General Certificate of Education
Advanced Level Examination
June 2014

Use of Mathematics (Pilot) USE3/PM

Mathematical Comprehension

Preliminary Material

Data Sheet

To be opened and issued to candidates between
Thursday 1 May 2014 and Thursday 15 May 2014

REMINDER TO CANDIDATES

YOU MUST **NOT** BRING THIS DATA SHEET
WITH YOU WHEN YOU SIT THE EXAMINATION.
A CLEAN COPY WILL BE MADE AVAILABLE.

Every penny counts!

You will probably have noticed that the prices of most goods and services seem to rise over time. This leads to a problem if you have some money now but do not want to buy something until a date in the future. Will you then have enough money to buy the item, as its price will probably have risen? One way of coping with the problem is to invest your money in a bank, in a building society, or in some other way, so that you earn interest.

Imagine that you had £1000 at the start of the year 2010 and invested it in an account so that it earned 4% interest at the end of each year. Assuming that you left your account untouched for five years, how much would you have at the end of 2014?

Figure 1 *High street banks*

At the end of the first year there would be $£1000\left(1+\frac{4}{100}\right) = £1040.00$. At the end of the second year there would be $£1000\left(1+\frac{4}{100}\right)\left(1+\frac{4}{100}\right) = £1000\left(1+\frac{4}{100}\right)^2 = £1081.60$, and so on.

Remember that you earn interest in the second year on the interest you earned in the first year as well as on the original investment.

By the end of 2014, you would have $£1000\left(1+\frac{4}{100}\right)^5 = £1216.65$.

In general, if a sum of money, $£P$, is invested at an interest rate of $r\%$ compounded annually for t years, it will become $£P\left(1+\frac{r}{100}\right)^t$.

You may have to do the reverse calculation if you want to ensure that you have a certain amount of money at a date in the future. For example, you may wish to make a major purchase in the future, such as a house or car. Suppose that you wish to have an amount, $£A$, in t years' time, you need to invest, $£P$ now, where

$$P = A\left(1+\frac{r}{100}\right)^{-t}$$

It is possible to invest money so that the interest is compounded more frequently than once a year. For example, it may be that instead of the 4% interest being added to your account once a year, 2% may be added at the end of every six months, or 1% at the end of every three months.

If you receive $\frac{4}{n}\%$ interest, n times per year, then at the end of t years, the amount originally invested, $£P$, will have become $£A$, where

$$A = P\left(1+\frac{4}{100n}\right)^{nt}$$

So in the case where the interest is added every month, £1000 would become £1040.74 after one year, rather than the £1040.00 it would become if interest were added only once at the end of the year, thus giving a slightly better deal for you as an investor!

If interest at an annual rate of r% is compounded many times in a year, so that it may be considered to be compounded continuously, then the amount originally invested, £P, becomes £A, where

$$A = P\mathrm{e}^{\frac{r}{100}t}$$

after t years.

Such a system, where interest is compounded continuously, means that after one year an investment of £1000 will have become £1040.81 – an even better deal for your investment.

What if you were able to add further savings to your account as time passes? For example, what if you were able to invest £1000 at the start of each year 2010, 2011, 2012, 2013 and 2014?

This would mean, with a fixed interest rate of 4%, at the end of 2014, you would have a final sum, £S, where

$$S = 1000(\mathrm{e}^{0.04\times 5} + \mathrm{e}^{0.04\times 4} + \mathrm{e}^{0.04\times 3} + \mathrm{e}^{0.04\times 2} + \mathrm{e}^{0.04}) = 5646.51$$

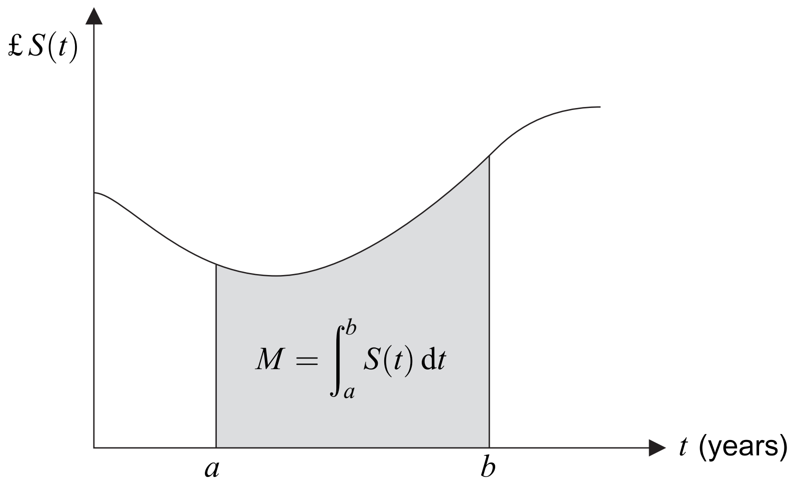
The idea of regularly adding money to an investment is particularly important to someone who has a stream of money which they can deposit into an account.

