Arc length and area of surface of revolution about the x-axis

Specifications

Arc length and Area of surface of revolution about the x–axis

Calculation of the arc length of a curve and the area of a surface of revolution using Cartesian or parametric coordinates.

Use of the following formulae will be expected:

$$s = \int_{x_1}^{x_2} \left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{\frac{1}{2}} dx \quad = \int_{t_1}^{t_2} \left[\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 \right]^{\frac{1}{2}} dt$$

$$S = 2\pi \int_{x_1}^{x_2} y \left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{\frac{1}{2}} dx = 2\pi \int_{t_1}^{t_2} y \left[\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 \right]^{\frac{1}{2}} dt$$

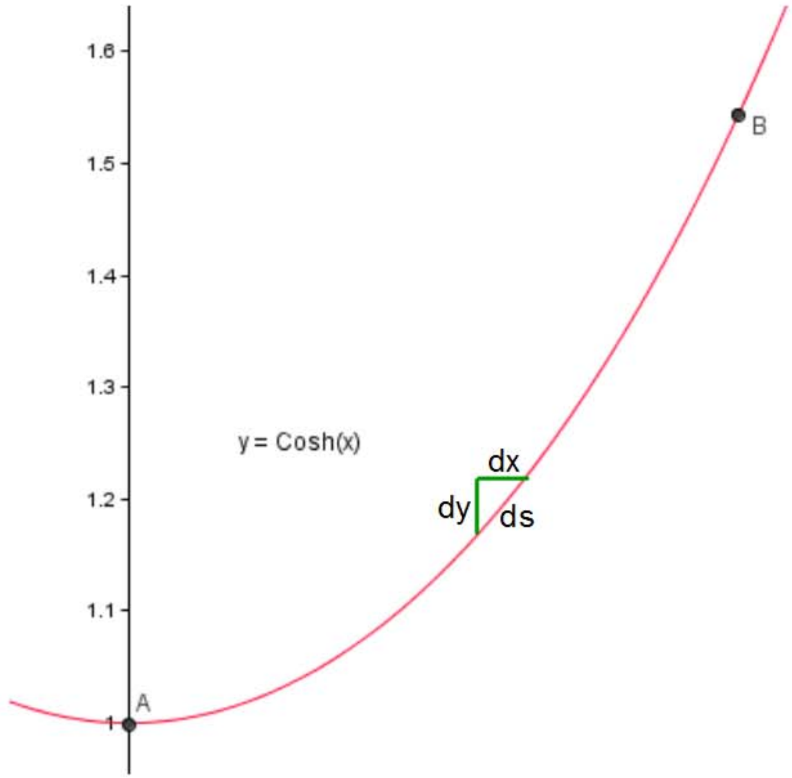
These formulae are given

Introduction

Consider the graph with equation y=f(x)
and two points A and B belonging to the curve.

The part of the curve between A and B is called an arc.

We want to work out the length of this arc.



Consider, on a very small scale an arc of this curve:ds

Using the pythagoras theorem,

$$(ds)^2 = (dx)^2 + (dy)^2 \quad (\div (dx)^2)$$

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$$

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \quad \textit{and by } \text{integrating}$$

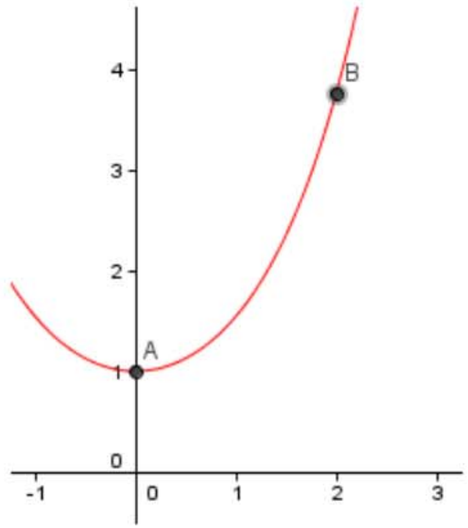
$$S = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Summary

