AQA - Further pure maths 2

Name: ……………………………………………………………………………

Tutor group: ....

Key dates

Further pure 2 exam: **6th June 2013 am**

Term dates:

Term 1: Monday 3 September 2012 - Wednesday 24 October 2012 (38 teaching days)	Term 4: Monday 18 February 2013 - Friday 22 March 2013 (25 teaching days)
Term 2: Monday 5 November 2012 - Friday 21 December 2012 (35 teaching days)	Term 5: Monday 8 April 2013 - Friday 24 May 2013 (34 teaching days)
Term 3: Monday 7 January 2013 - Friday 8 February 2013 (25 teaching days)	Term 6: Monday 3 June 2013 - Wednesday 24 July 2013 (38 teaching days)

F.Y.I: Further pure 1 (re-sit): 18th January 2013 pm

Scheme of Assessment *Further Mathematics*
Advanced Subsidiary (AS)
Advanced Level (AS + A2)

Candidates for AS and/or A Level Further Mathematics are expected to have already obtained (or to be obtaining concurrently) an AS and/or A Level award in Mathematics.

The Advanced Subsidiary (AS) award comprises three units chosen from the full suite of units in this specification, except that the Core units cannot be included. One unit must be chosen from MFP1, MFP2, MFP3 and MFP4. All three units can be at AS standard; for example, MFP1, MM1B and MS1A could be chosen. All three units can be in Pure Mathematics; for example, MFP1, MFP2 and MFP4 could be chosen.

The Advanced (A Level) award comprises six units chosen from the full suite of units in this specification, except that the Core units cannot be included. The six units must include at least two units from MFP1, MFP2, MFP3 and MFP4. All four of these units could be chosen. At least three of the six units counted towards A Level Further Mathematics must be at A2 standard.

All the units count for 33$_1$/$_3$% of the total AS marks
16$_2$/$_3$% of the total A level marks

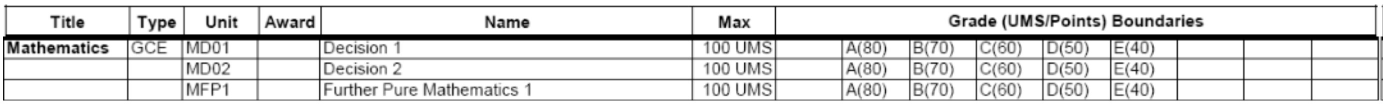
Written Paper
1hour 30 minutes
75 marks

Further Pure 1

All questions are compulsory. A graphics calculator may be used

Grading System

The AS qualifications will be graded on a five-point scale: A, B, C, D and E. The full A level qualifications will be graded on a six-point scale: A*, A, B, C, D and E.
To be awarded an A* in Further Mathematics, candidates will need to achieve grade A on the full A level qualification and 90% of the maximum uniform mark on the aggregate of the best three of the A2 units which contributed towards Further Mathematics. For all qualifications, candidates who fail to reach the minimum standard for grade E will be recorded as U (unclassified) and will not receive a qualification certificate.



Title	Type	Unit	Award	Name	Max	Grade (UMS/Points) Boundaries								
Mathematics	GCE	MD01		Decision 1	100 UMS		A(80)	B(70)	C(60)	D(50)	E(40)			
		MD02		Decision 2	100 UMS		A(80)	B(70)	C(60)	D(50)	E(40)			
		MFP1		Further Pure Mathematics 1	100 UMS		A(80)	B(70)	C(60)	D(50)	E(40)			

Further pure 2 subject content

Roots of polynomials
Complex numbers
De Moivre's theorem
Proof by induction
Finite series
The calculus of inverse trigonometric functions
Hyperbolic functions
Arc length and area of surface of revolution about the *x*-axis

Further pure 2 specifications

Candidates will be expected to be familiar with the knowledge, skills and understanding implicit in the modules Core 1, Core 2, Core 3, Core 4 and Further Pure 1.

Candidates may use relevant formulae included in the formulae booklet without proof except where proof is required in this module and requested in a question.

Roots of Polynomials

The relations between the roots and the coefficients of a polynomial equation; the occurrence of the non-real roots in conjugate pairs when the coefficients of the polynomial are real.	

Complex Numbers

The Cartesian and polar coordinate forms of a complex number, its modulus, argument and conjugate.	$x+iy \text{ and } r(\cos\theta+i\sin\theta)$
The sum, difference, product and quotient of two complex numbers.	The parts of this topic also included in module Further Pure 1 will be examined only in the context of the content of this module.
The representation of a complex number by a point on an Argand diagram; geometrical illustrations.	
Simple loci in the complex plane.	For example, $\lvert z-2-i\rvert \le 5$, $\arg(z-2)=\frac{\pi}{3}$ Maximum level of difficulty $\lvert z-a\rvert=\lvert z-b\rvert$ where $a \text{ and } b$ are complex numbers.

De Moivre's Theorem

De Moivre's theorem for integral *n*.	Use of $z+\frac{1}{z}=2\cos\theta$ and $z-\frac{1}{z}=2i\sin\theta$ leading to, for example, expressing $\sin^5\theta$ in terms of multiple angles and $\tan 5\theta$ in term of powers of $\tan\theta$. Applications in evaluating integrals, for example, $\int\sin^5\theta\, d\theta$.
De Moivre.s theorem; the nth roots of unity, the exponential form of a complex number.	The use, without justification, of the identity $e^{i\theta}=\cos\theta+i\sin\theta$.
Solutions of equations of the form $z^n = a + ib$.	To include geometric interpretation and use, for example, in expressing $\cos\frac{5\pi}{12}$ in surd form.

Proof by Induction

Applications to sequences and series, and other problems.	E.g. proving that $7^n + 4^n + 1$ is divisible by 6, or $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ where n is a positive integer.

Finite Series

Summation of a finite series by any method such as induction, partial fractions or differencing.	E.g. $\sum_{r=1}^{n} r \times r! = \sum_{r=1}^{n}((r+1)! - r!)$

The calculus of inverse trigonometrical functions

	Use of the derivatives of $\sin^{-1} x, \cos^{-1} x, \tan^{-1} x$ as given in the formulae booklet. To include the use of the standard integrals. $\int \frac{1}{a^2+x^2}dx$; $\int \frac{1}{\sqrt{a^2-x^2}}dx$ given in the formulae booklet.

