Finite series - exam questions

Question 1: Jan 2008

(a) Show that

$$(2r+1)^3 - (2r-1)^3 = 24r^2 + 2$$ *(3 marks)*

(b) Hence, using the method of differences, show that

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$ *(6 marks)*

Question 2: Jan 2009

(a) Given that $f(r) = \frac{1}{4}r^2(r+1)^2$, show that

$$f(r) - f(r-1) = r^3$$ *(3 marks)*

(b) Use the method of differences to show that

$$\sum_{r=n}^{2n} r^3 = \frac{3}{4}n^2(n+1)(5n+1)$$ *(5 marks)*

Question 3: Jan 2010

The sum to r terms, S_r, of a series is given by

$$S_r = r^2(r+1)(r+2)$$

Given that u_r is the rth term of the series whose sum is S_r, show that:

(a) (i) $u_1 = 6$; *(1 mark)*

(ii) $u_2 = 42$; *(1 mark)*

(iii) $u_n = n(n+1)(4n-1)$. *(3 marks)*

(b) Show that

$$\sum_{r=n+1}^{2n} u_r = 3n^2(n+1)(5n+2)$$ *(3 marks)*

Question 4: Jan 2006

(a) Show that

$$\frac{1}{r^2} - \frac{1}{(r+1)^2} = \frac{2r+1}{r^2(r+1)^2}$$ *(2 marks)*

(b) Hence find the sum of the first n terms of the series

$$\frac{3}{1^2 \times 2^2} + \frac{5}{2^2 \times 3^2} + \frac{7}{3^2 \times 4^2} + \ldots$$ *(4 marks)*

Question 5: June 2007

(a) Given that $f(r) = (r-1)r^2$, show that

$$f(r+1) - f(r) = r(3r+1)$$ *(3 marks)*

(b) Use the method of differences to find the value of

$$\sum_{r=50}^{99} r(3r+1)$$ *(4 marks)*

Question 6: Jan 2007

(a) Use the identity $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$ with $A = (r+1)x$ and $B = rx$ to show that

$$\tan rx \tan(r+1)x = \frac{\tan(r+1)x}{\tan x} - \frac{\tan rx}{\tan x} - 1$$

(4 marks)

(b) Use the method of differences to show that

$$\tan\frac{\pi}{50}\tan\frac{2\pi}{50} + \tan\frac{2\pi}{50}\tan\frac{3\pi}{50} + \ldots + \tan\frac{19\pi}{50}\tan\frac{20\pi}{50} = \frac{\tan\frac{2\pi}{5}}{\tan\frac{\pi}{50}} - 20$$

(5 marks)

Question 7: June 2009

(a) Given that

$$\frac{1}{4r^2-1} = \frac{A}{2r-1} + \frac{B}{2r+1}$$

find the values of A and B. *(2 marks)*

(b) Use the method of differences to show that

$$\sum_{r=1}^{n} \frac{1}{4r^2-1} = \frac{n}{2n+1}$$

(3 marks)

(c) Find the least value of n for which $\sum_{r=1}^{n} \dfrac{1}{4r^2-1}$ differs from 0.5 by less than 0.001 .

(3 marks)

Question 8: June 2010

(a) Express $\dfrac{1}{r(r+2)}$ in partial fractions. *(3 marks)*

(b) Use the method of differences to find

$$\sum_{r=1}^{48} \frac{1}{r(r+2)}$$

giving your answer as a rational number. *(5 marks)*

Question 9: June2008

(a) Given that

$$\frac{1}{r(r+1)(r+2)} = \frac{A}{r(r+1)} + \frac{B}{(r+1)(r+2)}$$

show that $A = \frac{1}{2}$ and find the value of B. *(3 marks)*

(b) Use the method of differences to find

$$\sum_{r=10}^{98} \frac{1}{r(r+1)(r+2)}$$

giving your answer as a rational number. *(4 marks)*

Finite series - exam questions

Question 1: Jan 2008

$(2r+1)^3-(2r-1)^3=(8r^3+12r^2+6r+1)-(8r^3-12r^2+6r-1)$

$24r^2+2$

$b)\ 24\sum_{r=1}^{n} r^2=\sum_{r=1}^{n}\left((2r+1)^3-(2r-1)^3-2\right)=\sum_{r=1}^{n}\left((2r+1)^3-(2r-1)^3\right)-\sum_{r=1}^{n}2$