If $y = \mathrm{f}(x)$, the length of the arc of curve from the point where $x = a$ to the point where $x = b$ is given by

$$s = \int_a^b \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$$

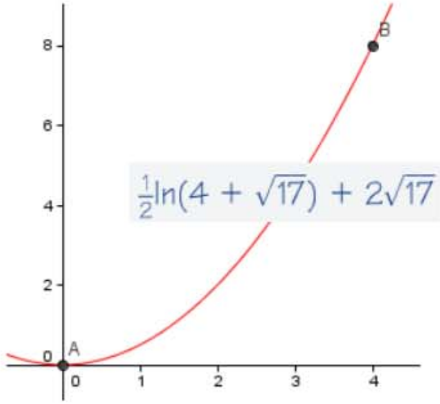
Find the length of the curve $y = \cosh x$ between the points where $x = 0$ and $x = 2$.



Arc length: sinh(2)

Exercises:

Find the exact length of the arc on the parabola with equation $y = \frac{1}{2}x^2$, from the origin to the point $P(4, 8)$.



1 Find the length of the arc of the curve with equation $y = \frac{1}{3}x^{\frac{3}{2}}$, from the origin to the point with x-coordinate 12.

2 The curve C has equation $y = \ln\cos x$. Find the length of the arc of C between the points with x-coordinates 0 and $\frac{\pi}{3}$.

3 Find the length of the arc on the catenary, with equation $y = 2\cosh\left(\frac{x}{2}\right)$, between the points with x-coordinates 0 and $\ln 4$.

5 The curve C has equation $y = \frac{1}{2}\sinh^2 2x$. Find the length of the arc on C from the origin to the point whose x-coordinate is 1, giving your answer to 3 significant figures.

6 The curve C has equation $y = \frac{1}{4}(2x^2 - \ln x)$, $x > 0$. The points A and B on C have x-coordinates 1 and 2 respectively. Show that the length of the arc from A to B is $\frac{1}{4}(6 + \ln 2)$.

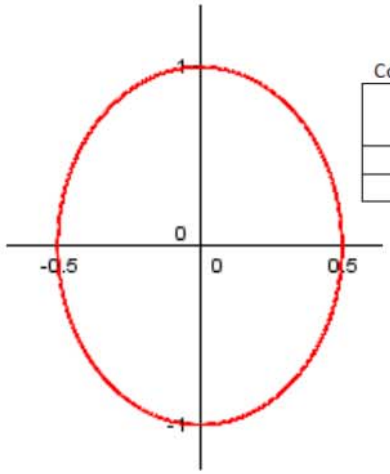
1 $\frac{56}{3}$ or $18\frac{2}{3}$
2 $\ln(2 + \sqrt{3})$
3 $\frac{3}{2}$
5 6.82 (s.f.)

Parametric equations

In parametric equations, x and y are functions of a parameter, usually t or θ.
(the position of a point (x and y coordinates) depends on the time)

Examples:

$$x(t) = Sin(t) \times Cos(t)$$
$$y(t) = Cos(2t) \qquad 0 \le t \le \pi$$



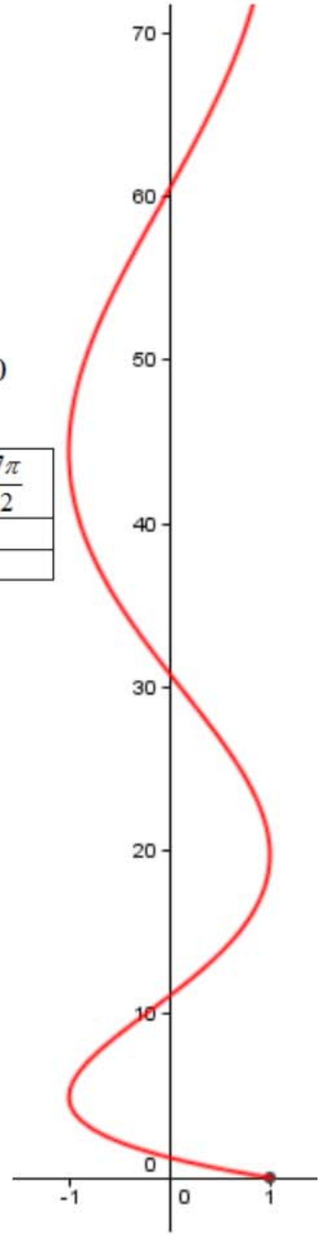
Complete the table and mark the point on the graph

