Vectors
Straight lines in 3D

Vectors and Three–Dimensional Coordinate Geometry	
Applications of vectors to two- and three-dimensional geometry, involving points, lines and planes.	Including the equation of a line in the form $(\mathbf{r}-\mathbf{a})\times\mathbf{b}=0$. Vector equation of a plane in the form $\mathbf{r.\,n} = d$ or $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$. Intersection of a line and a plane. Angle between a line and a plane and between two planes.
Cartesian coordinate geometry of lines and planes. Direction ratios and direction cosines.	To include finding the equation of the line of intersection of two non-parallel planes. Including the use of $l^2+m^2+n^2=1$ where l, m, n are the direction cosines. Knowledge of formulae other than those in the formulae booklet will not be expected.

In formulae book

- If A is the point with position vector $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$ and the direction vector $\mathbf{b}$ is given by $\mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$, then the straight line through A with direction vector $\mathbf{b}$ has cartesian equation

$$\frac{x-a_1}{b_1}=\frac{y-a_2}{b_2}=\frac{z-a_3}{b_3}=\lambda$$

- The plane through A with normal vector $\mathbf{n}=n_1\mathbf{i}+n_2\mathbf{j}+n_3\mathbf{k}$ has cartesian equation

$$n_1x+n_2y+n_3z=d \text{ where } d=\mathbf{a.n}$$

- The plane through non-collinear points A, B and C has vector equation

$$\mathbf{r}=\mathbf{a}+\lambda(\mathbf{b}-\mathbf{a})+\mu(\mathbf{c}-\mathbf{a})=(1-\lambda-\mu)\mathbf{a}+\lambda\mathbf{b}+\mu\mathbf{c}$$

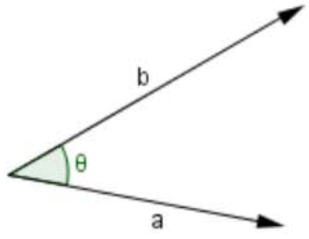
- The plane through the point with position vector $\mathbf{a}$ and parallel to $\mathbf{b}$ and $\mathbf{c}$ has equation

$$\mathbf{r}=\mathbf{a}+s\mathbf{b}+t\mathbf{c}$$

Reminder

Scalar and vector products
Relative position of vectors

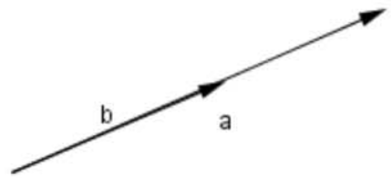
Consider the vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$

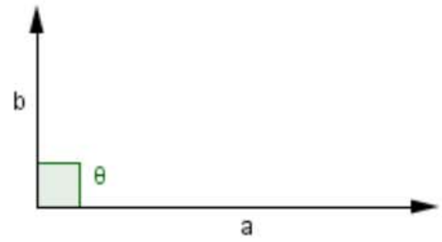


- **Angle between vectors.**

  If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, $\quad Cos\theta = \dfrac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$

  and $\quad Sin\theta = \dfrac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}||\mathbf{b}|}$



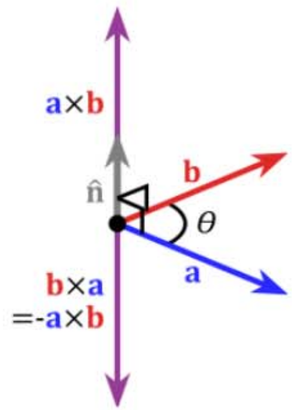


- **Parallel and perpendicular vectors.**

  If $\mathbf{a}$ and $\mathbf{b}$ are parallel, $\quad \mathbf{a}\times\mathbf{b} = 0$

  If $\mathbf{a}$ and $\mathbf{b}$ are perpendicular, $\quad \mathbf{a}.\mathbf{b} = 0$

  The vector $\mathbf{a}\times\mathbf{b}$ is a vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$

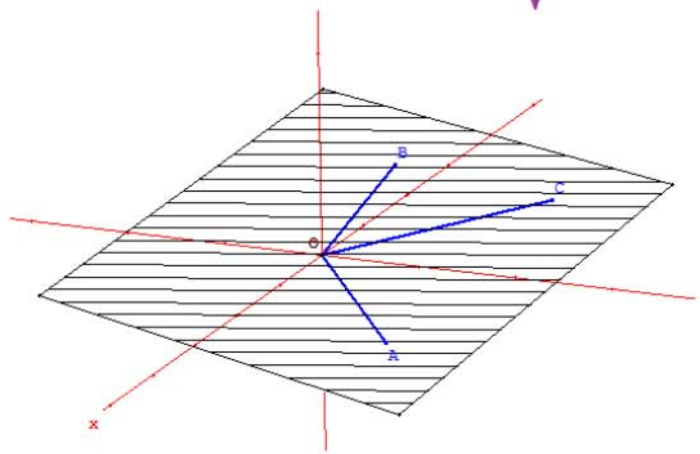


- **Coplanar vectors**

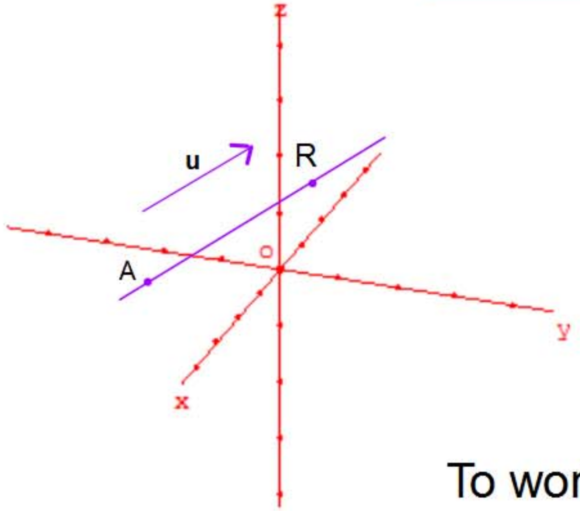
  The vectors $\mathbf{a}$,$\mathbf{b}$ and $\mathbf{c}$ are coplanar if it exists two real λ *and* μ so that $\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b}$.