$=3^3-1+5^3-3^3+7^3-5^3+...+$

$(2n-1)^3-(2n-3)^3+(2n+1)^3-(2n-1)^3-2n$

$=-1+(2n+1)^3-2n=-1+8n^3+12n^2+6n+1-2n$

$=8n^3+12n^2+4n$

$24\sum_{r=1}^{n} r^2=4n\left(2n^2+3n+1\right)=4n(2n+1)(n+1)$

$\sum_{r=1}^{n} r^2=\frac{1}{6}n(n+1)(2n+1)$

Question 2: Jan 2009

$a)\ f(r)-f(r-1)=\frac{1}{4}r^2(r+1)^2-\frac{1}{4}(r-1)^2r^2$

$=\frac{1}{4}r^2\left[(r+1)^2-(r-1)^2\right]$

$=\frac{1}{4}r^2\left(r^2+2r+1-r^2+2r-1\right)$

$=\frac{1}{4}r^2(4r)$

$f(r)-f(r-1)=r^3$

$b)\ \sum_{r=n}^{2n} r^3=\sum_{r=n}^{2n} f(r)-f(r-1)=f(n)-f(n-1)+$

$f(n+1)-f(n)+$

$f(n+2)-f(n+1)+$

$....+$

$f(2n-1)-f(2n-2)+$

$f(2n)-f(2n-1)$

all the terms cancel except $f(2n)-f(n-1)$

$\sum_{r=n}^{2n} r^3=f(2n)-f(n-1)$

$=\frac{1}{4}(2n)^2(2n+1)^2-\frac{1}{4}(n-1)^2n^2=\frac{1}{4}n^2\left[4(2n+1)^2-(n-1)^2\right]$

$=\frac{1}{4}n^2\left(16n^2+4+16n-n^2+2n-1\right)=\frac{1}{4}n^2\left(15n^2+18n+3\right)=\frac{3}{4}n^2\left(5n^2+6n+1\right)$

$\sum_{r=n}^{2n} r^3=\frac{3}{4}n^2(5n+1)(n+1)$

Question 3: Jan 2010

$S_r=r^2(r+1)(r+2)$

$a)\ i)\ S_r=u_1+u_2+u_3+...+u_{r-1}+u_r$

$so\ \ for\ \ r=1,\ \ S_1=u_1$

$u_1=1^2\times(2)\times(3)=u_1=6$

$ii)\ S_2=u_1+u_2=6+u_2$

$and\ \ S_2=2^2\times(3)\times(4)=48$

$u_2=48-6=$

$u_2=42$

$iii)\ u_n=S_n-S_{n-1}$

$=n^2(n+1)(n+2)-(n-1)^2(n)(n+1)$

$=n(n+1)\left[n(n+2)-(n-1)^2\right]$

$=n(n+1)\left(n^2+2n-n^2-1+2n\right)$

$u_n=n(n+1)(4n-1)$

$b)\ \sum_{r=n+1}^{2n} u_r=\sum_{r=n+1}^{2n} S_r-S_{r-1}=S_{n+1}-S_n+$

$S_{n+2}-S_{n+1}+$

$S_{n+3}-S_{n+2}+...$

$+S_{2n}-S_{2n-1}$

$\sum_{r=n+1}^{2n} u_r=S_{2n}-S_n=(2n)^2(2n+1)(2n+2)-n^2(n+1)(n+2)$

$=8n^2(2n+1)(n+1)-n^2(n+1)(n+2)$

$=n^2(n+1)\left[8(2n+1)-(n+2)\right]$

$=n^2(n+1)(15n+6)=3n^2(n+1)(5n+2)$

Question 4: Jan 2006

$a)\ \frac{1}{r^2}-\frac{1}{(r+1)^2}=\frac{(r+1)^2}{r^2(r+1)^2}-\frac{r^2}{r^2(r+1)^2}=\frac{r^2+2r+1-r^2}{r^2(r+1)^2}=\frac{2r+1}{r^2(r+1)^2}$

$b)\ \frac{3}{1^2\times2^2}+\frac{5}{2^2\times3^2}+\frac{7}{3^2\times4^2}+...+\frac{2n+1}{n^2(n+1)^2}=\sum_{r=1}^{n}\frac{2r+1}{r^2(r+1)^2}$ *and*