t	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
x					
y					

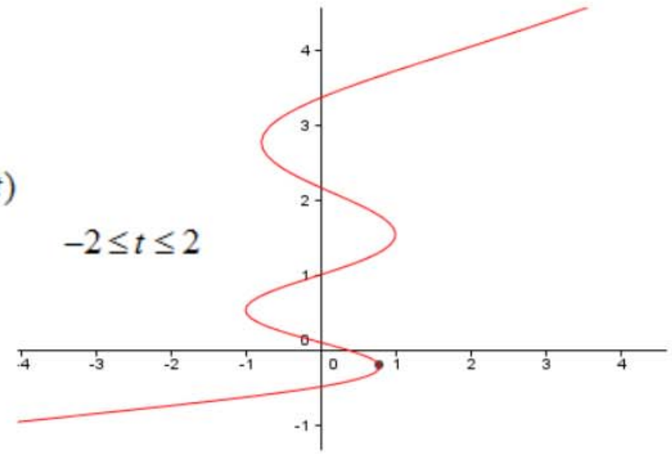
$$x(t) = \cos(t)$$
$$y(t) = \frac{1}{2}t^2 \qquad t \ge 0$$

Complete the table and mark the point on the graph

t	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π	$\frac{7\pi}{2}$
x								
y								



$$x(t) = t^5 + Sin(2\pi t)$$
$$y(t) = t + e^t \qquad -2 \le t \le 2$$

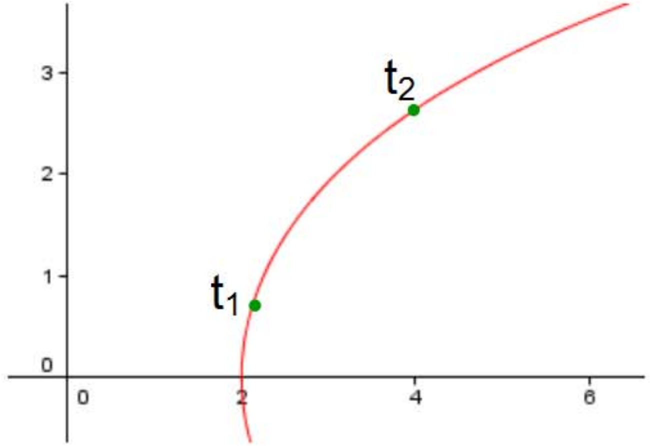


Length of arcs

The length of arc of a curve in terms of a parameter t is given by

$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt,$$

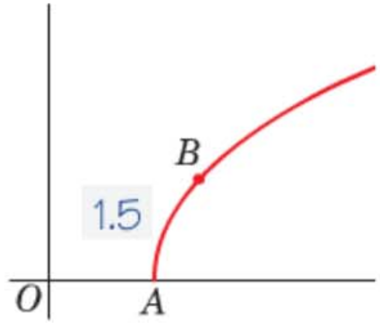
where t_1 and t_2 are the values of the parameter at each end of the arc



The curve C has parametric equations

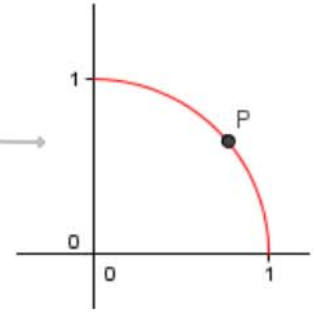
$$x = t + \frac{1}{t},\ y = 2\ln t,\ t > 0$$

Find the length of the arc between points A and B with $t = 1$ and $t = 2$ respectively.