  $\mathbf{a}, \mathbf{b}$ *and* $\mathbf{c}$ are coplanar if $\mathbf{a}.(\mathbf{b}\times\mathbf{c}) = 0$

  The vectors $\mathbf{a}$,$\mathbf{b}$ and $\mathbf{c}$ are coplanar is equivalent to say that the points O, A, B and C are coplanar.



Equations of a straight line in 3 dimensions



Like in 2D, to define a line in space, we need
a point and a direction
which is equivalent to : we need
a position vector and a direction vector

Let's call **a** the vector position of a point A and **u** a direction vector.
Consider the line going through A parallel to **u**.

To work out an equation of this line is to find a property satisfied by any point R, or vector position **r**, which belongs to this line.

The point R belongs to the line is equivalent to say that:

The vector $\overrightarrow{AR}$ is a multiple of $\mathbf{u}$

$$\overrightarrow{AR} = t\mathbf{u} \qquad \textit{where } t \in \mathbb{R}$$

$$\mathbf{r} - \mathbf{a} = t\mathbf{u}$$

$$\mathbf{r} = \mathbf{a} + t\mathbf{u} \qquad t \in \mathbb{R}$$

With components

$$\mathbf{r} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + t \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \quad t \in \mathbb{R}$$

This is a PARAMETRIC vector equation of the line

Examples:

- Find a vector equation of the straight line which passes through the point A, with position vector $3\mathbf{i} - 5\mathbf{j} + 4\mathbf{k}$, and is parallel to the vector $7\mathbf{i} - 3\mathbf{k}$.

- Find a vector equation of the straight line which passes through the points A and B, with coordinates $(4, 5, -1)$ and $(6, 3, 2)$ respectively.

- The straight line l has vector equation $\mathbf{r} = (3\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}) + t(\mathbf{i} - 6\mathbf{j} - 2\mathbf{k})$.
  Given that the point $(a, b, 0)$ lies on l, find the value of a and the value of b.

Exercises:

1) Find a vector equation of the straight line which passes through the point A, with position vector $\mathbf{a}$, and is parallel to the vector $\mathbf{b}$:

a $\mathbf{a} = 6\mathbf{i} + 5\mathbf{j} - \mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - 3\mathbf{j} - \mathbf{k}$

b $\mathbf{a} = 2\mathbf{i} + 5\mathbf{j}$, $\mathbf{b} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

c $\mathbf{a} = -7\mathbf{i} + 6\mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$

d $\mathbf{a} = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$

2) Find a vector equation for the line which passes through the points:

a (2, 1, 9) and (4, −1, 8)

b (−3, 5, 0) and (7, 2, 2)

c (1, 11, −4) and (5, 9, 2)

d (−2, −3, −7) and (12, 4, −3)

3) The point $(1, p, q)$ lies on the line l. Find the values of p and q, given that the equation is l is:

a $\mathbf{r} = (2\mathbf{i} - 3\mathbf{j} + \mathbf{k}) + t(\mathbf{i} - 4\mathbf{j} - 9\mathbf{k})$

b $\mathbf{r} = (-4\mathbf{i} + 6\mathbf{j} - \mathbf{k}) + t(2\mathbf{i} - 5\mathbf{j} - 8\mathbf{k})$

c $\mathbf{r} = (16\mathbf{i} - 9\mathbf{j} - 10\mathbf{k}) + t(3\mathbf{i} + 2\mathbf{j} + \mathbf{k})$

1 a $\mathbf{r} = \begin{pmatrix} 6 \\ 5 \\ -1 \end{pmatrix} + t\begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix}$ **b** $\mathbf{r} = \begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

c $\mathbf{r} = \begin{pmatrix} -7 \\ 6 \\ 2 \end{pmatrix} + t\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$ **d** $\mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} + t\begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$

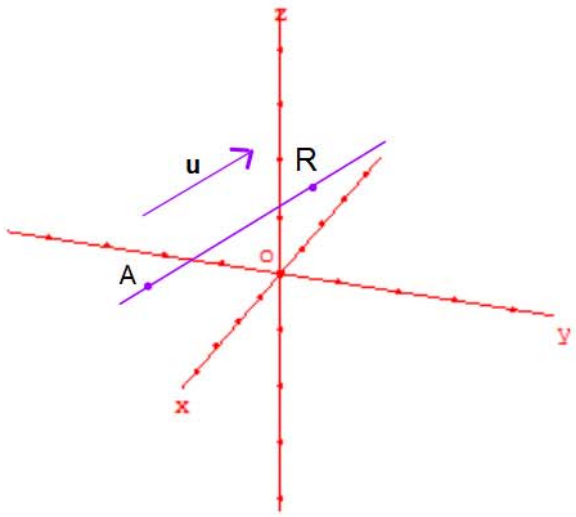
e $\mathbf{r} = \begin{pmatrix} 6 \\ -11 \\ 2 \end{pmatrix} + t\begin{pmatrix} 0 \\ 5 \\ -2 \end{pmatrix}$

a $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 9 \end{pmatrix} + t\begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}$ **b** $\mathbf{r} = \begin{pmatrix} -3 \\ 5 \\ 0 \end{pmatrix} + t\begin{pmatrix} 10 \\ -3 \\ 2 \end{pmatrix}$

c $\mathbf{r} = \begin{pmatrix} 1 \\ 11 \\ -4 \end{pmatrix} + t\begin{pmatrix} 4 \\ -2 \\ 6 \end{pmatrix}$ **d** $\mathbf{r} = \begin{pmatrix} -2 \\ -3 \\ -7 \end{pmatrix} + t\begin{pmatrix} 14 \\ 7 \\ 4 \end{pmatrix}$

a $p = 1, q = 10$
b $p = -6\frac{1}{2}, q = -21$
c $p = -19, q = -15$

Equation of a line - vector product



The point R belongs to the line is equivalent to say that:

The vector $\overrightarrow{AR}$ is parallel to the vector $\mathbf{u}$

so $\overrightarrow{AR} \times \mathbf{u} = 0$

$$(\mathbf{r} - \mathbf{a}) \times \mathbf{u} = 0$$

This can also be written : $\mathbf{r} \times \mathbf{u} - \mathbf{a} \times \mathbf{u} = 0$

$$\mathbf{r} \times \mathbf{u} = \mathbf{a} \times \mathbf{u}$$

With components

$$\left(\mathbf{r} - \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}\right) \times \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = 0$$

This is the VECTOR PRODUCT FORM form of the equation.