Consider someone who has a stream of money £$S(t)$, where S is a function of t years. Then the total amount of this money, £M, without investment, from $t = a$ to $t = b$ is given by

$$M = \int_a^b S(t)\,\mathrm{d}t$$

This can be represented by the area under the graph of $S(t)$ plotted against t, as shown in **Figure 2**.

Figure 2 *Total amount of money, £M, without investment, for a stream of money £$S(t)$, where S is a function of t years from $t = a$ to $t = b$ is given by $M = \int_a^b S(t)\,\mathrm{d}t$*



Turn over ▶

As the money streams in, the extra amount of money arriving during the small interval from t to $t + \delta t$ is given by $S(t) \times \delta t$. If we assume that this is invested at a continuous compound interest rate, $r\%$, for the time remaining up to some given time $t = T$, this amount will be invested for $T - t$ years. At the end of this time, it will have become $S(t)\delta t \mathrm{e}^{\frac{r}{100}(T-t)}$.

If we sum all the small increments of money that accumulate, the final value of the investment of the stream of money is £A, where

$$A = \int_0^T S(t)\mathrm{e}^{\frac{r}{100}(T-t)}\ \mathrm{d}t = \mathrm{e}^{\frac{r}{100}T}\int_0^T S(t)\mathrm{e}^{\frac{-rt}{100}}\ \mathrm{d}t$$

To get a feel for the effect of investing a stream of money in this way, assume that someone has such a stream of money which arrives at a constant rate throughout the year where $S = 1000$.

After five years, during which the money has been invested at an interest rate of 4% compounded continuously, the amount, to the nearest penny, in their account will be £5535.07 .

We can compare this with the value we found for a discrete sum of £1000 invested at the start of each of five years. This suggests that if we invested £1000 at the start of each year, after five years we would have £5646.51 . This is slightly more than the value given by integrating the continuous stream of money. This might be expected, because when investing the discrete sums, we assumed that each £1000 is available to invest at the start of each year.

The calculus method has the advantage of allowing us to consider cases where the stream of money can be modelled as a function.

For example, if $S = 1000 + 10t$, the stream of money is increasing linearly over the same period so that, after five years at an interest rate of 4%, the amount in the account will be given by

$$\begin{aligned}
A &= \mathrm{e}^{0.2}\int_0^5 (1000 + 10t)\mathrm{e}^{-0.04t}\ \mathrm{d}t \\
&= \mathrm{e}^{0.2}\int_0^5 1000\mathrm{e}^{-0.04t}\ \mathrm{d}t + 10\mathrm{e}^{0.2}\left(\int_0^5 t\mathrm{e}^{-0.04t}\ \mathrm{d}t\right) \\
&= \mathrm{e}^{0.2}\left[\frac{1000\mathrm{e}^{-0.04t}}{-0.04}\right]_0^5 + 10\mathrm{e}^{0.2}\left(\left[\frac{\mathrm{e}^{-0.04t}}{-0.04}\times t\right]_0^5 - \int_0^5 \frac{\mathrm{e}^{-0.04t}}{-0.04}\ \mathrm{d}t\right) \\
&= \mathrm{e}^{0.2}\left[\frac{1000\mathrm{e}^{-0.04t}}{-0.04}\right]_0^5 + 10\mathrm{e}^{0.2}\left(\left[\frac{\mathrm{e}^{-0.04t}}{-0.04}\times t\right]_0^5 - \left[\frac{\mathrm{e}^{-0.04t}}{0.04^2}\right]_0^5\right)
\end{aligned}$$

If you evaluate this, you find that it makes relatively little difference to the value we found previously, that is £5535.07 , because the extra amount invested is small.

As you can see, even these models of relatively simple ideas in economics already require the use of some complex mathematical concepts. It is little wonder, therefore, that there is often disagreement over economic policy which involves many more complicated and interacting factors.

END OF DATA SHEET

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