Exercises:

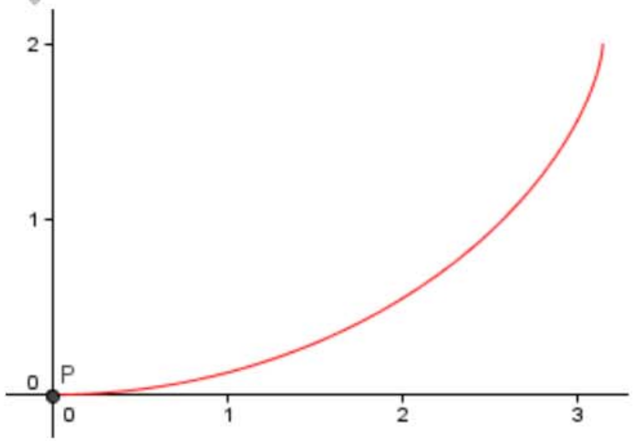
11 Calculate the length of the arc on the curve with parametric equations $x = \tanh u$, $y = \operatorname{sech} u$, between the points with parameters $u = 0$ and $u = 1$.

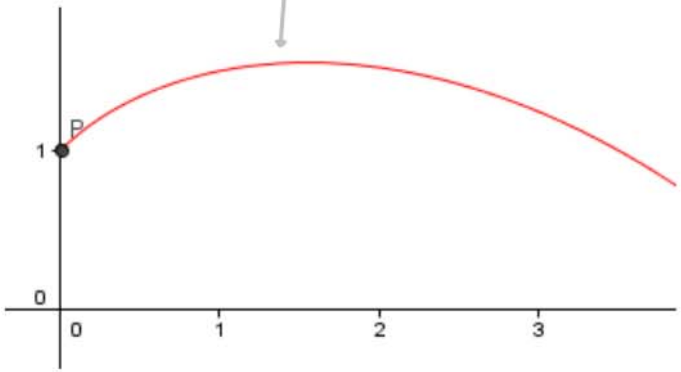


12 The cycloid has parametric equations $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$. Find the length of the arc from $\theta = 0$ to $\theta = \pi$.

13 Show that the length of the arc, between the points with parameters $t = 0$ and $t = \frac{\pi}{3}$ on the curve defined by the equations $x = t + \sin t$, $y = 1 - \cos t$, is 2.

14 Find the length of the arc of the curve given by the equations $x = e^t \cos t$, $y = e^t \sin t$, between the points with parameters $t = 0$ and $t = \frac{\pi}{4}$.



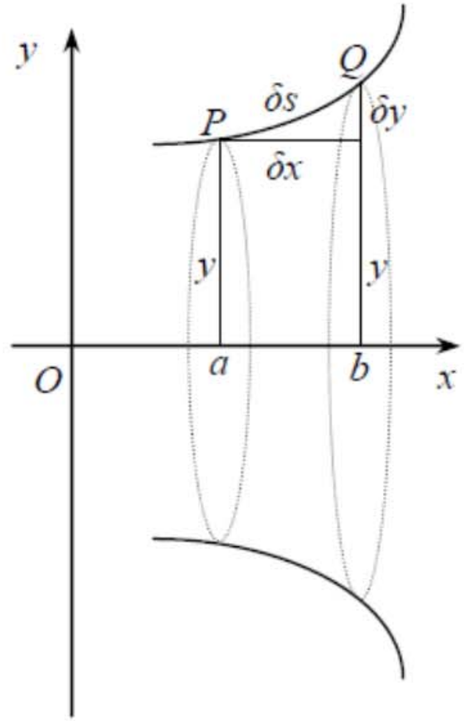


11 $2\arctan(e) - \frac{\pi}{2}$ or 0.866 (3 s.f.)

12 $4a$

14 $\sqrt{2}\,[e^{\frac{\pi}{4}} - 1]$ or 1.69 (3 s.f.)