Example:

- Find the vector equation of the line through the points (1, 2, −1) and (3, −2, 2) in the form $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = 0$.

Exercises:

1 Find an equation of the straight line passing through the point with position vector **a** which is parallel to the vector **b**, giving your answer in the form $\mathbf{r} \times \mathbf{b} = \mathbf{c}$, where **c** is evaluated:

a $\mathbf{a} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ $\qquad \mathbf{b} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$

b $\mathbf{a} = 2\mathbf{i} - 3\mathbf{k}$ $\qquad \mathbf{b} = \mathbf{i} + \mathbf{j} + 5\mathbf{k}$

c $\mathbf{a} = 4\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ $\qquad \mathbf{b} = -\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$

3 Find, in the form $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = 0$, an equation of the straight line passing through the points with coordinates

a (1, 3, 5), (6, 4, 2) **b** (3, 4, 12), (4, 3, 5)

c (−2, 2, 6), (3, 7, 11) **d** (4, 2, −4), (1, 1, 1)

7 Given that the point with coordinates $(p, q, 1)$ lies on the line with equation

$\mathbf{r} \times \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 \\ -7 \\ -3 \end{pmatrix}$, find the values of p and q.

8 Given that the equation of a straight line is

$$\mathbf{r} \times \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

Hint: Let $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and set up simultaneous equations.

find an equation for the line in the form $\mathbf{r} = \mathbf{a} + t\mathbf{b}$, where t is a scalar parameter.

1 a $\mathbf{r} \times (3\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = -4\mathbf{i} + 10\mathbf{j} - \mathbf{k}$

b $\mathbf{r} \times (\mathbf{i} + \mathbf{j} + 5\mathbf{k}) = 3\mathbf{i} - 13\mathbf{j} + 2\mathbf{k}$

c $\mathbf{r} \times (-\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) = -4\mathbf{i} - 13\mathbf{j} - 10\mathbf{k}$

3 a $\left[\mathbf{r} - \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}\right] \times \begin{pmatrix} 5 \\ 1 \\ -3 \end{pmatrix} = 0$

b $\left[\mathbf{r} - \begin{pmatrix} 3 \\ 4 \\ 12 \end{pmatrix}\right] \times \begin{pmatrix} 1 \\ -1 \\ -7 \end{pmatrix} = 0$

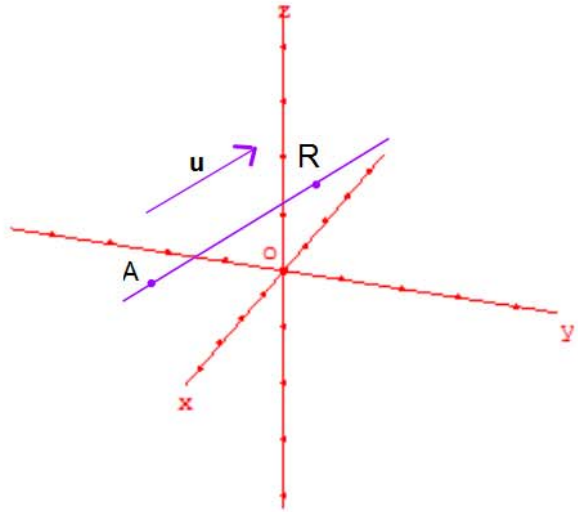
c $\left[\mathbf{r} - \begin{pmatrix} -2 \\ 2 \\ 6 \end{pmatrix}\right] \times \begin{pmatrix} 5 \\ 5 \\ 5 \end{pmatrix} = 0$

d $\left[\mathbf{r} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right] \times \begin{pmatrix} -3 \\ -1 \\ 5 \end{pmatrix} = 0$

7 $p = 3$ and $q = 3$

8 $\mathbf{r} = -\mathbf{j} + 2\mathbf{k} + t(\mathbf{i} + \mathbf{j} - \mathbf{k})$

Equation of a line - Cartesian equations - direction ratios



The parametric equation of a line can be written

$$\mathbf{r} = \mathbf{a} + t\mathbf{u} \qquad t \in \mathbb{R}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + t\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \text{ this gives}$$

$$\begin{cases} x = a_1 + tu_1 \\ y = a_2 + tu_2 \\ z = a_3 + tu_3 \end{cases} \quad \textit{now make } t \textit{ the subject:} \qquad \begin{cases} t = \dfrac{x-a_1}{u_1} \\ t = \dfrac{y-a_2}{u_2} \\ t = \dfrac{z-a_3}{u_3} \end{cases}$$

$t = \dfrac{x-a_1}{u_1} = \dfrac{y-a_2}{u_2} = \dfrac{z-a_3}{u_3}$ are the CARTESIAN equations of the lines.

This is also called the DIRECTION RATIO form of the equation because it can be obtained simply from the ratios of the x,y and z components of the direction vector of the line:

$$u_1 : u_2 : u_3 = x - a_1 : y - a_2 : z - a_3$$

Example:

Find the direction ratio form of the equation of the line through $A(1,-1, 4)$ and $B(2, 2, 3)$.

Exercises:

1 Find the cartesian equations for each of the line passing through the point with position vector **a** which is parallel to the vector **b**.

a $\mathbf{a} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ $\quad$ $\mathbf{b} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$

b $\mathbf{a} = 2\mathbf{i} - 3\mathbf{k}$ $\quad$ $\mathbf{b} = \mathbf{i} + \mathbf{j} + 5\mathbf{k}$

c $\mathbf{a} = 4\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ $\quad$ $\mathbf{b} = -\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$