Hyperbolic Functions

Hyperbolic and inverse hyperbolic functions and their derivatives; applications to integration.	The proofs mentioned below require expressing hyperbolic functions in terms of exponential functions. To include solution of equations of the form $a\sinh x + b\cosh x = c$. Use of basic definitions in proving simple identities. Maximum level of difficulty: sinh (x + y) ≡ sinh x cosh y + cosh x sinh y . The logarithmic forms of the inverse functions, given in the formulae booklet, may be required. Proofs of these results may also be required. Proofs of the results of differentiation of the hyperbolic functions, given in the formula booklet, are included. Knowledge, proof and use of: $\cosh^2 x - \sinh^2 x = 1$ $1 - \tanh^2 x = \text{sech}^2 x$ $\coth^2 x - 1 = \text{cosech}^2 x$ Familiarity with the graphs of sinh x , cosh x , tanh x , $\sinh^{-1}$ x , $\cosh^{-1}$ x , $\tanh^{-1}$ x.

Arc length and Area of surface of revolution about the x-axis

Calculation of the arc length of a curve and the area of a surface of revolution using Cartesian or parametric coordinates.	Use of the following formulae will be expected: $s = \int_{x_1}^{x_2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$ $S = 2\pi\int_{x_1}^{x_2} y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = 2\pi\int_{t_1}^{t_2} y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

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The formulae booklet

Mensuration

Surface area of sphere $= 4\pi r^2$

Area of curved surface of cone $= \pi r \times$ slant height

Arithmetic series

$$u_n = a + (n-1)d$$

$$S_n = \tfrac{1}{2}n(a+l) = \tfrac{1}{2}n[2a+(n-1)d]$$

Geometric series

$$u_n = ar^{n-1}$$

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_\infty = \frac{a}{1-r} \text{ for } |r| < 1$$

Summations

$$\sum_{r=1}^{n} r = \tfrac{1}{2}n(n+1)$$

$$\sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1)$$

$$\sum_{r=1}^{n} r^3 = \tfrac{1}{4}n^2(n+1)^2$$

Trigonometry – the Cosine rule

$$a^2 = b^2 + c^2 - 2bc\cos A$$

Binomial Series

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n \qquad (n \in \mathbb{N})$$

$$\text{where } \binom{n}{r} = {}^{n}\mathrm{C}_r = \frac{n!}{r!(n-r)!}$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{1.2\ldots r}x^r + \ldots \qquad (|x| < 1,\ n \in \mathbb{R})$$

Logarithms and exponentials

$$a^x = \mathrm{e}^{x\ln a}$$

Complex numbers

$$\{r(\cos\theta + \mathrm{i}\sin\theta)\}^n = r^n(\cos n\theta + \mathrm{i}\sin n\theta)$$

$$\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$$

The roots of $z^n = 1$ are given by $z = \mathrm{e}^{\frac{2\pi k\mathrm{i}}{n}}$, for $k = 0, 1, 2, \ldots, n-1$

Maclaurin's series

$$\mathrm{f}(x) = \mathrm{f}(0) + x\,\mathrm{f}'(0) + \frac{x^2}{2!}\mathrm{f}''(0) + \ldots + \frac{x^r}{r!}\mathrm{f}^{(r)}(0) + \ldots$$

$$\mathrm{e}^x = \exp(x) = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^r}{r!} + \ldots \qquad \text{for all } x$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{r+1}\frac{x^r}{r} + \ldots \qquad (-1 < x \leqslant 1)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^r \frac{x^{2r+1}}{(2r+1)!} + \ldots \qquad \text{for all } x$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^r \frac{x^{2r}}{(2r)!} + \ldots \qquad \text{for all } x$$

Hyperbolic functions

$$\cosh^2 x - \sinh^2 x = 1$$

$$\sinh 2x = 2\sinh x \cosh x$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

$$\cosh^{-1} x = \ln\left\{x + \sqrt{x^2 - 1}\right\} \qquad (x \geqslant 1)$$

$$\sinh^{-1} x = \ln\left\{x + \sqrt{x^2 + 1}\right\}$$

$$\tanh^{-1} x = \tfrac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad (|x| < 1)$$

Conics

	Ellipse	Parabola	Hyperbola	Rectangular hyperbola
Standard form	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$y^2 = 4ax$	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$xy = c^2$
Asymptotes	none	none	$\frac{x}{a} = \pm\frac{y}{b}$	$x = 0, y = 0$

Trigonometric identities

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \qquad \left(A \pm B \neq (k + \tfrac{1}{2})\pi\right)$$

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

Differentiation

$\mathbf{f}(x)$	$\mathbf{f}'(x)$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\tan kx$	$k\sec^2 kx$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\operatorname{sech}^2 x$
$\sinh^{-1} x$	$\frac{1}{\sqrt{1+x^2}}$
$\cosh^{-1} x$	$\frac{1}{\sqrt{x^2-1}}$
$\tanh^{-1} x$	$\frac{1}{1-x^2}$
$\frac{\mathrm{f}(x)}{\mathrm{g}(x)}$	$\frac{\mathrm{f}'(x)\,\mathrm{g}(x)-\mathrm{f}(x)\,\mathrm{g}'(x)}{(\mathrm{g}(x))^2}$

Integration

(+ constant; $a > 0$ where relevant)

$\mathbf{f}(x)$	$\int \mathbf{f}(x)\,\mathrm{d}x$
$\tan x$	$\ln\lvert\sec x\rvert$
$\cot x$	$\ln\lvert\sin x\rvert$
$\operatorname{cosec} x$	$-\ln\lvert\operatorname{cosec} x+\cot x\rvert = \ln\lvert\tan(\tfrac{1}{2}x)\rvert$
$\sec x$	$\ln\lvert\sec x+\tan x\rvert = \ln\lvert\tan(\tfrac{1}{2}x+\tfrac{1}{4}\pi)\rvert$
$\sec^2 kx$	$\frac{1}{k}\tan kx$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\ln\cosh x$

$\dfrac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\dfrac{x}{a}\right) \quad (\lvert x\rvert < a)$
$\dfrac{1}{a^2+x^2}$	$\dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)$
$\dfrac{1}{\sqrt{x^2-a^2}}$	$\cosh^{-1}\left(\dfrac{x}{a}\right)$ or $\ln\{x+\sqrt{x^2-a^2}\} \quad (x>a)$
$\dfrac{1}{\sqrt{a^2+x^2}}$	$\sinh^{-1}\left(\dfrac{x}{a}\right)$ or $\ln\{x+\sqrt{x^2+a^2}\}$
$\dfrac{1}{a^2-x^2}$	$\dfrac{1}{2a}\ln\left\lvert\dfrac{a+x}{a-x}\right\rvert = \dfrac{1}{a}\tanh^{-1}\left(\dfrac{x}{a}\right) \quad (\lvert x\rvert < a)$
$\dfrac{1}{x^2-a^2}$	$\dfrac{1}{2a}\ln\left\lvert\dfrac{x-a}{x+a}\right\rvert$

