General Certificate of Education
June 2008
Advanced Subsidiary Examination

APPLYING MATHEMATICS **UOM4/1PM**
Paper 1

PRELIMINARY MATERIAL

DATA SHEET

To be issued to candidates between Friday 2 May 2008 and Friday 9 May 2008

REMINDER TO CANDIDATES

YOU MUST **NOT** BRING THIS DATA SHEET
WITH YOU WHEN YOU SIT THE EXAMINATION.
A CLEAN COPY WILL BE MADE AVAILABLE.

Population growth

For many years, countries around the world have been interested in the growth of their populations for both economic and social reasons. At a global level, this is an important matter as it is clear that the world cannot support an ever-growing population. Some basic mathematical models have been developed to help us make sense of how populations grow and allow us to make predictions about their future size. Initially, these models were based on the ideas put forward by Thomas Malthus, a political economist who wrote an important book, 'An Essay on the Principle of Population', first published in 1798. In this, Malthus wrote about his concerns that the supply of food could not be increased at a sufficient rate to keep pace with the increase in population.

Figure 1

Thomas Malthus

Malthus suggested that populations grow exponentially. He wrote of this growth in terms of a series in which, to get to the next term (in this case, population size), you multiply by a constant growth factor. In this way each successive term is greater than the previous term and, consequently, each successive increase will be greater than the last, as each time you are starting with a larger value than you did previously.

This can be expressed mathematically by a recurrence relation of the form $P_{n+1} = (1 + r)P_n$, where P_n is the population at the start of a time period, r is the growth rate (population increase per unit time) and P_{n+1} is the population after one time period.

You might recognise this as being the same idea as that associated with compound interest. For example, imagine that you invest £100 in a bank account that gives interest at a rate of 5% per year. After one year the amount of money you have in the bank will become £105, after a second year this will become £110.25, after a third year £115.76, and so on.

Another mathematical way of expressing growth of this type is to use an exponential function of the form $P = P_0 e^{kt}$, where P is the population size at time t, P_0 is the initial population (at time $t = 0$) and k is a factor that depends on the growth rate.

One way of finding how the factor k used in the exponential function is linked to the growth rate, r, is to consider growth in the first year.

For example, consider again the case of compound interest above. For the first year of growth the recurrence relation gives $P_1 = 1.05 \times P_0$ and the exponential function gives $P_1 = P_0 \times e^k$. Equating these two expressions gives $1.05 = e^k$. So $k = 0.0488$.

In his essay, Malthus argued his case by referring to how long it takes populations to double in size, suggesting that human populations typically double in size every 25 years. To find the annual growth rate that this implies, you can substitute some values into the exponential function associated with such growth.

For example, if the population is P_0 when $t = 0$, then twenty five years later (ie when $t = 25$), $P = 2P_0$.

This gives $2P_0 = P_0 \times e^{25\times k}$ leading to $k = 0.0277$.

The annual growth rate can be found in this case by again using $(1+r) = e^k$, so r, the growth rate, is approximately 2.8%.

Figure 2

Year (AD)	World population in billions
1000	0.31
1250	0.40
1500	0.50
1750	0.79
1800	0.98
1850	1.26
1900	1.65
1910	1.75
1920	1.86
1930	2.07
1940	2.30
1950	2.52
1960	3.02
1970	3.70
1980	4.44
1990	5.27
2000	6.06
2010	6.79
2020	7.50
2030	8.11
2040	8.58
2050	8.91
2100	9.46
2150	9.75

This mathematics leads to the 'rule of 70' which links the growth rate to the time it takes a population to double. In the case above, $25 \times 2.8 = 70$. This suggests that to find an approximate value for the time it takes a population (or an amount of money gaining compound interest) to double, you should divide 70 by the growth rate expressed as a percentage. So, if you were to invest an amount of money with interest compounded at 5%, it would take approximately 14 years to double in value.