2 Find, in the form **i** $\mathbf{r} \times \mathbf{b} = \mathbf{c}$, and also in the form **ii** $\mathbf{r} = \mathbf{a} + t\mathbf{b}$, where t is a scalar parameter, the equation of the straight line with Cartesian equation

$$\frac{(x-3)}{2} = \frac{(y+1)}{5} = \frac{(2z-3)}{3} = \lambda.$$

a $\frac{x-2}{3} = \frac{y-1}{1} = \frac{z-2}{-2} = \lambda$

b $\frac{x-2}{1} = \frac{y}{1} = \frac{z+3}{5} = \lambda$

c $\frac{x-4}{-1} = \frac{y+2}{-2} = \frac{z-1}{3} = \lambda$

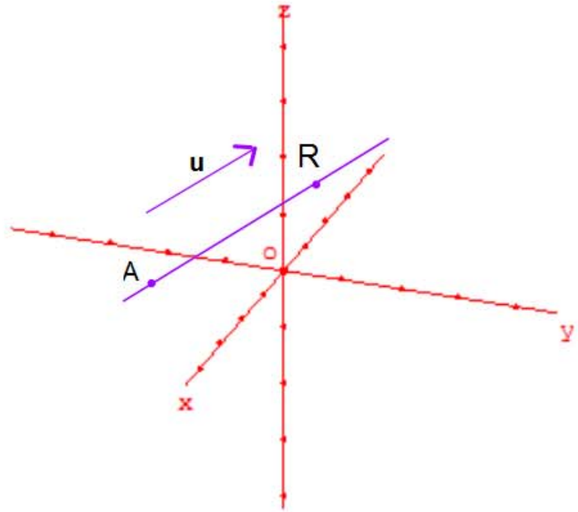
i $\mathbf{r} \times (2\mathbf{i} + 5\mathbf{j} + \frac{3}{2}\mathbf{k}) = -9\mathbf{i} - \frac{3}{2}\mathbf{j} + 17\mathbf{k}$

ii $\mathbf{r} = 3\mathbf{i} - \mathbf{j} + \frac{3}{2}\mathbf{k} + t(2\mathbf{i} + 5\mathbf{j} + \frac{3}{2}\mathbf{k})$

or $\mathbf{r} = 3\mathbf{i} - \mathbf{j} + \frac{3}{2}\mathbf{k} + s(4\mathbf{i} + 10\mathbf{j} + 3\mathbf{k})$

Summary

Equations of a line



A is a point with vector position $\mathbf{a}\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}$ *and* $\mathbf{u}\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}$ is a (direction) vector.

The point R has position vector $\mathbf{r}\begin{pmatrix}x\\y\\z\end{pmatrix}$.

We have seen that the line going through A following the direction of **u** has three type of equations.

R belongs to the line if and only if:

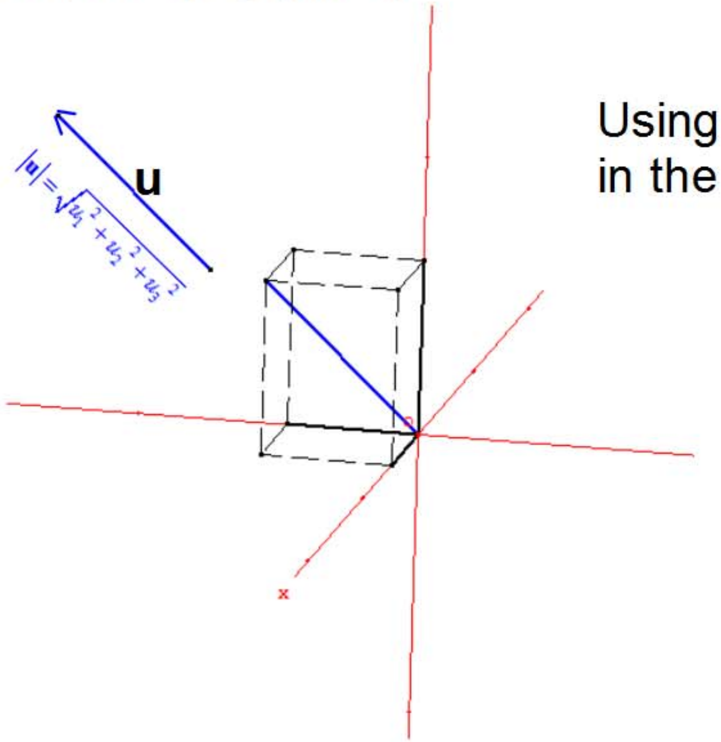
- $\mathbf{r}=\mathbf{a}+t\mathbf{u}$ *meaning* $\mathbf{r}=\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}+t\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}$ $t\in\mathbb{R}$ (*Parametric vector equation*)
- $(\mathbf{r}-\mathbf{a})\times\mathbf{u}=0$ *meaning* $\left(\begin{pmatrix}x\\y\\z\end{pmatrix}-\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\right)\times\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}=0$ (*Vector product equation*)
- $\dfrac{x-a_1}{u_1}=\dfrac{y-a_2}{u_2}=\dfrac{z-a_3}{u_3}$ (*Cartesian equations or direction ratios*)

Direction cosines

A line L has direction vector $\mathbf{u}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$.

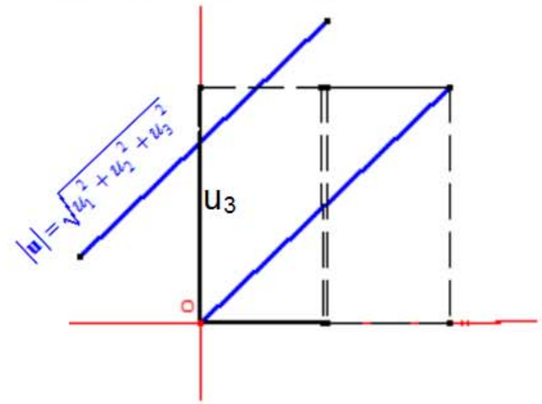
We consider the angles that the vector **u** makes with the x-, y- and z- axes.

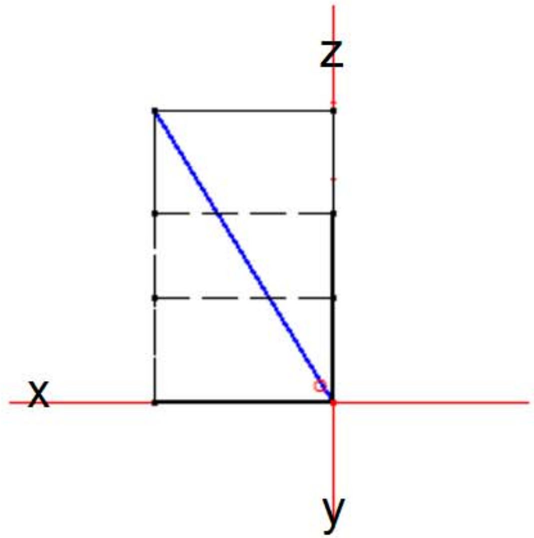
Respectively, θ_1, θ_2 *and* θ_3.



