitimately obtained
(b) (iii)	$\frac{d^2t}{dx^2} = -\left\{\frac{N}{D}\right\}$ where $D = (a-x)^2+b^2$ and $N = (-1)\sqrt{(a-x)^2+b^2} - (a-x)\left(-\frac{a-x}{\sqrt{(a-x)^2+b^2}}\right)$	M1	Attempt at 2nd derivative (may be minor slips but it should look plausible)
	$\frac{d^2t}{dx^2} = \frac{(a-x)^2+b^2-(a-x)^2}{\left((a-x)^2+b^2\right)^{3/2}} = \frac{b^2}{\left((a-x)^2+b^2\right)^{3/2}}$	A1	Correct unsimplified answer.
	> 0 hence a Minimum	A1 **(3)**	Correctly concluded from correct second derivative.
(c) (i)	Then $t = \frac{a}{\lambda} - \frac{b}{\lambda\sqrt{\lambda^2-1}} + \sqrt{\frac{b^2}{\lambda^2-1}+b^2}$ $= \frac{a}{\lambda} - \frac{b}{\lambda\sqrt{\lambda^2-1}} + \frac{b\lambda}{\sqrt{\lambda^2-1}}$	M1	Substituting for x
	$= \frac{1}{\lambda}\left(a + \frac{b(\lambda^2-1)}{\sqrt{\lambda^2-1}}\right) = \frac{a+b\sqrt{\lambda^2-1}}{\lambda}$	A1 **(2)**	Correct in terms of $\sqrt{\lambda^2-1}$
(c) (ii)	t minimised when $x = 0$	M1	Finds time for $x = 0$ from their equation.
	$\Rightarrow t = \sqrt{a^2+b^2}$	A1 **(2)**	Correct answer.

(d)	$t_A = \dfrac{a+b}{\lambda}$ $t_A < t_x$ provided $\dfrac{a+b}{\lambda} < \dfrac{1}{\lambda}\left(a + b\sqrt{\lambda^2 - 1}\right)$ i.e. $1 < \sqrt{\lambda^2 - 1}$ i.e. $\lambda > \sqrt{2}$	B1 M1 A1 **(3)**	
S2	Award S1 for a clear and concise solution that is EITHER - fully correct and is succinct in (a) and (b) OR - that scores 11+ in (a) and (b) and includes an S+ point in (b) Award S2 for a solution that - satisfies the S1 award and in addition scores at least 5 marks from (c) and (d) and includes at least one S+ point		**(2)**
	S+ opportunities: for justifying – or just noting – the taking of a positive square-root in **(b) (i)** for considering the $x \le a$ end in (b)(ii). for properly handled inequalities in **(d)** for any innovative ways used throughout the question.		
			Total 20 + 2 marks

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