Teacher Support Materials

AS Use of Mathematics 5351

Paper Reference UOM 4/1

DATA SHEET INFORMATION

Watch your speed

Over recent years, motorists have had their speed monitored by a range of speed cameras. Different types of camera work on different principles – but, as you might expect, maths always has a role to play.

Some speed cameras rely on an important scientific principle called the Doppler Effect. This principle has many diverse applications and is used by people working in scientific and technological areas ranging from astronomers trying to discover new planets to surgeons investigating the structure of patients' hearts. You may have noticed the Doppler Effect in action when standing by the side of the road as an emergency vehicle has driven past sounding its siren. Perhaps you noticed a change in frequency of the sound of the siren as the vehicle passed by, with the frequency higher as it approached and lower as it drove away?

To understand this effect, first of all you need to know how the speed of a wave is related to its frequency (number of waves in unit time) and wavelength. **Figure 1** shows a graph of $y = \sin 1440t°$, a wave with a period of $\frac{360}{1440} = \frac{1}{4}$ second and therefore a frequency of 4 waves per second.

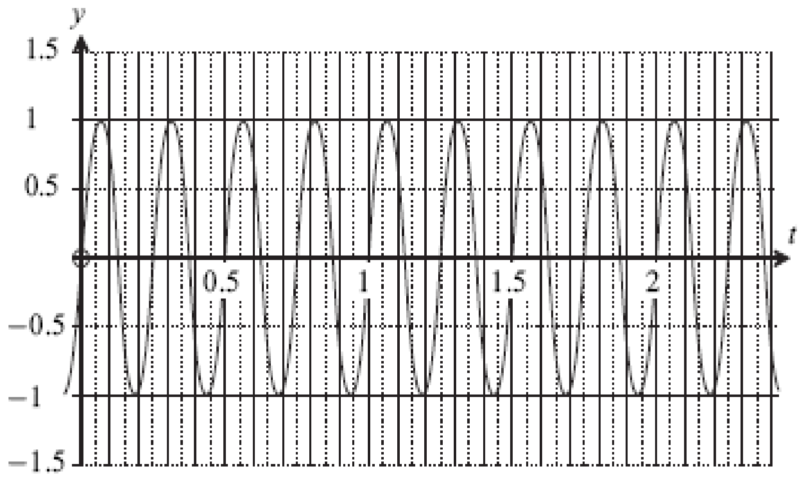


Figure 1

Imagine you are standing at a distance from a stationary siren that is emitting waves at a frequency f Hz (that is, f waves per second). These waves travel at a speed v m s^{-1}. In one second, therefore, f waves would reach you and these f waves would occupy a distance of v metres (**Figure 2**) as they travel through the air. The length of each wave, its wavelength, λ metres, would therefore be given by $\lambda = \frac{v}{f}$.