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

Area of a sector

$$A = \frac{1}{2}\int r^2\,\mathrm{d}\theta \qquad \text{(polar coordinates)}$$

Arc length

$$s = \int\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x \qquad \text{(cartesian coordinates)}$$

$$s = \int\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t \qquad \text{(parametric form)}$$

Surface area of revolution

$$S_x = 2\pi\int y\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x \qquad \text{(cartesian coordinates)}$$

$$S_x = 2\pi\int y\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t \qquad \text{(parametric form)}$$

Numerical integration

The trapezium rule: $\int_a^b y\,\mathrm{d}x \approx \frac{1}{2}h\{(y_0+y_n)+2(y_1+y_2+\ldots+y_{n-1})\}$, where $h=\dfrac{b-a}{n}$

The mid-ordinate rule: $\int_a^b y\,\mathrm{d}x \approx h(y_{\frac{1}{2}}+y_{\frac{3}{2}}+\ldots+y_{n-\frac{3}{2}}+y_{n-\frac{1}{2}})$, where $h=\dfrac{b-a}{n}$

Simpson's rule: $\int_a^b y\,\mathrm{d}x \approx \frac{1}{3}h\{(y_0+y_n)+4(y_1+y_3+\ldots+y_{n-1})+2(y_2+y_4+\ldots+y_{n-2})\}$

where $h=\dfrac{b-a}{n}$ and n is even

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Roots of polynomials

A polynomial $ax^3+bx^2+cx+d=0$ has roots α,β,γ

- The sum of the roots: $\sum\alpha=\alpha+\beta+\gamma=-\dfrac{b}{a}$
- The sum of the double products: $\sum\alpha\beta=\alpha\beta+\alpha\gamma+\beta\gamma=\dfrac{c}{a}$
- The product of the roots: $\alpha\beta\gamma=-\dfrac{d}{a}$

If we are given the values of

a) the sums of the roots,

b) the sum of the double products and

c) the product of all the roots,

then we can form the corresponding cubic equation :

$$x^3-(\text{sum of roots})x^2+(\text{sum of the double products})x-(\text{product of roots})=0$$

or using the notations

$$x^3-\left(\sum\alpha\right)x^2+\left(\sum\alpha\beta\right)x-(\alpha\beta\gamma)=0$$

Identities to remember:

- $\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)$

  Using the notations:

$$\sum(\alpha^2)=\left(\sum\alpha\right)^2-2\sum\alpha\beta$$

- $\alpha^3+\beta^3+\gamma^3=(\alpha+\beta+\gamma)^3-3(\alpha\beta+\alpha\gamma+\beta\gamma)(\alpha+\beta+\gamma)-3\alpha\beta\gamma$

If all the coefficients of the polynomial (of order 3) are REAL numbers, there are either:

- 3 real roots
- 1 real root and 2 complex CONJUGATE roots

If the coefficients of the polynomial are complex numbers, there are no rules.

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Roots of polynomials – General case

Consider a polynomial of order n (degree n):

$$a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \ldots + a_2 z^2 + a_1 z + a_0 = 0$$

$where\ a_n, a_{n-1}, .., a_1, a_0\ are\ numbers$

This polynomial has roots $\alpha_1, \alpha_2, \ldots \alpha_n$

- The sum of the roots: $\sum \alpha_i = \alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_n = -\frac{a_{n-1}}{a_n}$
- The sum of the double products: $\sum \alpha\beta = \frac{a_{n-2}}{a_n}$
- The sum of the triple products: $\sum \alpha\beta\gamma = -\frac{a_{n-3}}{a_n}$

……..

- The product of the roots: $\alpha_1 \alpha_2 \alpha_3 \ldots \alpha_n = (-1)^n \frac{a_0}{a_n}$

When all the coefficients of the polynomial are REAL numbers,
If a root α is a complex number, then its conjugate α^* is also a root.

Roots of polynomials - exercises

Question 1:

The following equations have roots α, β, γ.

In each case, work out $i) \sum \alpha = \alpha + \beta + \gamma$

$ii) \sum \alpha\beta = \alpha\beta + \alpha\gamma + \beta\gamma$

$iii) \alpha\beta\gamma$

$a) z^3 + 2z^2 - 4z + 6 = 0$ $\quad b) 2z^3 + 6z^2 - 4 = 0$ $\quad c) z^3 + (2+i)z^2 - iz + 3 - i = 0$

$iv)$ For each of the above equations, work out $\alpha^2 + \beta^2 + \gamma^2$

Question 2:

The equation $z^3 - 9z^2 + pz - 36 = 0$ *has roots* α, β *and* γ

$a)$ Write down the value of

$i) \alpha + \beta + \gamma$

$ii) \alpha\beta\gamma$

$b)$ It is given that $\alpha^2 + \beta^2 + \gamma^2 = 73$

$i)$ *Work out the value of* p

$c)$ It is also known that α is of the form ki, *where* k *is a positive real number.*

$i)$ *work out* α *and* β

$ii)$ *work out* γ

Question 3: AQA June 2005

The cubic equation $x^3 - 11x - 150 = 0$ *has roots* α, β, γ.

$a)$ Write the value of $\alpha + \beta + \gamma$

$b)$ $i)$ Explain why $\alpha^3 = 11\alpha + 150$

$ii)$ Hence or ortherwise show that $\alpha^3 + \beta^3 + \gamma^3 = 450$

$c)$ Given that $\alpha = -3 + 4i$, write down the other non-real root β and find the third root γ.

d) Show that $(3-4i)^3 + (3+4i)^3 = -234$

Question 4:

The cubic equation $x^3 + px^2 + qx + 30 = 0$, where p *and* q are real numbers,

has a root $\alpha = 1 + 2i$

$a)$ Write down the other non-real root, β, of the equation.

b) Find

$i) \alpha\beta$

$ii)$ the third root, γ, of the equation.

c) Hence, or otherwise, find the value of p *and* q.

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Question 5:

The cubic equation $z^3 + z^2 + pz + 15 = 0$ has roots α, β, γ

It is given that $\alpha^3 + \beta^3 + \gamma^3 = -49$

$a)$ Write down the value of $\alpha + \beta + \gamma$.