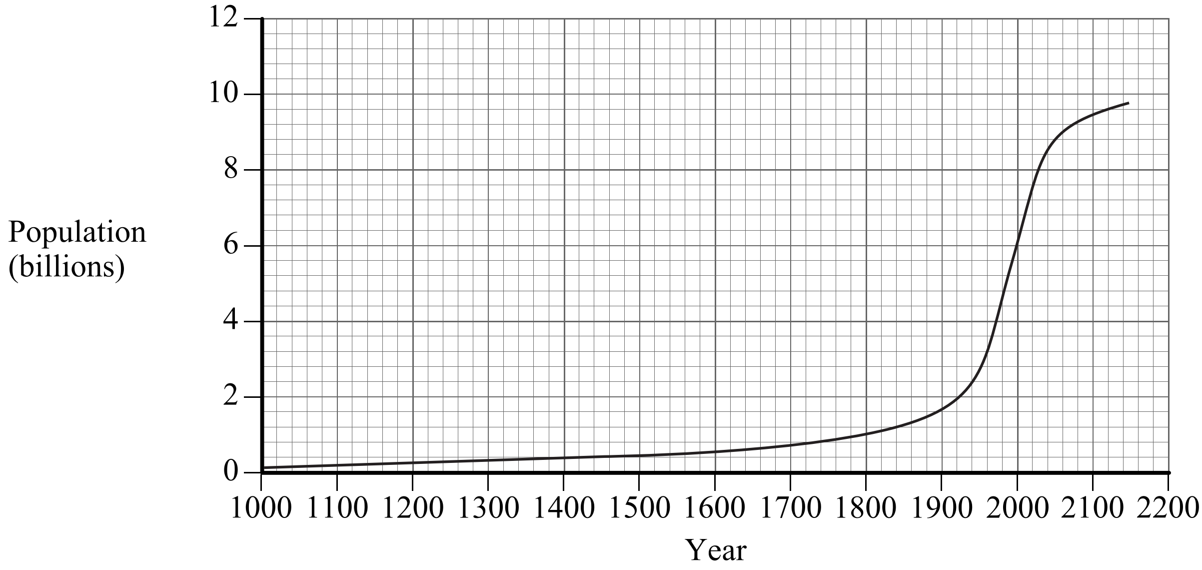
However, as you might suspect, the growth of human populations is not as simple as this: the growth rate varies from year to year, although for large populations this variation is not too great. Malthus realised this and wrote about variable population growth rates in later editions of his essay.

Let us now have a look at how these ideas might apply to the growth of the population of the world.

The data in **Figure 2** show the growth of the world's population over time and its predicted growth to 2150. It is clear that early values are estimates and future values are predictions.

Figure 3

Graph showing growth of the world's population and predicted growth to 2150

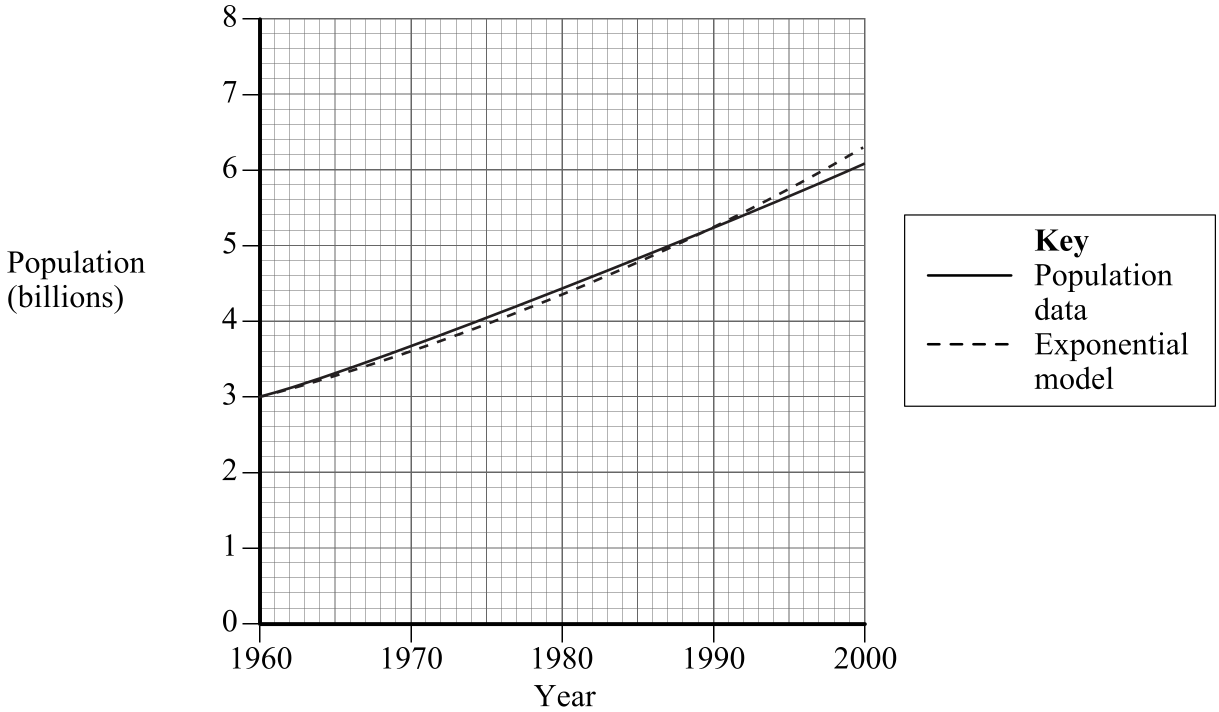


As the graph in **Figure 3** shows there are some periods where the growth of the world's population might be modelled effectively by an exponential function, but at other times a linear model might perhaps be equally effective.