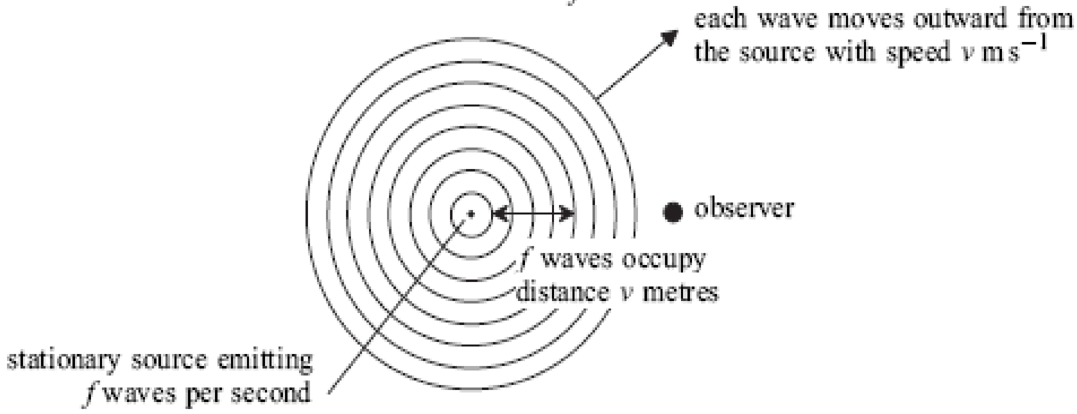


Figure 2 Stationary source emitting waves

If the source of the siren is moving towards you with speed v_s m s^{-1}, then in one second the f waves would occupy a shorter distance $v - v_s$ metres (**Figure 3**). So the apparent wavelength, λ', is given by $\lambda' = \dfrac{v - v_s}{f}$, and the apparent frequency, f', is given by

$$f' = \frac{\text{speed of the wave}}{\text{apparent wavelength}} = \frac{v}{(v - v_s)/f} = \frac{vf}{v - v_s}$$

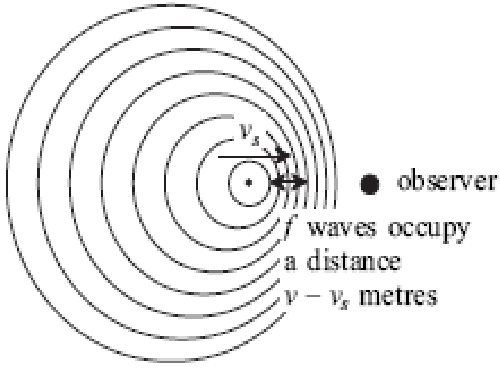


Figure 3 Source moving towards observer

A similar argument applies if the source of the siren is moving away from you, but now the f waves per second would occupy a longer distance $v + v_s$ metres (**Figure 4**), and the apparent frequency, f', is given by $f' = \dfrac{\text{speed of the wave}}{\text{apparent wavelength}} = \dfrac{v}{(v + v_s)/f} = \dfrac{vf}{v + v_s}$.

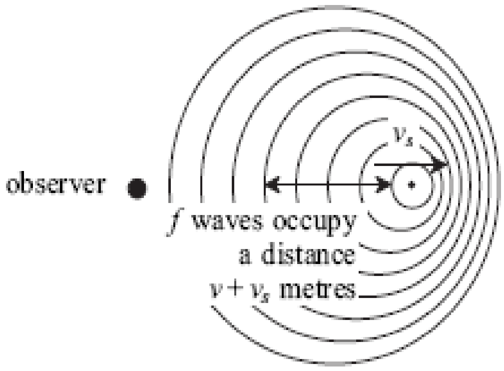


Figure 4 Source moving away from observer

When the source is moving away, the difference between the frequency of the transmitted wave and that received by the observer is therefore given by

$$\begin{aligned} f - f' &= f - \frac{v}{(v + v_s)} f \\ &= \frac{v_s}{v + v_s} f \end{aligned}$$

Some speed cameras use this effect by transmitting microwave radiation which bounces off a car travelling away from the camera and measuring the change in frequency between the transmitted wave and the wave that bounces back to the camera. In this case, the speed of the transmitted wave is the same as for all electromagnetic waves, $c = 300\,000\,000 = 3 \times 10^8$ metres per second. Because c is clearly very much greater than v_{car}, the measured speed of the car, then

$$f - f' = \frac{v_{\text{car}}}{c + v_{\text{car}}} f \approx \frac{v_{\text{car}}}{c} f$$

Because the speed camera measures the frequency of the wave that is bounced off a car, it effectively detects two shifts, so in this case the difference in frequency, f_{diff}, that the camera measures is given by $f_{\text{diff}} = 2\frac{v_{\text{car}}}{c}f$.