Using trigonometry
in the right-angle triangles, we have:

Alternative views





Conclusion:

A line L has direction vector $\mathbf{u}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$.

We consider the angles that the vector **u** makes with the x-, y- and z- axes.
Respectively, θ_1, θ_2 *and* θ_3.

$$Cos\ \theta_1 = \frac{u_1}{|\mathbf{u}|} = l \qquad Cos\ \theta_2 = \frac{u_2}{|\mathbf{u}|} = m \qquad Cos\ \theta_3 = \frac{u_3}{|\mathbf{u}|} = n$$

The quantities l, m *and* n are called the DIRECTION COSINES of the line.

Property:

If l, m *and* n are the direction cosines then $l^2 + m^2 + n^2 = 1$

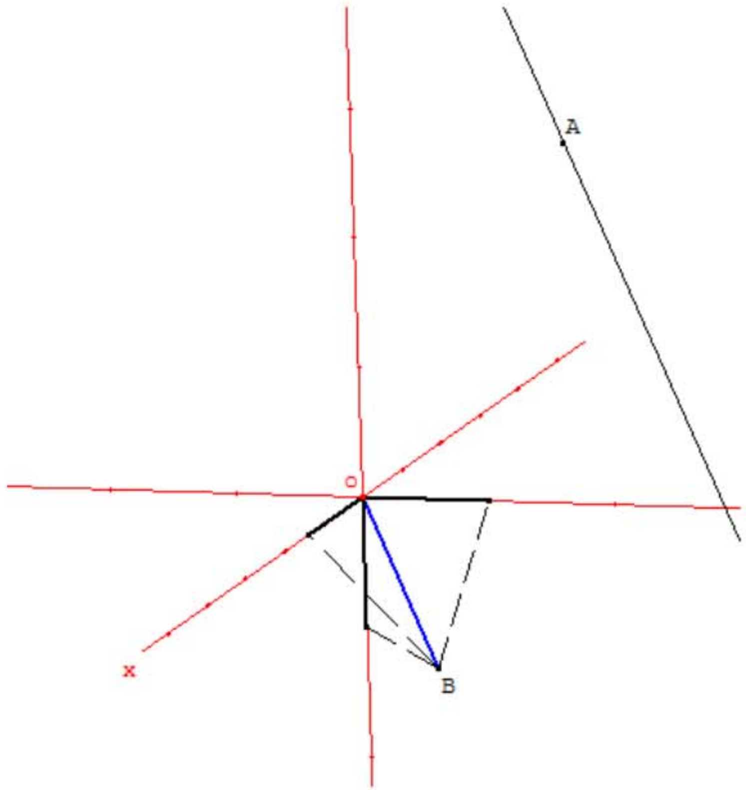
Proof:

Exercise:

Find the angles made by the line

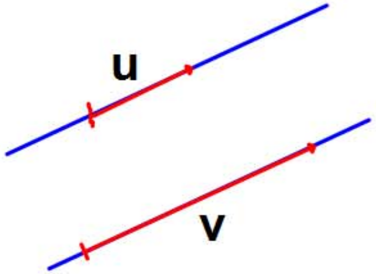
$$\frac{x-1}{\sqrt{2}} = \frac{y-2}{1} = \frac{z-3}{-1}$$

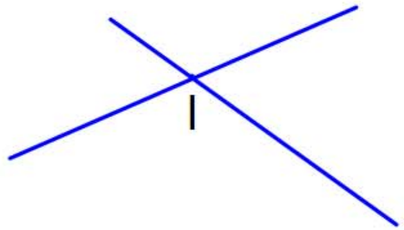
with the coordinate axes.

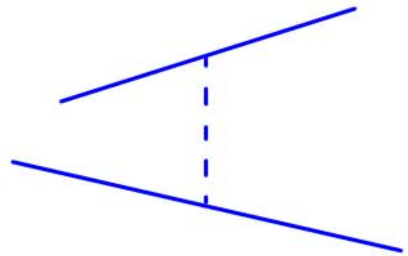


45º, 60º, 120º

Relative positions of two lines in space







Two lines can be parallel:
u is a multiple of **v**

Two lines can intersect.

Two lines can be SKEW

These two cases are treated the same way:
Solve simultaneously the equations

example:

$$\mathbf{r}=\begin{pmatrix}1\\2\\3\end{pmatrix}+t\begin{pmatrix}2\\-1\\4\end{pmatrix}\ \text{and}\ \mathbf{r}=s\begin{pmatrix}6\\-3\\12\end{pmatrix}$$

are parallel

$$\mathbf{v}=\begin{pmatrix}6\\-3\\12\end{pmatrix}=3\mathbf{u}=3\times\begin{pmatrix}2\\-1\\4\end{pmatrix}$$

If a solution exists, the line intersect

$$L_1:\mathbf{r}=\begin{pmatrix}2\\1\\2\end{pmatrix}+t\begin{pmatrix}2\\3\\-1\end{pmatrix}\ \text{and}$$

$$L_2:\mathbf{r}=\begin{pmatrix}3\\-2\\1\end{pmatrix}+s\begin{pmatrix}6\\0\\4\end{pmatrix}\ \text{intersect.}$$

Find the position vector of the point of intersection.

If there is no solution they are skew

$$\text{Show that}\ L_1:\mathbf{r}=\begin{pmatrix}3\\1\\-1\end{pmatrix}+t\begin{pmatrix}2\\3\\0\end{pmatrix}\ \text{and}$$

$$L_2:\mathbf{r}=\begin{pmatrix}1\\1\\1\end{pmatrix}+s\begin{pmatrix}2\\0\\3\end{pmatrix}\ \text{are skew.}$$

Exercises:

1) State which of the following lines are parallel to the line $r = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})$

l_1: $\mathbf{r} = -\mathbf{i} + \mathbf{j} + 2\mathbf{k} + \lambda(2\mathbf{i} - 4\mathbf{j} + 6\mathbf{k})$

l_2: $\mathbf{r} = 6\mathbf{i} - 4\mathbf{j} + 2\mathbf{k} + \lambda(-\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$

l_3: $\mathbf{r} = (2 - \lambda)\mathbf{i} + (-4 + 2\lambda)\mathbf{j} + (1 - 3\lambda)\mathbf{k}$

l_4: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

2) Show that the lines with equations $\mathbf{r} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k} + \lambda(2\mathbf{i} + 4\mathbf{j} + 3\mathbf{k})$ and $\mathbf{r} = \mathbf{i} + 2\mathbf{k} + \mu(2\mathbf{i} + \mathbf{j} + \mathbf{k})$ intersect and find the point of intersection.

a Check the direction vectors are not equal so the lines are not parallel.

b Equate the coefficients of **i** and **j** to find λ and μ.

c Show that with these values of λ and μ, the coefficients of **j** are equal.

d Find the point of intersection of the lines.