$b) i)$ Explain why $\alpha^3 + \alpha^2 + p\alpha + 15 = 0$

$ii)$ *Hence*, show that $\alpha^2 + \beta^2 + \gamma^2 = p + 4$

$iii)$ Deduce that $p = -1$

$c) i)$ Find the REAL root α of the equation

$ii)$ Find β and γ

Question 6:

The cubic equation $z^3 - (8+4i)z^2 + qz - 30i = 0$ *has roots* α, β, γ

$a)$ Write down the value of

$i) \alpha + \beta + \gamma$

$ii) \alpha\beta\gamma$

$b)$ It is given that $\alpha = \beta + \gamma$

Show that

$i) \alpha = 4 + 2i$

$ii) \beta\gamma = 3 + 6i$

$iii) q = 15 + 22i$

$c)$ Show that β *and* γ are the roots of the equation

$$z^2 - (4+2i) + (3+6i) = 0$$

$d)$ Given that β is a real number, find β *then* γ

Question 7:

The cubic equation $x^3 + px^2 + qx + r = 0$ *with* p, q *and* r *real numbers*

has roots α, β, γ

$a) \alpha + \beta + \gamma = 7$ *and* $\alpha^2 + \beta^2 + \gamma^2 = 31$

Find p *and* q

$b)$ Given that one root is $4 - i$, find r

Question 8:

The cubic equation $z^3 + pz^2 + 49z + q = 0$ has roots α, β, γ.

$a)$ Write down the value of $\alpha\beta + \alpha\gamma + \beta\gamma$

$b)$ Given that q *and* p are *positive real numbers and that* $\alpha^2 + \beta^2 + \gamma^2 = -17$

$i)$ Explain why the cubic equation has 2 non-real roots

$ii)$ Find p.

$c)$ One root is $-2 + 5i$

$i)$ Find the other root

ii) Find q

Roots of polynomials - exercises - answers

Question 1:

	$\alpha+\beta+\gamma$	$\alpha\beta+\alpha\gamma+\beta\gamma$	$\alpha\beta\gamma$	$\alpha^2+\beta^2+\gamma^2$
$z^3+2z^2-4z+6=0$	-2	-4	-6	12
$2z^3+6z^2-4=0$	-3	0	2	9
$z^3+(2+i)z^2-iz+3-i=0$	-2-i	-i	-3+i	3+6i

Question 2:

$a)i)\sum\alpha=9$

$ii)\alpha\beta\gamma=36$

$b)i)\alpha^2+\beta^2+\gamma^2=73=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)$

$73=(-9)^2-2p$

$p=4$

$c)\alpha=ki\ so\ (ki)^3-9(ki)^2+4(ki)-36=0$

$-ik^3+9k^2+4ki-36=0$

$(9k^2-36)+i(4k-k^3)=0$

$k^2=4$

$k=2\ or\ k=-2\qquad and\ check\ that: 4\times(\pm2)-(\pm2)^3=0$

This gives $\alpha=2i,\ \beta=-2i$

$ii)\alpha\beta\gamma=36=-2i\times2i\times\gamma\qquad\gamma=9$

Question 3:

$a)\alpha+\beta+\gamma=0\qquad(x^3+0x^2-11x-150=0)$

$b)i)\alpha$ is a root so it satisfies the equation:

$\alpha^3-11\alpha-150=0$

$\alpha^3=11\alpha+150$

$ii)$For the same reason $\beta^3=11\beta+150\ and\ \gamma^3=11\gamma+150$

Adding the three equalities, we have

$\alpha^3+\beta^3+\gamma^3=11\alpha+150+11\beta+150+11\gamma+150$

$\alpha^3+\beta^3+\gamma^3=11(\alpha+\beta+\gamma)+450$

$\alpha^3+\beta^3+\gamma^3=450\qquad because\ \alpha+\beta+\gamma=0$

$c)\alpha=-3+4i\ so\ \beta=\alpha^*$ (because the coefficients of the equation are real)

$\beta=-3-4i$

$from\ question\ a)\ we\ have\ \alpha+\beta+\gamma=0$

$-3+4i-3-4i+\gamma=0$

$\gamma=6$

$d)\ from\ question\ b)i)\alpha^3+\beta^3+\gamma^3=450$

$(-3+4i)^3+(-3-4i)^3+6^3=450$

$-(3-4i)^3-(3+4i)^3=450-216$

$(3-4i)^3+(3+4i)^3=-234$

Question 4:

$a)$ All the coefficients of the equation are real, so

if $\alpha=1+2i$ is a root then $\alpha^*$is also a root

$\beta=1-2i$

$b)i)\alpha\beta=(1+2i)(1-2i)=1+4=5$

$ii)\alpha\beta\gamma=-30\qquad5\gamma=-30\qquad\gamma=-6$

$c)\alpha+\beta+\gamma=-p$

$1+2i+1-2i-6=-p\qquad p=4$

$\alpha\beta+\alpha\gamma+\beta\gamma=5-6(1+2i)-6(1-2i)=q$

$q=-7$

Question 5:

$a)\,\alpha+\beta+\gamma=-1$

$b)\,i)\,\alpha$ is a root is it satisfies the equation:

$$\alpha^3+\alpha^2+p\alpha+15=0$$

$ii)$ For the same reason $\beta^3+\beta^2+p\beta+15=0$ *and* $\gamma^3+\gamma^2+p\gamma+15=0$

By adding the equalities

$$\alpha^3+\alpha^2+p\alpha+15+\beta^3+\beta^2+p\beta+15+\gamma^3+\gamma^2+p\gamma+15=0$$

$$(\alpha^3+\beta^3+\gamma^3)+(\alpha^2+\beta^2+\gamma^2)+p(\alpha+\beta+\gamma)+45=0$$

$$-49+\alpha^2+\beta^2+\gamma^2+p\times-1+45=0$$

$$\alpha^2+\beta^2+\gamma^2=p+4$$

$iii)\,\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)$

$$p+4=(-1)^2-2p$$

$$3p=-3 \qquad p=-1$$

$c)\,i)\,z^3+z^2-z+15=0$ (Test the values from -3 to 3)

$\alpha=-3$ is a root of the equation

$$z^3+z^2-z+15=(z+3)(z^2-2z+5)=0$$

Discriminant:$(-2)^2-4\times1\times5=-16=(4i)^2$

$$roots\ are\ \beta=\frac{2+4i}{2}=\beta=1+2i\ and\ \gamma=\beta^*=1-2i$$

Question 8:

$a)\,\alpha\beta+\alpha\gamma+\beta\gamma=49$

$b)\,\alpha^2+\beta^2+\gamma^2=-17<0$ *so one of the root is complex*

and all the coefficients of the equation are real

soif one root is complex then its CONJUGATE is also a root.