A typical transmitted frequency of radar wave is $24\ \text{GHz} = 2.4 \times 10^{10}\ \text{Hz}$. For a car travelling at 40 mph the corresponding shift in frequency is given by

$$f_{\text{diff}} = 2\frac{v_{\text{car}}}{c}f = \frac{2 \times 40 \times 0.447 \times 2.4 \times 10^{10}}{3 \times 10^{8}} = 2860.8\ \text{Hz}$$

The factor 0.447 is used to convert a speed in miles per hour into metres per second, the units necessary for v_s to be consistent with the other values used in the equation.

One assumption that has been made so far in applying equations to the case of a speed camera is that the vehicle was moving directly away from the speed camera. As the photograph in **Figure 5** shows, this will not be strictly true: the camera will almost always be mounted to one side of the road. What effect does this have on the accuracy of the measured speed of the car?

Figure 5

As **Figure 6** shows, the camera will measure the speed, v_{car}, in the direction of a line joining the camera and the point where the transmitted waves hit the car (ignoring any difference in height between the camera and the car's number plate). This is a fraction of its actual speed, v_{actual}. These values are related by $v_{\text{car}} = v_{\text{actual}} \cos\theta$, where θ is the angle that the direction of travel of the car makes with the line joining the speed camera and the car.

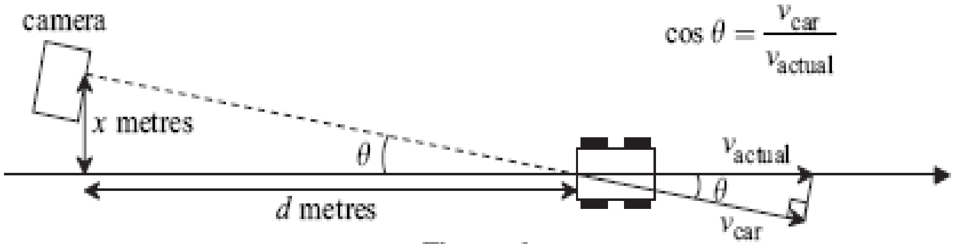


Figure 6

The cosine of angle θ can be found by measuring the distance, d metres, along the line of travel of the car from the camera to the point on which it is focused, and the sideways displacement, x metres, of the camera from the line of travel of the car (see **Figure 6**). In this case

$$\cos\theta = \frac{d}{\sqrt{d^2 + x^2}}$$

This means that v_{car} will always be less than v_{actual}, so what is known as the "cosine error" is always to the advantage of the motorist.

However, don't rely on the "cosine error" – at all times make sure that you drive, or are driven, within the speed limit!

Question 1

1 A sine wave of the form $y = \sin nt°$ has a period of 0.1 seconds.

(a) Write down its frequency (number of waves in unit time). *(1 mark)*

(b) State the value of n. *(1 mark)*

(c) How would the wave $y = \cos nt°$ differ from the wave $y = \sin nt°$? *(1 mark)*

Student Response

Leave blank

1a) $0.1 = \frac{1}{10}$. so this is 10 waves ~~in unit time~~. ✓ 1

~~x 360 = 3600~~

b) $n = 10 \times 360 = 3600$ ✓ 1

c) ~~It would~~ $\cos nt°$ would have a phase shift of 90° to the left. ✓ 1

3

Commentary

Question 1 tested candidates' understanding of the properties of the general sine wave, $y = \sin nt°$ and how these relate to the numerical value of *n*. This was exemplified in the first part of the text of the pre-release data sheet, and some quoted the results for $n = 1440$. Candidates should expect to understand the general principles and apply or reinterpret these in different situations.

A wide variety of responses were produced to the question of part (c), with many candidates seemingly unable to work with transformations as required by the specifications. Credit was given for less technically accurate descriptions of the differences between sine and cosine waves based on transformational geometry. For example as show here those who recognised ideas of "phase shift" were rewarded as were those who described the different points of intersections of the two waves with the vertical axis.