3) Show that the following lines intersect and find the position vector of the point of intersection.

a $\mathbf{r} = 4\mathbf{i} - 3\mathbf{j} + 2\mathbf{k} + \lambda(\mathbf{i} + 4\mathbf{j} + 3\mathbf{k})$
$\mathbf{r} = 3\mathbf{i} - \mathbf{j} + 5\mathbf{k} + \mu(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$

b $\mathbf{r} = \begin{pmatrix} -3 \\ 4 \\ -2 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

$\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + \mu\begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$

c $\mathbf{r} = (2 + \lambda)\mathbf{i} + (1 + 3\lambda)\mathbf{j} + (4 - 2\lambda)\mathbf{k}$
$\mathbf{r} = (7 - \mu)\mathbf{i} + (-8 + 3\mu)\mathbf{j} + (-6 + 2\mu)\mathbf{k}$

4) Find the coordinates of the point where the line $\mathbf{r} = \begin{pmatrix} 4 \\ 1 \\ 6 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$ meets the xy-plane.

5) Find the coordinates of the point where the line $\mathbf{r} = \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix} + \lambda\begin{pmatrix} 8 \\ 10 \\ 4 \end{pmatrix}$ meets the yz-plane.

6) Line l cuts the xy-plane at (4, 5, 0) and the yz-plane at (0, 15, 4).

a Find a vector equation of line l.

b Find the coordinates of the points where line l meets the xz-plane.

7) In parts **a**, **b** and **c** you are given a pair of lines. Determine whether the lines are parallel, skew or intersect at a point.

a $\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ 8 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix}$, $\mathbf{r} = \begin{pmatrix} 5 \\ 1 \\ 3 \end{pmatrix} + \mu\begin{pmatrix} -1 \\ 10 \\ 2 \end{pmatrix}$

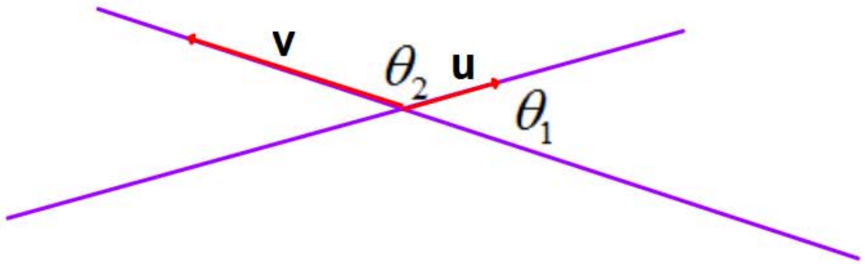
b $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + \lambda\begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix}$, $\mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \mu\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$

c $\mathbf{r} = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$, $\mathbf{r} = \begin{pmatrix} 1 \\ -5 \\ 2 \end{pmatrix} + \mu\begin{pmatrix} 6 \\ 9 \\ -3 \end{pmatrix}$

1) l_1 and l_3
2) $\lambda = 1, \mu = 2$, $5\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$
3) a) $\begin{pmatrix} 6 \\ 5 \\ 8 \end{pmatrix}$ b) $\begin{pmatrix} 3 \\ 7 \\ 1 \end{pmatrix}$ c) $\begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}$
4) (7, 10, 0)
5) (0, −2, −4)
6) a) $\mathbf{r} = \begin{pmatrix} 4 \\ 5 \\ 0 \end{pmatrix} + t\begin{pmatrix} 2 \\ -5 \\ -2 \end{pmatrix}$
b) (6, 0, −2)
7) a) skew
b) intersect (7, 2, −6)
c) parallel

Angles between two intersecting lines

If two lines intersect, you can work out the angle(s) between the lines



Note that when two lines cross, they form two angles:

an acute θ_1 and an obtuse angle θ_2

$$\theta_1 + \theta_2 = 180^\circ$$

(In the exam, they will tell you which one is required.)

To work out the ACUTE angle between the lines,

using the scalar product of **u** and **v**, work out $\quad \text{Cos}\,(\alpha) = \dfrac{\mathbf{u}.\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$

If α is acute, then $\theta_1 = \alpha$

If α is obtuse, then $\theta_1 = 180^\circ - \alpha$

Example:

Find the coordinates of the point of intersection of the lines l_1 and l_2 where

l_1 has equation $\mathbf{r} = 3\mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} - \mathbf{k})$ and

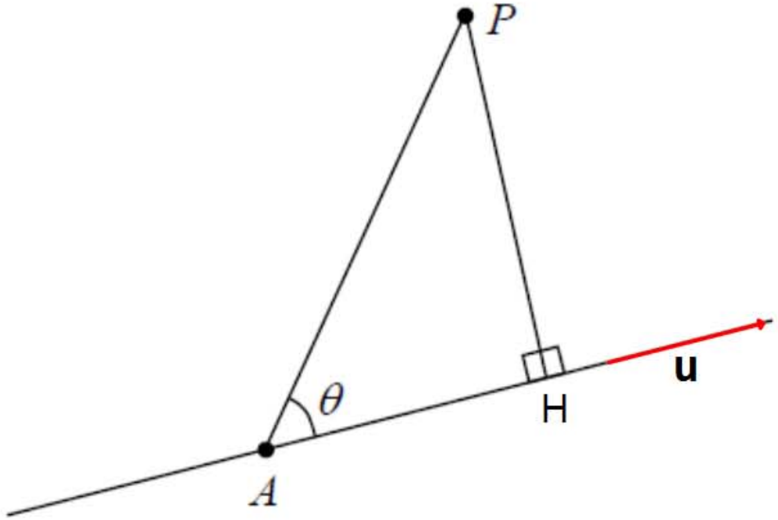
l_2 has equation $\mathbf{r} = -2\mathbf{j} + 3\mathbf{k} + \mu(-5\mathbf{i} + \mathbf{j} + 4\mathbf{k})$

Find the acute angle between the lines. Give your answer rounded to two decimal places

(5,-3,-1)

46.14°

Shortest distance between a point and a line



The line L goes through the point A, with vector position **a** and is parallel to the vector **u**.