Area of surface of revolution



The area of surface of revolution obtained by rotating an arc of the curve $y = f(x)$ through 2π radians about the x-axis between the points where $x = a$ and $x = b$ is given by

$$A = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

In parametric form

$$S = 2\pi \int_{t_A}^{t_B} y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

the curve $y = \cosh x$ between the points where $x = 0$ and $x = 2$ is rotated through 2π radians about the x-axis

Find the area of surface of revolution

$$\pi\left[2 + \frac{1}{2}\sinh 4\right]$$

A curve has parametric representation

$$x = \theta + \sin\theta, \quad y = 1 + \cos\theta, \qquad 0 \le \theta \le 2\pi.$$

(a) Prove that $\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = 4\cos^2\frac{\theta}{2}$.

(b) The arc of this curve, between the points when $\theta = 0$ and $\theta = \frac{\pi}{2}$ is rotated about the x-axis through 2π radians. The area of the surface generated is denoted by S.
Determine the value of the constant k for which

$$S = k\int_0^{\frac{\pi}{2}}\left(1 - \sin^2\frac{\theta}{2}\right)\cos\frac{\theta}{2}\, d\theta,$$

and hence evaluate S exactly.

2 The arc of the curve $y = x^3$, between the origin and the point (1, 1), is rotated through 4 right-angles about the x-axis. Find the area of the surface generated.

3 The arc of the curve $y = \frac{1}{2}x^2$, between the origin and the point (2, 2), is rotated through 4 right-angles about the y-axis. Find the area of the surface generated.

4 The points A and B, in the first quadrant, on the curve $y^2 = 16x$ have x-coordinates 5 and 12 respectively. Find, in terms π, the area of the surface generated when the arc AB is rotated completely about the x-axis.

5 The curve C has equation $y = \cosh x$. The arc s on C, has end points (0, 1) and (1, cosh 1).

a Find the area of the surface generated when s is rotated completely about the x-axis.

b Show that the area of the surface generated when s is rotated completely about the y-axis is $2\pi\left(\frac{e-1}{e}\right)$.

6 The curve C has equation $y = \frac{1}{2x} + \frac{x^3}{6}$.

a Show that $\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \frac{1}{2}\left(x^2 + \frac{1}{x^2}\right)$.

The arc of the curve between points with x-coordinates 1 and 3 is rotated completely about the x-axis.

b Find the area of the surface generated.

9 The finite arc of the parabola with parametric equations $x = at^2$, $y = 2at$, where a is a positive constant, cut off by the line $x = 4a$, is rotated through 180° about the x-axis. Show that the area of the surface generated is $\frac{8}{3}\pi a^2(5\sqrt{5} - 1)$.

10 The arc, in the first quadrant, of the curve with parametric equations $x = \operatorname{sech} t$, $y = \tanh t$, between the points where $t = 0$ and $t = \ln 2$, is rotated completely about the x-axis. Show that the area of the surface generated is $\frac{2\pi}{5}$.

11 The arc of the curve given by $x = 3t^2$, $y = 2t^3$, from $t = 0$ and $t = 2$, is completely rotated about the y-axis.

a Show that the area of the surface generated can be expressed as $36\pi\int_0^2 t^3\sqrt{1 + t^2}\,dt$.

b Using integration by parts, find the exact value of this area.

12 The arc of the curve with parametric equations $x = t^2$, $y = t - \frac{1}{3}t^3$, between the points where $t = 0$ and $t = 1$, is rotated through 360° about the x-axis. Calculate the area of the surface generated.

13 The astroid C has parametric equations $x = a\cos^3 t$, $y = a\sin^3 t$, where a is a positive constant. The arc of C, between $t = \frac{\pi}{6}$ and $t = \frac{\pi}{2}$, is rotated through 2π radians about the x-axis. Find the area of the surface of revolution formed.

2 $\frac{\pi}{27}[10\sqrt{10} - 1]$ or 3.56 (3 s.f.)

3 $\frac{2\pi}{3}[5\sqrt{5} - 1]$

4 $\frac{592}{3}\pi$

5 a 8.84 (3 s.f.)

6 b $23\frac{1}{9}\pi$

11 b $\frac{24\pi}{5}[25\sqrt{5} + 1]$

12 $\frac{11\pi}{9}$

13 $\frac{93\pi a^2}{80}$