Mark scheme

(a)	10 (waves per second)	**B1**	
(b)	3600	**B1**	
(c)	Same wave translated horizontally by $t = \frac{-90}{n}$	**B1**	Accept "translation to the left by $\frac{90}{n}$" Or when $t = 0$ $\sin nt = 0$ whereas $\cos nt = 1$ Condone translation of – 90
	TOTAL	**3**	

Question 2

2 A siren on a police car has a frequency of 1100 Hz. Sound travels through air at approximately $330\,m\,s^{-1}$.

(a) Find the wavelength of the sound waves if both the police car and observer are stationary. *(2 marks)*

(b) If the observer is stationary and the police car is moving **towards** the observer at a speed of $30\,m\,s^{-1}$, find:

(i) the apparent wavelength of the sound waves; *(2 marks)*

(ii) the apparent frequency of the sound waves; *(2 marks)*

(iii) the difference between the frequency of the transmitted wave and that received by the observer. *(1 mark)*

Student response

(Next Page)

Leave blank

2) Wavelength = λ

(a) Distance = v

Waves per second = f Hz

$\lambda = \frac{v}{f}$ (Where f = 1100, and v = 330 m/s^{-1})

$\lambda = \frac{330}{1100}$

$\lambda = 0.3$

Wavelength = 0.3

✓ 2

(b)

(i) Apparent Wavelength = $\lambda' = \frac{v - v_s}{f}$ (f = 1100, v = 330 m/s; V_s = 30 ~~[illegible]~~ ~~[illegible]~~)

$\lambda' = \frac{330 - 30}{1100}$

$\lambda' = \frac{300}{1100}$

Apparent Wavelength: $\lambda' = 0.272$

✓ 2

(ii) Apparent frequency = $f' = \frac{vf}{v - v_s}$

$f' = \frac{330 \times 1100}{330 - 30}$

$f' = \frac{363000}{300}$

Apparent Wavelength = $f' = 1210$

✓ 2

Leave blank

(iii) Frequency of wave .= Apparent wavelength was 1210.

$\frac{360}{1210}$: 0.3 [gives 18 waves per seconds ($\frac{3}{10} \times 60 = 18$)]

Difference = Transmitted - Recieved

= 0.3 - 0.3

difference = 0

✗ 0/1

Commentary

This question required candidates to work carefully with formulae given in the text of the article in the order that they were given.
Overall, the response illustrated here shows how a candidate did this effectively. The work throughout is exemplary in setting out very clearly at each stage the method used. This is very much to be encouraged as when a mistake is made in manipulating numerical values it may be possible to award marks for a correct method when there is a subsequent mistake made in calculation.
In response to the final part of the question this candidate appears to muddle a number of the key ideas working with $v + v_s$ rather than the previously calculated $v - v_s$ and additionally confusing apparent frequency and apparent wavelength.

Mark Scheme

(a)	$\lambda = \frac{v}{f} = \frac{330}{1100} = \frac{3}{10} = 0.3(\text{m})$	**M1** **A1**	M1 for $\frac{v}{f}$ or $\frac{330}{1100}$
(b)(i)	$\lambda' = \frac{v-v_s}{f} = \frac{330-30}{1100} = \frac{300}{1100} = 0.273(\text{m})$	**M1** **A1**	330 – 30 or 300 Accept 0.27
(ii)	$f' = \frac{\text{speed of wave}}{\text{apparent wavelength}} = \frac{330}{0.2727} = 1210(\text{Hz})$	**M1** **A1**	M1 for $330 \div (\text{b})(\text{i})$ Accept 1205 – 1222
(iii)	Difference $= 1210 - 1100 = 110(\text{Hz})$	**A1ft**	Their (b)(ii) – 1100
	TOTAL	**7**	

Question 3

3 (a) Show that a speed of 45 mph is $20.1\ \mathrm{m\,s^{-1}}$ to three significant figures. *(2 marks)*