$\sum_{r=1}^{n}\frac{2r+1}{r^2(r+1)^2}=\sum_{r=1}^{n}\frac{1}{r^2}-\frac{1}{(r+1)^2}$

$=1-\frac{1}{4}+\frac{1}{4}-\frac{1}{9}+\frac{1}{9}-\frac{1}{16}+....+\frac{1}{(n-1)^2}-\frac{1}{n^2}+\frac{1}{n^2}-\frac{1}{(n+1)^2}$

All the terms cancel out except the first and the last one:

$\sum_{r=1}^{n}\frac{2r+1}{r^2(r+1)^2}=1-\frac{1}{(n+1)^2}$

Question 5: June 2007

a) $f(r)=(r-1)r^2$

$$f(r+1)-f(r)=r(r+1)^2-(r-1)r^2$$
$$=r\left[(r+1)^2-r(r-1)\right]$$
$$=r\left[r^2+2r+1-r^2+r\right]$$
$$=r(3r+1)$$

b) $\sum_{r=50}^{99} r(3r+1)=\sum_{r=50}^{99} f(r+1)-f(r)=\cancel{f(51)}-f(50)$

$$+\cancel{f(52)}-\cancel{f(51)}$$
$$+\cancel{f(53)}-\cancel{f(52)}$$
$$+....+\cancel{f(99)}-\cancel{f(98)}$$
$$+f(100)-\cancel{f(99)}$$

All the terms cancel except $f(100)-f(50)=99\times100^2-49\times50^2$

$$\sum_{r=50}^{99} r(3r+1)=867500$$

Question 6: Jan 2007

a) $Tan(A-B)=Tan((r+1)x-rx)=Tan(x)$

and $Tan((r+1)x-rx)=\dfrac{Tan((r+1)x)-Tan(rx)}{1+Tan((r+1)x)Tan(rx)}$

So $Tan\,x=\dfrac{Tan((r+1)x)-Tan(rx)}{1-Tan((r+1)x)Tan(rx)}$

$$1+Tan((r+1)x)Tan(rx)=\frac{Tan((r+1)x)-Tan(rx)}{Tan\,x}$$

$$Tan((r+1)x)Tan(rx)=\frac{Tan((r+1)x)}{Tan\,x}-\frac{Tan(rx)}{Tan\,x}-1$$

b) $Tan\frac{\pi}{50}Tan\frac{2\pi}{50}+Tan\frac{2\pi}{50}Tan\frac{3\pi}{50}+...+Tan\frac{19\pi}{50}Tan\frac{20\pi}{50}$

$$=\sum_{r=1}^{19}Tan(r\frac{\pi}{50})Tan((r+1)\frac{\pi}{50})=\sum_{r=1}^{19}\frac{Tan((r+1)\frac{\pi}{50})}{Tan\frac{\pi}{50}}-\frac{Tan(r\frac{\pi}{50})}{Tan\frac{\pi}{50}}-1$$

$$=\frac{\cancel{Tan(\frac{2\pi}{50})}}{Tan\frac{\pi}{50}}-\frac{Tan(\frac{\pi}{50})}{Tan\frac{\pi}{50}}-1+\cancel{\frac{Tan(\frac{3\pi}{50})}{Tan\frac{\pi}{50}}}-\cancel{\frac{Tan(\frac{2\pi}{50})}{Tan\frac{\pi}{50}}}-1+\cancel{\frac{Tan(\frac{4\pi}{50})}{Tan\frac{\pi}{50}}}-\cancel{\frac{Tan(\frac{3\pi}{50})}{Tan\frac{\pi}{50}}}-1$$

$$+...+\cancel{\frac{Tan(\frac{19\pi}{50})}{Tan\frac{\pi}{50}}}-\cancel{\frac{Tan(\frac{18\pi}{50})}{Tan\frac{\pi}{50}}}-1+\frac{Tan(\frac{20\pi}{50})}{Tan\frac{\pi}{50}}-\cancel{\frac{Tan(\frac{19\pi}{50})}{Tan\frac{\pi}{50}}}-1$$

all the terms cancel except $-\dfrac{Tan(\frac{\pi}{50})}{Tan\frac{\pi}{50}}+\dfrac{Tan(\frac{20\pi}{50})}{Tan\frac{\pi}{50}}-1-1-1-1..-1$

$$=-1+\frac{Tan(\frac{20\pi}{50})}{Tan\frac{\pi}{50}}-19=\frac{Tan(\frac{2\pi}{5})}{Tan\frac{\pi}{50}}-20$$