The point P, with vector position **p** does not belong to the line.

The perpendicular distance from the point P to the line is $d = PH = AP \times \text{Sin}\,\theta$ where θ is the angle between $\overrightarrow{AP}$ and **u**

$$d = \frac{\left|\overrightarrow{AP}\right| \times |\mathbf{u}| \times Sin\theta}{|\mathbf{u}|} \quad = \quad d = \frac{\left|\overrightarrow{AP} \times \mathbf{u}\right|}{|\mathbf{u}|}$$

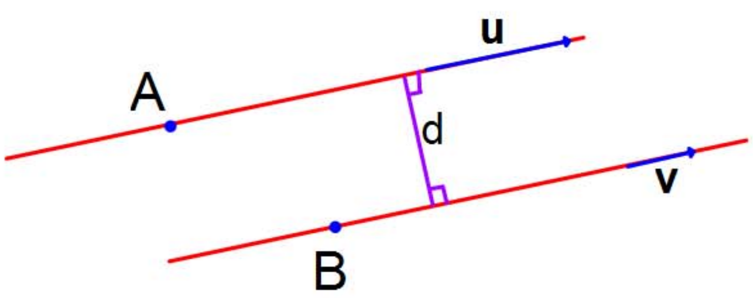
Example:

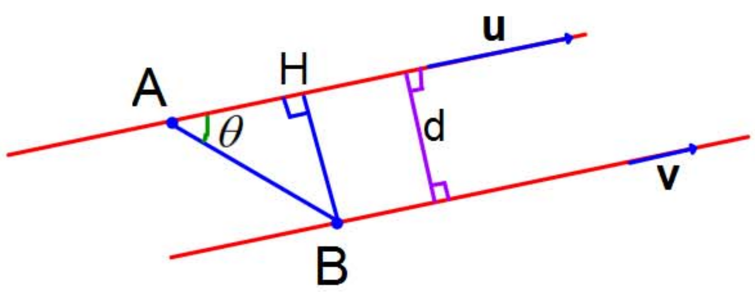
Find the perpendicular distance from the point $P(2, -1, 3)$ to the straight line with the equation

$$x - 2 = \frac{y+2}{3} = \frac{z-1}{2}.$$

$$\sqrt{\frac{3}{2}}$$

Shortest distance between two parallel lines





Using trigonometry, we establish

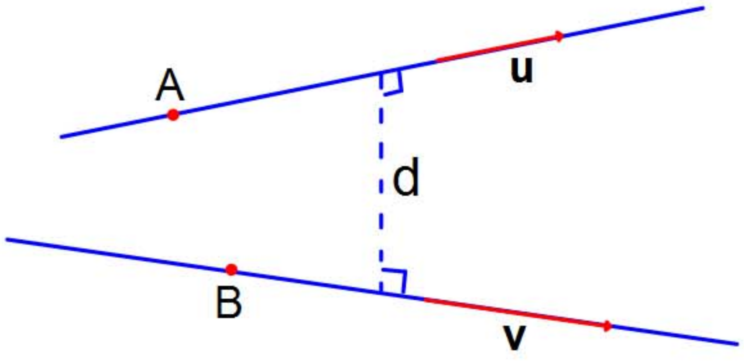
that $d = BH = AB \times Sin\theta$ *where* θ is the angle between $\overrightarrow{AB}$ and **u**

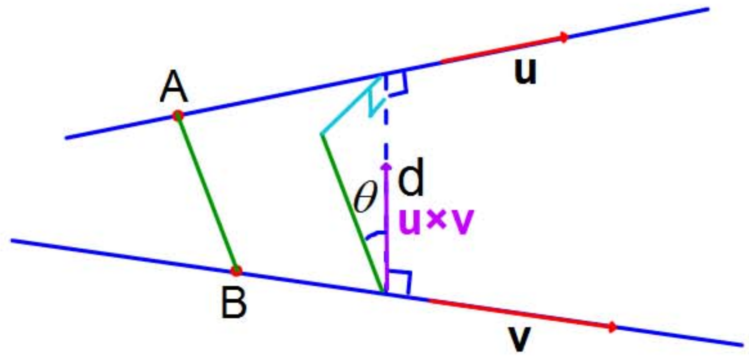
$$d = \frac{\left|\overrightarrow{AB} \times \mathbf{u}\right|}{|\mathbf{u}|} \text{ also } d = \frac{\left|\overrightarrow{AB} \times \mathbf{v}\right|}{|\mathbf{v}|}$$

Try it:

Show that the shortest distance between the parallel lines with equations $\mathbf{r} = \mathbf{i} + 2\mathbf{j} - \mathbf{k} + \lambda(5\mathbf{i} + 4\mathbf{j} + 3\mathbf{k})$ and $\mathbf{r} = 2\mathbf{i} + \mathbf{k} + \mu(5\mathbf{i} + 4\mathbf{j} + 3\mathbf{k})$, where λ and μ are scalars, is $\frac{21\sqrt{2}}{10}$.

Shortest distance between two skew lines





Using trigonometry, we establish

that d $= AB \times Cos\theta$ *where* θ is the angle between $\overrightarrow{AB}$ and $\mathbf{u} \times \mathbf{v}$

$$d = \frac{\left|\overrightarrow{AB}.(\mathbf{u} \times \mathbf{v})\right|}{|\,\mathbf{u} \times \mathbf{v}\,|}$$

Try it

Find the shortest distance between the two skew lines with equations $\mathbf{r} = \mathbf{i} + \lambda(\mathbf{j} + \mathbf{k})$ and $\mathbf{r} = -\mathbf{i} + 3\mathbf{j} - \mathbf{k} + \mu(2\mathbf{i} - \mathbf{j} - \mathbf{k})$, where λ and μ are scalars.