(b) The article states that, for speed cameras, the difference in frequency, f_{diff}, that the camera measures is given by

$$f_{\mathrm{diff}} = 2\frac{v_{\mathrm{car}}}{c}f$$

(i) For the case of a car travelling **away** from a speed camera at 45 mph show that f_{diff} expressed as a fraction of the transmitted frequency is 1.34×10^{-7}.
(3 marks)

(ii) Find f_{diff} in Hz when the frequency of waves transmitted from this speed camera is 24 GHz. *(2 marks)*

Student Response

3. a) $45 \times 0.447 = 20.1\ \mathrm{ms^{-1}}$ ✓ 2.

b) $f_{diff} = 2\frac{V_{car}}{c}f$

i) $f_{diff} = \frac{2 \times 45 \times 0.447 \times f}{3 \times 10^8} = \frac{40.23 f}{3 \times 10^8} = \frac{13.41 f}{10^8}$ ✓

$f_{diff} = \frac{1.341 f}{10^7} = 1.34 \times 10^{-7} f$ ✓ 3

ii) $f_{diff} = \frac{2 \times 45 \times 0.447 \times 2.4 \times 10^{10}}{3 \times 10^8} = \frac{96.55 \times 10^{10}}{3 \times 10^8} = 3218.4\ \mathrm{Hz}$ ✓ 2

7

Commentary

Candidates had mixed fortunes with this question, particularly in part (b) in which they were asked to show how a numerical result is arrived at. The vast majority, as exemplified here, did not understand how to show one value expressed as a fraction of another. Credit was given to work of the form shown here as it was interpreted that the numerical value when clearly expressed in this way indicates the required fraction.

Part (a) was most easily answered by reference to the conversion factor given in the text (as demonstrated here) although a significant number of candidates demonstrated their own methods of conversion. Unfortunately some of these gave unacceptably inaccurate answers. Candidates should be encouraged to refer closely to the original text as questions are always derived from this.

Even where candidates had misinterpreted what was required in earlier parts of the question they often made a fresh start with part (c) and gained full credit. This is to be encouraged.

Mark Scheme

(a)	$45 \text{ m.p.h.} = 45\times 0.447 = 20.115 \approx 20.1 \text{ ms}^{-1}$	**M1** **A1**	**SC1** 20 (with calculation)
(b)(i)	$\frac{f_{\text{diff}}}{f} = 2\frac{v_{\text{car}}}{c} = 2\left(\frac{20.1}{3\times 10^8}\right) = 1.34\times 10^{-7}$	**M1** **M1** **A1**	M1 for $\frac{f_{\text{diff}}}{f}$ M1 for inserting 20.1, 3×10^8
(ii)	$f_{\text{diff}} = 1.34\times 10^{-7} \times 2.4\times 10^{10} = 3216 = 3220$	**M1** **A1ft**	M1 for their (b)(i) $\times 2.4\times 10^{10}$ **FT** from (b)(i) Accept 3218.4 rounding to 3220
	TOTAL	**7**	

Question 4

4 Another speed camera transmits radar waves with a frequency of 35 GHz. It detects a difference in frequency, f_{diff}, of 4000 Hz when a car is travelling **away** from the camera.

The speed limit on the stretch of road is 40 mph.

Would the motorist be prosecuted for speeding?
Show calculations to support your answer. *(5 marks)*

Student Response

3×10 $\quad$ 2.7×10

A0

(4) $\quad 4000 = \dfrac{2 \times 40 \times 0.447 \times 3.5 \times 10^{10}}{3 \times 10^8}$ $\quad$ (4)

$\dfrac{2 \times 40 \times 0.447 \times 3.5 \times 10^{10}}{3 \times 10^8} = 4172 - 4000 = 172$ $\quad$ 4

0 (4)

Commentary

Although this question is unstructured involving multiple steps it was answered well by a relatively large number of candidates who recognised the method which was exemplified in the text. There were two approaches that were used by approximately equal proportions of candidates: one of which involved substitution of all known values and the direct calculation of the speed, the other exemplified here followed most closely the method shown in the pre-release data where the shift in frequency was calculated for a car travelling at 40 mph. Unfortunately a significant number of candidates misinterpreted the resulting value of 4172 Hz: because this is greater than the value of 4000 Hz quoted in the article it was assumed that the car was speeding. Candidates need to interpret mathematical statements and numerical values carefully in terms of the situation to which they relate.