For example, consider the period from 1960 to 2000. We have data for the world's population every ten years during this period and this is shown as a graph in **Figure 4**, together with an exponential model for the period. However, it does appear that, for many purposes, the data for the period 1960 to 2000 might equally well be modelled by a linear function.

Figure 4

Graph showing growth of the world's population and exponential model for the period 1960 to 2000



As you can see from the graph in **Figure 3**, the predicted values of the world's population look as though they are tending towards a limit. In some ways, this is inevitable: the Earth will not be able to sustain exponential growth of the human population as there are not enough resources in terms of food and energy.

To take account of this, a better model is to use a recurrence relation explored by the mathematician Verhulst in the mid-nineteenth century. This is known as the logistic equation and, in the form $P_{n+1} = P_n + 0.035P_n\left(1 - \frac{P_n}{9.8}\right)$, gives a good approximation to the world population data (and predicted growth) from 1960 onwards as shown in **Figure 5**.

Figure 5

Graph of world population data and predicted growth modelled by the logistic equation

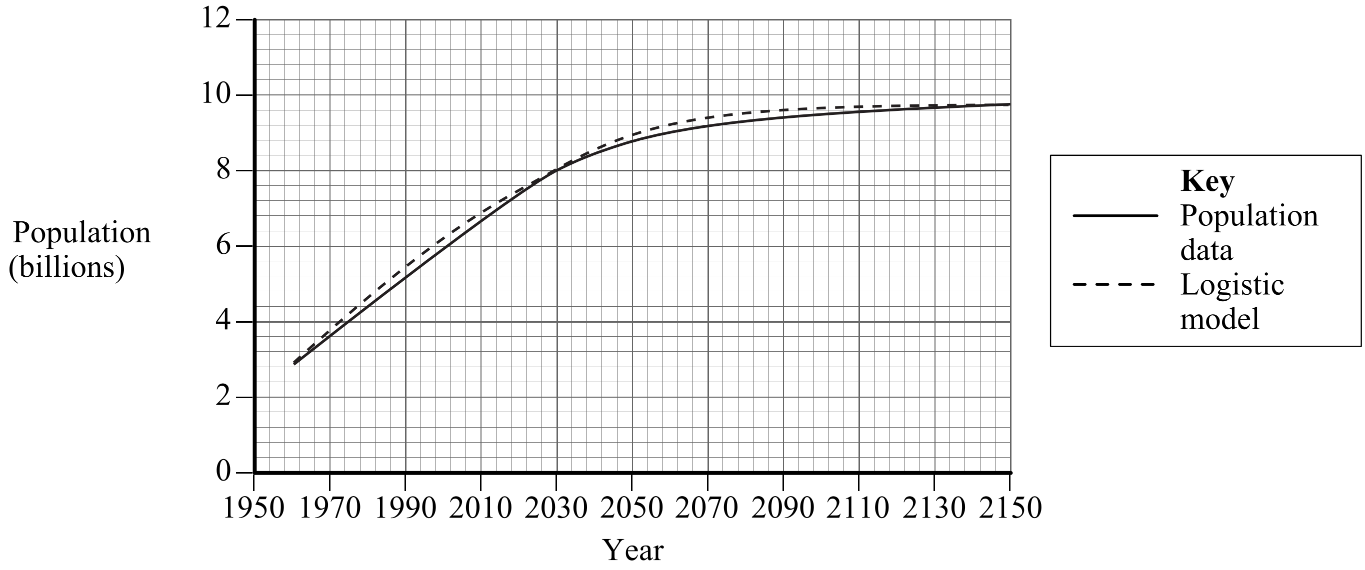


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Figure 1: Portrait of Thomas Robert Malthus (1766–1834) engraved by Fournier for the 'Dictionary of Political Economics', 1853 (engraving) (b/w photo) by Linnell, John (1792–1882) (after) © Bibliotheque Nationale, Paris, France / The Bridgeman Art Library Nationality / Copyright status: English / out of copyright.