$ii)\,\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)$

$$-17=(\alpha+\beta+\gamma)^2-2\times49$$

$$81=(\alpha+\beta+\gamma)^2$$

$$p=\pm9 \quad p\ \text{is positive} \quad p=9$$

$c)\,-2+5i-2-5i+\gamma=-9$

$$\gamma=-5$$

$$q=-\alpha\beta\gamma=-(-2+5i)(-2-5i)\times-5=145$$

$$q=145$$

Question 6:

$a)\,i)\,\alpha+\beta+\gamma=8+4i$

$ii)\,\alpha\beta\gamma=30i$

$b)\,\alpha=\beta+\gamma$

$i)\,\alpha+\beta+\gamma=8+4i$

$$\alpha+\alpha=8+4i \qquad \alpha=4+2i$$

$ii)\,\alpha\beta\gamma=30i$

$$\beta\gamma=\frac{30i}{\alpha}=\frac{30i}{4+2i}=\frac{15i}{2+i}=\frac{15i(2-i)}{(2+i)(2-i)}=\frac{15+30i}{5}=3+6i$$

$iii)\,q=\alpha\beta+\alpha\gamma+\beta\gamma=\alpha(\beta+\gamma)+\beta\gamma=\alpha^2+\beta\gamma$

$$q=(4+2i)^2+3+6i=16+16i-4+3+6i$$

$$q=15+22i$$

$c)$ the sum $\beta+\gamma=4+2i$ *and* the product $\beta\gamma=3+6i$

so β and γ are roots of the equation $z^2-(\beta+\gamma)z+\beta\gamma=0$

$$z^2-(4+2i)z+3+6i=0$$

$\beta=x$ *is real so* $x^2-4x-2ix+3+6i=0$

$$(x^2-4x+3)+i(6-2x)=0$$

$$so\ \ 6-2x=0\ and\ x^2-4x+3=0$$

$$x=3 \qquad and\ 3^2-4\times3+3=9-12+3=0$$

$\beta=3$ *and* $\gamma=4+2i-\beta=\gamma=1+2i$

Question 7:

$a)\,-p=\alpha+\beta+\gamma=7 \qquad p=7$

$$q=\alpha\beta+\alpha\gamma+\beta\gamma$$

and $\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)$

$$31=(7)^2-2q$$

$$q=-5$$

$b)$ All the coefficients are real so if $4-i$ is a root then $4+i$ is also a root

$$\alpha+\beta+\gamma=4+i+4-i+\gamma=7$$

$$\gamma=-1$$

Complex numbers

Basic operations with complex numbers

$z_1 = x_1 + iy_1 \quad and \quad z_2 = x_2 + iy_2$

$Addition / Subtraction: z_1 \pm z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2)$

$Multiplication: z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$

$Division: \dfrac{z_1}{z_2} = \dfrac{z_1 z_2^*}{z_2 z_2^*} \quad because\ z_2 z_2^* = |z_2|^2\ is\ a\ real\ number.$

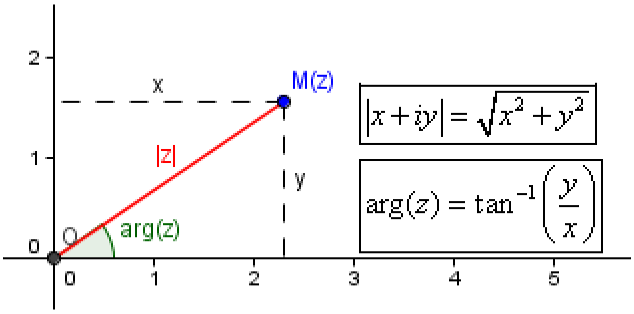
Cartesian and polar form of a complex number

$z = x + iy$ can be represented by a point $M(x, y)$ in a set of axis (Argand diagram)

The distance OM is called the MODULUS of z, noted $|z|$

The angle made by OM with the x-axis is called the ARGUMENT of z, noted arg(z)

- $|z| = |x + iy| = \sqrt{x^2 + y^2} \quad (Pythagoras)$
- $\arg(z) = \arg(x + iy) = \tan^{-1}\left(\dfrac{y}{x}\right) \quad when\ x > 0$



$$\boxed{|x + iy| = \sqrt{x^2 + y^2}}$$

$$\boxed{\arg(z) = \tan^{-1}\left(\frac{y}{x}\right)}$$

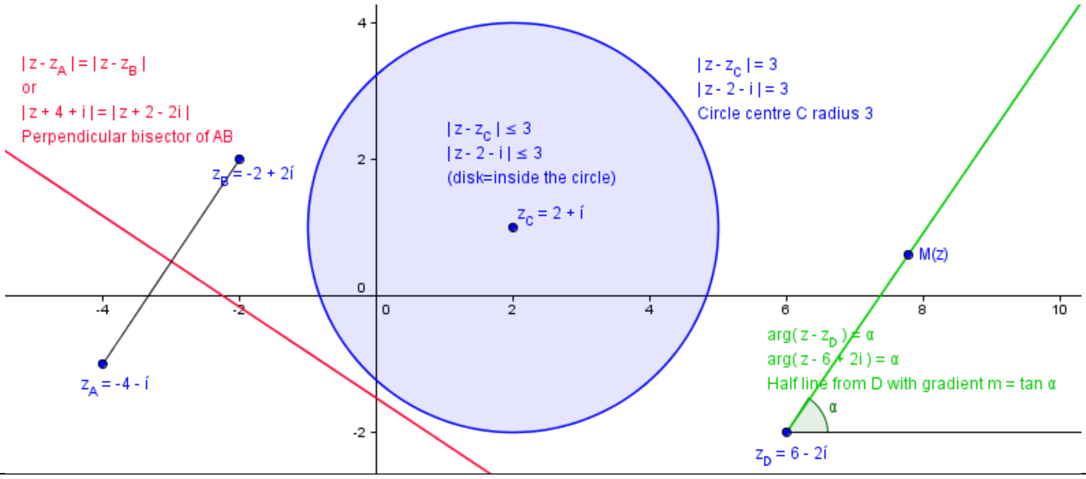
$\arg(z) = \tan^{-1}\left(\dfrac{y}{x}\right) \pm \pi$ when $x < 0$.

because the function $\tan^{-1}$ returns angles only between $-\dfrac{\pi}{2}\ and\ \dfrac{\pi}{2}$.

Loci

- $|z - z_A| = r$ is the circle centre A(z_A) and radius r.
- $\arg(z - z_A) = \alpha$ is the half line from A with gradient $m = \tan\alpha$.

  (this half line make an angle α with the "horizontal")
- $|z - z_A| = |z - z_B|$ is the perpendicular bisector of the segment line AB.