	$f_{\text{diff}} = 4000 = 2\left(\frac{v_{\text{car}}}{c}\right) \times f$	**M1**	Use of any equation equivalent to $f_{\text{diff}} = 2\left(\frac{v_{\text{car}}}{c}\right) \times f$
	$= 2\left(\frac{v_{\text{car}}}{3\times10^8}\right) \times 3.5\times10^{10} = 233.33\, v_{\text{car}}$	**A1**	Substitution for *c* and *f*
	$\therefore v_{\text{car}} = \frac{4000}{233.333} = 17.14 \text{ ms}^{-1}$		
		M1	Rearrange to make equation $v_{\text{car}} = \ldots$
	$= \frac{17.14}{0.447} = 38.4 \text{ m.p.h}$	**A1**	Accept 38.3 or 38
	No – below speed limit	**B1ft**	Dependent on at least M1 above
	ALTERNATIVE METHOD		
		(M1)	For 40×0.447
	$f_{\text{diff}} = \frac{2\times40\times0.447\times3.5\times10^{10}}{3\times10^8}$	**(M1)**	Use of 3.5×10^{10}
		(A1)	Correct equation structure (may be implied by working)
	$= 4172$	**(A1)**	Answer
	No - f_{diff} needs to be greater than 4172 to be breaking the speed limit	**(B1ft)**	Dependent on at least M1 above
	TOTAL	**5**	

Question 5

5 Because speed cameras are positioned to the side of the road, the speed of a car as measured by the camera is a fraction of its actual speed given by $v_{\text{car}} = v_{\text{actual}} \cos\theta$. The positioning of the camera affects $\cos\theta$.

A camera is positioned so that the point on which it is focused is 10 metres away along the line of travel of the car and the sideways displacement of the camera is 2.5 metres.

(a) Show that $\dfrac{v_{\text{car}}}{v_{\text{actual}}} = 0.970$. *(2 marks)*

(b) For this camera position find the angle θ. *(2 marks)*

Student Response

Leave blank

5)a) $\dfrac{V_{car}}{V_{actual}} = 0.970$

$\cos\theta = \dfrac{V_{car}}{V_{actual}} \Rightarrow \cos\theta = \dfrac{d}{\sqrt{d^2+x^2}}$

$\dfrac{V_{car}}{V_{actual}} = \dfrac{10}{\sqrt{10^2+2.5^2}} = \dfrac{10}{\sqrt{106.25}} = \underline{0.970}$ ✓ 2

b) $\cos\theta = 0.970$

$\therefore \theta = \cos^{-1} 0.970$

$\theta = \underline{14.0^\circ}$ ✓ 2

4

Commentary

This candidate's response to question 5 shows correct solutions to both parts of the question that were unfortunately rare. Again candidates were required to substitute given values into a formula found in the text. In part (b) candidates were expected to use the inverse cosine function on their graphic calculator to determine the angle. This simple action was completed by only very few candidates, although a small number who had been unable to answer part (a) successfully were able to use the result given to successfully calculate the appropriate angle

Mark Scheme

(a)	$\frac{v_{\text{car}}}{v_{\text{actual}}} = \frac{d}{\sqrt{d^2+x^2}} = \frac{10}{\sqrt{10^2+2.5^2}}$ $= 0.970(1425)$	**M1** **A1**	M1 for $\frac{10}{\sqrt{10^2+2.5^2}}$
(b)	$\theta = \cos^{-1}(0.970) = 14.1^{\circ}$	**M1** **A1ft**	$\cos^{-1}$ their (a) Accept 14 and 14.0…
	TOTAL	**4**	