Question 7: June 2009

$$a)\ \frac{A}{2r-1}+\frac{B}{2r+1}=\frac{A(2r-1)+B(2r+1)}{(2r+1)(2r-1)}$$

$$=\frac{(2A+2B)r+A-B}{4r^2-1}$$

this is equal to $\frac{1}{4r^2-1}$ *when* $A-B=1$ *and* $2A+2B=0$

$$A=\frac{1}{2}\ and\ B=-\frac{1}{2}$$

$$b)\ \sum_{r=1}^{n}\frac{1}{4r^2-1}=\sum_{r=1}^{n}\frac{1}{2(2r-1)}-\frac{1}{2(2r+1)}=\frac{1}{2}-\frac{1}{6}+\frac{1}{6}-\frac{1}{10}+$$

$$\frac{1}{10}-\frac{1}{14}+\ldots+$$

$$\frac{1}{4n-6}-\frac{1}{4n-2}+$$

$$\frac{1}{4n-2}-\frac{1}{4n+2}$$

All the terms cancel except $\frac{1}{2}-\frac{1}{4n+2}$

$$\sum_{r=1}^{n}\frac{1}{4r^2-1}=\frac{1}{2}-\frac{1}{4n+2}=\frac{4n+2-2}{2(4n+2)}=\frac{4n}{4(2n+1)}=\frac{n}{2n+1}$$

$$c)\ \frac{1}{2}-\sum_{r=1}^{n}\frac{1}{4r^2-1}<0.001$$

$$\frac{1}{2}-\left(\frac{1}{2}-\frac{1}{4n+2}\right)<0.001 \qquad \frac{1}{4n+2}<0.001$$

$$4n+2>\frac{1}{0.001} \qquad 4n>998$$

$$n>249.5 \qquad n=250$$

Question 8: June 2010

$$a)\ \frac{1}{r(r+2)}=\frac{A}{r}+\frac{B}{(r+2)} \qquad \times r(r+2)$$

$$1=A(r+2)+Br$$

$r=0$ *gives* $2A=1$ $\qquad A=\frac{1}{2}$

$r=-2$ *gives* $-2B=1$ $\qquad B=-\frac{1}{2}$

$$\frac{1}{r(r+2)}=\frac{1}{2r}-\frac{1}{2(r+2)}$$

$$b)\ \sum_{r=1}^{48}\frac{1}{r(r+2)}=\sum_{r=1}^{48}\frac{1}{2r}-\frac{1}{2(r+2)}$$

$$=\frac{1}{2}-\frac{1}{6}+\frac{1}{4}-\frac{1}{8}+\frac{1}{6}-\frac{1}{10}+\ldots$$

$$\frac{1}{92}-\frac{1}{96}+\frac{1}{94}-\frac{1}{98}+\frac{1}{96}-\frac{1}{100}$$

$$\sum_{r=1}^{48}\frac{1}{r(r+2)}=\frac{1}{2}+\frac{1}{4}-\frac{1}{98}-\frac{1}{100}=\frac{894}{1225}$$

Question 9: June 2008

$$\frac{A}{r(r+1)}+\frac{B}{(r+1)(r+2)}=\frac{A(r+2)+Br}{r(r+1)(r+2)}=\frac{(A+B)r+2A}{r(r+1)(r+2)}$$

This expression is equal to $\frac{1}{r(r+1)(r+2)}$ *for*

$$A=\frac{1}{2}\ and\ B=-\frac{1}{2}$$

$$b)\ \sum_{r=10}^{98}\frac{1}{r(r+1)(r+2)}=\sum_{r=10}^{98}\frac{1}{2r(r+1)}-\frac{1}{2(r+1)(r+2)}$$

$$=\frac{1}{220}-\frac{1}{264}+\frac{1}{264}-\frac{1}{312}+\frac{1}{312}-\frac{1}{364}+$$

$$\ldots+\frac{1}{19012}-\frac{1}{19404}+\frac{1}{19404}-\frac{1}{19800}$$

All the terms cancel except $\frac{1}{220}-\frac{1}{19800}=\frac{89}{19800}$