Complex numbers - exercises

Question 1:

On a Argand diagram, represent the locus of the points satisfying

a) $|z| = 5$

$b)\ |z-2-3i| = 2$

$c)\ |z-2i| \le 4$

$d)\ |z-3| = |z-1-i|$

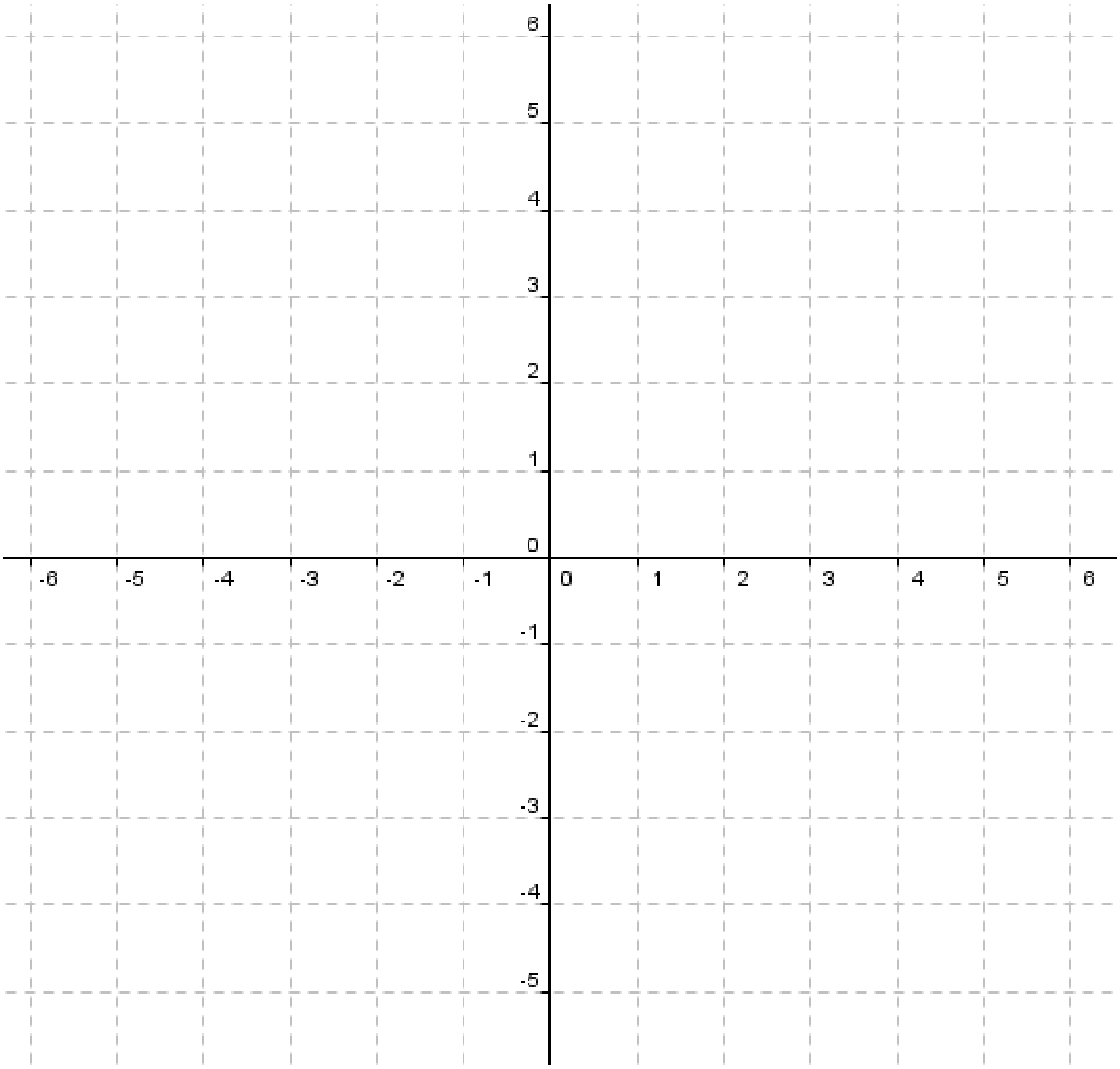
$e)\ |z| = |z-5+3i|$

$f)\ \arg(z) = \dfrac{\pi}{4}$

$g)\ \arg(z+4-5i) = \dfrac{\pi}{6}$

$h)\ 0 \le \arg(z+3+2i) \le \dfrac{\pi}{3}$

$i)$ *shade the locus*: $|z+5i| \le 3$ *and* $|z+5i| \ge |z-2+5i|$



Question 2:

The complex number z satisfies the relation

$$|z+2-2i|=2$$

$a)$ Sketch, on an Argand diagram, the locus of z.

$b)$ Show that the greatest value of $|z|$ is $2(1+\sqrt{2})$

$c)$ Find the value of z for which $\arg(z+2-2i)=\frac{1}{3}\pi$

Question 3:

$a)$ On one Argand diagram. sketch the locus of the points satisfying

$i)\ |z-4+i|=3$

$ii)\ \arg(z-2)=-\frac{\pi}{4}$

$b)$ Indicate on your sketch the set of points satisfying both

$$|z-4+i|\le 3 \quad and \quad \arg(z-2)=-\frac{\pi}{4}$$

Question 4:

$a)$ Sketch on one Argand diagram

$i)$ the locus of points satisfying $|z-3+2i|=3$

$ii)$ the locus of points satisfying $|z-2|=|z-2+4i|$

$b)$ Shade on your sketch the region in which

both $|z-3+2i|\le 3 \ \ AND \ \ |z-2|\le|z-2+4i|$

Question 5:

$a)$ Indicate on an Argand diagram the region for which $|z+6i|\le 3$

$b)$ The complex number z satisfies $|z+6i|\le 3$.Find the range of possible values of arg z.

Question 6:

A circle C and a half-line L have equations

$$|z-1-2i\sqrt{3}|=4$$

and $\arg(z+1)=\frac{\pi}{3}$ *respectively*

$a)$ Show that:

$i)$ The circle C passes through the point where $z=-1$

$ii)$ the half line L passes through the centre of C.

$b)$ On one Argand diagram, sketch C ad L.