Question 6

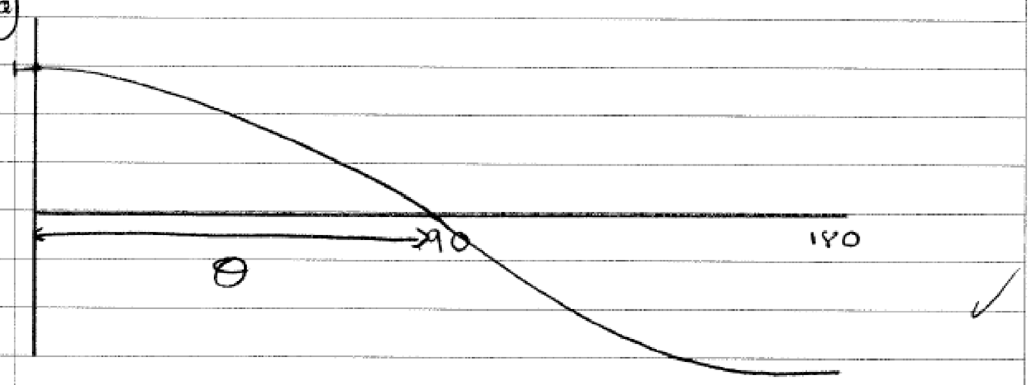
(a) Sketch a graph of $y = \cos\theta$ for $0° \leqslant \theta \leqslant 90°$.
Show clearly intercepts with both axes. *(2 marks)*

(b) Use your sketch to explain why the "cosine error" is always to the advantage of the motorist. *(2 marks)*

Student Response

Leave blank

6a)



✓ 2

b) Since cos of any number is always less than one except where θ cos 0 = 1 multiplying any number by a number less than 0 always makes the number smaller. ✓ 2

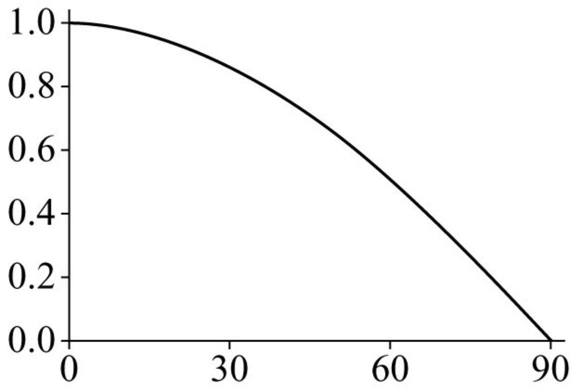
4

Commentary

The response shown here illustrates one of the better responses of candidates. The sketch graph in answer to part (a) shows the shape of a graph of $y = \cos\theta$ for θ varying between 0 and 90 and it also shows clearly where this cuts both axes. This is all that is required: this level of detail could be easily determined using a graphic calculators even if candidates are not familiar with the function. Candidates would benefit from having practised sketching a range of functions with reference to their graphic calculators.

The explanation given in part (b) shows understanding of the mathematics underpinning the reasoning referred to in the text of the article. Note that the slip in referring to a "number less than 0" is condoned as it is clear from the candidate's opening words that he or she understands that this is because $\cos\theta$ gives a fractional value.

Mark Scheme

(a)	 General shape passing through $(0,1)$ $(90,0)$	**B1** **B1**	Could sketch greater range
(b)	$\cos\theta<1,$ so that v_{car} will always be a fraction of v_{actual} meaning that the speed measured by the camera is less than the actual speed	**B1** **B1**	$\cos\theta<1$ Dependent on first B1
	TOTAL	**4**	
	TOTAL MARK FOR PAPER	**30**	