$2\sqrt{2}$

Exercises:

1) Find the shortest distance between the two skew lines with equations
$\mathbf{r} = \mathbf{i} + \lambda(-3\mathbf{i} - 12\mathbf{j} + 11\mathbf{k})$ and $\mathbf{r} = 3\mathbf{i} - \mathbf{j} + \mathbf{k} + \mu(2\mathbf{i} + 6\mathbf{j} - 5\mathbf{k})$, where λ and μ are scalars.

2) Find the shortest distance between the parallel lines with equations
$\mathbf{r} = 2\mathbf{i} - \mathbf{j} + \mathbf{k} + \lambda(-3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k})$ and $\mathbf{r} = \mathbf{j} + \mathbf{k} + \mu(-3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k})$, where λ and μ are scalars.

3) Determine whether the lines l_1 and l_2 meet. If they do, find their point of intersection. If they do not, find the shortest distance between them. (In each of the following cases λ and μ are scalars.)

a l_1 has equation $\mathbf{r} = \mathbf{i} + \mathbf{j} + \lambda(2\mathbf{i} - \mathbf{j} + 5\mathbf{k})$ and
l_2 has equation $\mathbf{r} = -\mathbf{i} + \mathbf{j} + 2\mathbf{k} + \mu(2\mathbf{i} - 5\mathbf{j} + \mathbf{k})$

b l_1 has equation $\mathbf{r} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k} + \lambda(2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$ and
l_2 has equation $\mathbf{r} = \mathbf{i} - \mathbf{j} + 3\mathbf{k} + \mu(\mathbf{i} - \mathbf{j} + \mathbf{k})$

c l_1 has equation $\mathbf{r} = \mathbf{i} + \mathbf{j} + 5\mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j} - 2\mathbf{k})$ and
l_2 has equation $\mathbf{r} = -\mathbf{i} - \mathbf{j} + 2\mathbf{k} + \mu(\mathbf{i} + \mathbf{j} + \mathbf{k})$

4) Find the shortest distance between the point with coordinates $(4, 1, -1)$ and the line with equation
$\mathbf{r} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k} + \mu(2\mathbf{i} - \mathbf{j} - \mathbf{k})$, where μ is a scalar.

$\frac{13}{11}$

$x = \frac{\sqrt{198}}{5}$ or 2.81 (3 s.f.)

a The lines do not meet.
Distance $= \frac{4\sqrt{11}}{11}$ or 1.21

b Lines do not meet.
$x = 3\sqrt{2}$ or 4.24 (3 s.f.)

c Lines do not meet.
Shortest distance = 0.196 (3 s.f.)

3.54 (3 s.f.)

Miscellaneous questions

2. Find the distance of the point (1, 1, 2) from the line $\mathbf{r} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + \lambda \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$.

3. A line has the equation

$$\frac{x-1}{-2} = \frac{y-3}{4}, \quad z = 1.$$

Express this in the form $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \mathbf{0}$.

13. The line L passes through the point $A(4, 4, -3)$ and has vector equation

$$\mathbf{r} = \begin{bmatrix} 4 \\ 4 \\ -3 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}.$$

(a) Show that the line M which passes through the points $B(4, 6, 1)$ and $C(6, 7, 3)$ is parallel to the line L.

(b) (i) Given that angle ACB is θ, show that $\cos\theta = \frac{19}{21}$.

(ii) Express $\sin\theta$ in surd form.

(c) Hence, or otherwise, show that the shortest distance between the lines L and M is $k\sqrt{5}$, where k is a rational number to be determined.

[AQA-NEAB, 2001]

6. Four points are given by A (1, –2, 0), B (3, –3, –1), C (2, 3, –1) and D (3, 4, –5).

(a) Calculate $\overrightarrow{AB} \times \overrightarrow{CD}$.

(b) Hence find the shortest distance between AB and CD.

12. The lines L_1 and L_2 have vector equations $\mathbf{r} = (2+\lambda)\mathbf{i} + (-2-\lambda)\mathbf{j} + (7+\lambda)\mathbf{k}$ and $\mathbf{r} = (4+4\mu)\mathbf{i} + (26+14\mu)\mathbf{j} + (-3-5\mu)\mathbf{k}$, respectively, where λ and μ are scalar parameters.

(a) The vector $\mathbf{n} = -\mathbf{i} + a\mathbf{j} + b\mathbf{k}$, where a and b are integers, is perpendicular to both L_1 and L_2. Find the value of a and the value of b.

(b) The point P on L_1 and the point Q on L_2 are such that $\overrightarrow{PQ} = m\mathbf{n}$ for some scalar constant m.

(i) Determine the value of m.

(ii) Deduce the shortest distance between L_1 and L_2.

[AQA-NEAB, 2000]

Answers

2. $\frac{1}{6}\sqrt{30}$

3. $\left(\mathbf{r}-\begin{bmatrix}1\\3\\1\end{bmatrix}\right)\times\begin{bmatrix}-2\\4\\0\end{bmatrix}=\mathbf{0}$

6. (a) $3\mathbf{i}+9\mathbf{j}+3\mathbf{k}$ (b) $\frac{15}{11}\sqrt{11}$

12. (a) $\begin{bmatrix}1\\-1\\1\end{bmatrix}\times\begin{bmatrix}4\\14\\-5\end{bmatrix}=9\begin{bmatrix}-1\\1\\2\end{bmatrix} \Rightarrow a=1,\quad b=2$

(b)(i) $\overrightarrow{PQ}.\mathbf{n}=m\,\mathbf{n}.\mathbf{n}$ (ii) $\sqrt{6}$ units

$$\begin{bmatrix}2\\28\\-10\end{bmatrix}.\begin{bmatrix}-1\\1\\2\end{bmatrix}=6m \quad\Rightarrow m=1$$

13. (a) $\overrightarrow{BC}=\begin{bmatrix}2\\1\\2\end{bmatrix}$

(b)(i) $\begin{bmatrix}-2\\-1\\-2\end{bmatrix}.\begin{bmatrix}-2\\-3\\-6\end{bmatrix}=21\cos\theta \quad\Rightarrow\cos\theta=\frac{19}{21}$ (ii) $\sin\theta=\frac{4\sqrt{5}}{21}$

(c) $|AC|\sin\theta=\frac{4\sqrt{5}}{3}$