$c)$ Shade on your sketch the set of points satisfying both

$$|z-1-2i\sqrt{3}|\le 4$$

and $$0\le\arg(z+1)\le\frac{\pi}{3}$$

Complex numbers – exercises - answers

Question 1:

$a)\, Circle\ centre\,(0,0)\, radius\, 5$

$b)\, Circle\ centre\,(2,3)\, radius\, 2$

$c)\, Disc\ centre\,(0,2)\, radius\, 4$

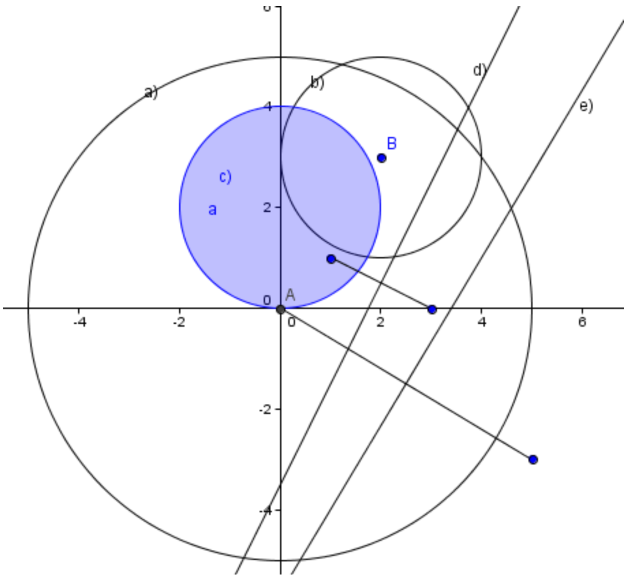
$d)\, Perpendicular\ \text{bisector}\ of\ (3,0)\ and\ (1,1)$

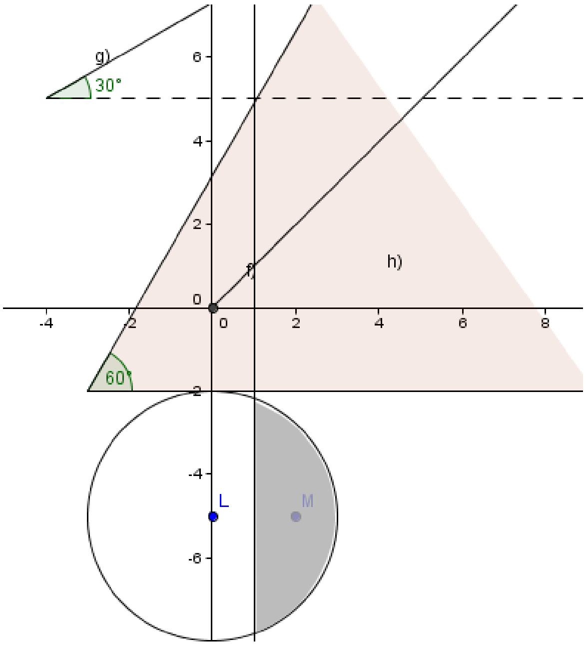
$e)\, Perpendicular\ \text{bisector}\ of\ (0,0)\ and\ (5,-3)$

$f)\, Half\ line\ from\,(0,0)\, gradient\, \tan\frac{\pi}{4}=1\, with\ x>0$

$g)\, Half\ line\ from\,(-4,5)\, gradient\, \tan\frac{\pi}{6}=\frac{1}{\sqrt{3}}\, with\ x>-4$

$h)\, and\ e)\, see\ diagram$





Question 2:

$a)$ The locus is the circle centre (-2,2) radius $r=2$

$b)$ The greatest value of $|z|\, is\, |-2+2i|+r=\sqrt{(-2)^2+(2)^2}+2$

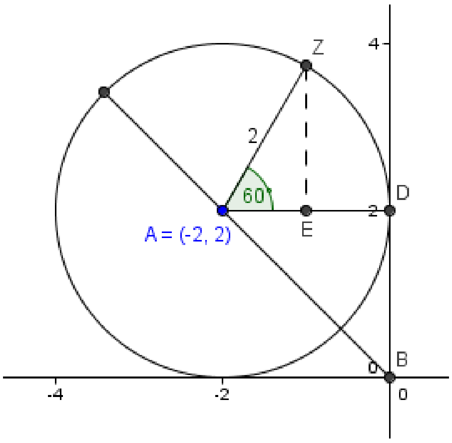
$$=2\sqrt{2}+2=2(1+\sqrt{2})$$

$c)$ We need to find the coordinates of the point Z.

Using trigonometry in the right-angled triangle AEZ,

we have: $EZ=2\times\sin\frac{\pi}{3}=2\times\frac{\sqrt{3}}{2}=\sqrt{3}\ \ and\ \ AE=2\times\cos\frac{\pi}{3}=1$

This gives Z(-1,2+$\sqrt{3}$)

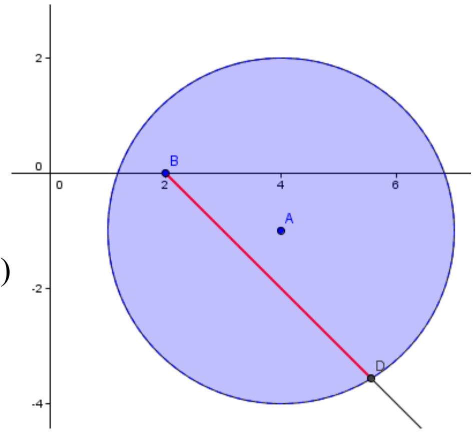


Question 3:

$a)i)$ The locus is the circle centre (4,1) radius $r=3$

$ii)\, Half\ line\ from\,(2,0)\, with\ gradient\, \tan\left(-\frac{\pi}{4}\right)=-1\ with\ x>2$

$b)$ The set of points which are solutions contitute the segment line intersection between the half line and the disc (inside of the circle)



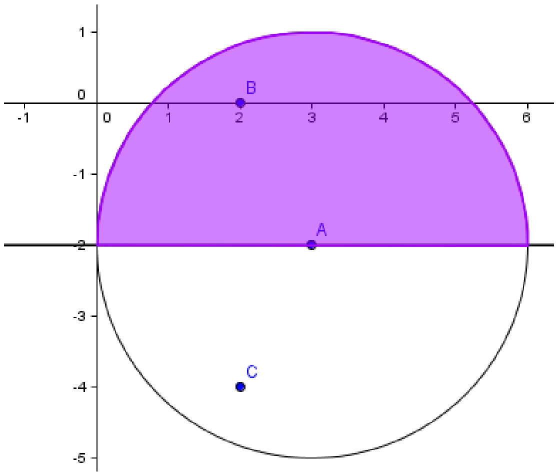
Question 4:

$a)\,i)\,Circle\ centre\,(3,-2)\ and\ radius\ r=3$

$ii)$ Perpendicular bisector of $(2,0)$ and $(2,-4)$

$b)\,|z-3+2i|\le 3$ is the disc $(3,-2)\ and\ radius\ r=3$

$|z-2|\le|z-2+4i|$ is the half plane containing the point $(2,0)$



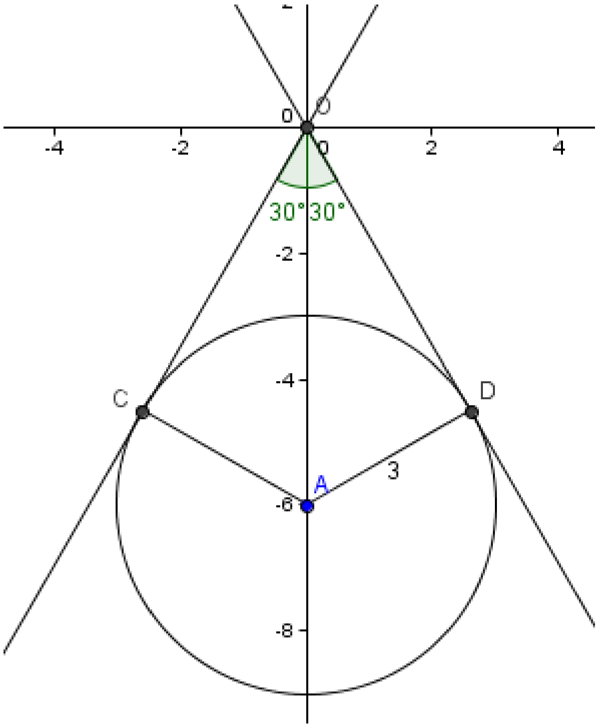
Question 5:

$a)$ The locus is the disc centre (0,-6) radius r = 3

$b)$ The range of the arg(z) is given by drawing the tangents to the circle going through the origin (0,0). Knowing that the radius is perpendicular to the tangent, we can use trigonometry to work out the angles needed.

In OAD, angle AOD $=\text{Sin}^{-1}\left(\frac{opp}{hyp}\right)=Sin^{-1}\left(\frac{3}{6}\right)=\frac{\pi}{6}$

The arg(z) goes from $-\frac{2\pi}{3}\ to-\frac{\pi}{3}\ \ (-120^{o}\ to-60^{o})$



Question 6:

$a)i)\ |z-1-2i\sqrt{3}|=|-1-1-2i\sqrt{3}|=|-2-2i\sqrt{3}|=\sqrt{(-2)^2+(-2\sqrt{3})^2}=\sqrt{16}=4$

$z=-1\ belongs\ to\ C$

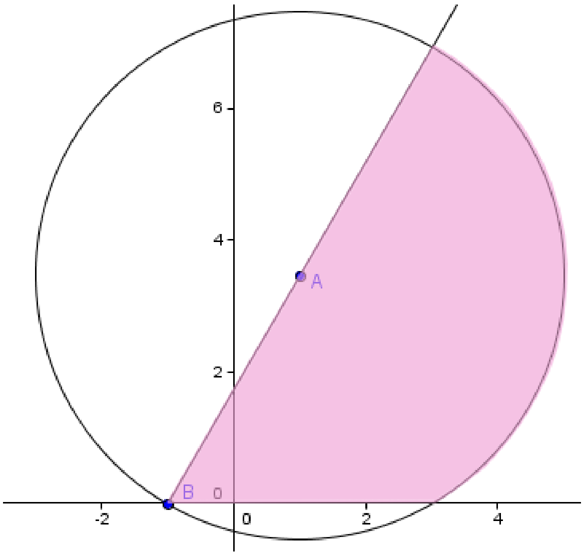
$ii)\ \arg(z+1)=\arg(1+2i\sqrt{3}+1)=\arg(2+2i\sqrt{3})=\theta$

$$\tan\theta=\frac{2\sqrt{3}}{2}=\sqrt{3}\qquad \tan^{-1}\sqrt{3}=\frac{\pi}{3}$$

$z=1+2i\sqrt{3}\ belongs\ to\ L$

$b)$

$c)\ see\ \ diagram.$



De Moivre’s theorem

Trigonometric form of a complex number

$z = x + iy \;\; and \;\; |z| = r \;, \; \arg(z) = \theta$

The trigonometric form of z is : $z = r(Cos\,\theta + iSin\,\theta)$

De Moivre's Theorem

For all n $\in \mathbb{Q}$, and all $\theta \in \mathbb{R}$, $\left(Cos\,\theta + i\,Sin\theta\right)^n = Cos\,(n\theta) + i\,Sin\,(n\theta)$

Expressions of $\sin\theta$ and $\cos\theta$

$If \;\; z = \cos\theta + i\sin\theta$

$$then \qquad z + \frac{1}{z} = 2\cos\theta \quad and \quad z - \frac{1}{z} = 2i\sin\theta$$

AND

$$z^n + \frac{1}{z^n} = 2\cos(n\theta) \quad and \quad z^n - \frac{1}{z^n} = 2i\sin(n\theta)$$

Exponential form of a complex number

$z = r(\cos\theta + i\sin\theta)\, is \;\; noted \;\; z = r\,e^{i\theta}$

The rules of calculation with exponential remain valid with complex numbers.

i.e: $e^{i\theta} \times e^{i\alpha} = e^{i(\theta+\alpha)}$

$$\frac{e^{i\theta}}{e^{i\alpha}} = e^{i(\theta-\alpha)}$$

$$\left(e^{i\theta}\right)^n = e^{in\theta} \qquad (De\,moivre's\,theorem)$$

Expressions of $\sin\theta$ and $\cos\theta$

$If \;\; z = e^{i\theta}$

$$then \qquad \cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) \;\; and \quad \sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$$

AND

$$\cos(n\theta) = \frac{1}{2}\left(e^{in\theta} + e^{-in\theta}\right) \;\; and \quad \sin(n\theta) = \frac{1}{2i}\left(e^{in\theta} - e^{-in\theta}\right)$$

The nth roots of the unity: $z^n = 1$

$$z = e^{i\frac{2k\pi}{n}} \qquad k = 0, 1, 2, \ldots, n-1$$

Solving $z^n = x + iy$

$z^n = r^n e^{in\theta} \;\; and \;\; x + iy = r_0\,e^{i\theta_0}$

$$z = r_0^{\frac{1}{n}}\,e^{i\frac{\theta_0 + 2k\pi}{n}} \qquad k = 0, 1, 2, \ldots, n-1$$

De Moivre's theorem - exercises

Question 1:

Work out the exact value of the following complex numbers:

a) $\left(\cos\frac{3\pi}{7}+i\sin\frac{3\pi}{7}\right)^7$ $\quad$ $b)\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)^5$

$c)\dfrac{\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}}{\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}}$ $\quad$ $d)\dfrac{\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}}{\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)^6}$

Question 2:

By finding the modulus and argument first,

work out the exact value of the power of these complex numbers

a) $\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)^5$ $\quad$ $b)\left(1+i\sqrt{3}\right)^6$ $\quad$ $c)\left(4-4i\right)^8$

$d)\left(3+\sqrt{3}\,i\right)^6$ $\quad$ $e)\left(-2-2i\sqrt{3}\right)^9$ $\quad$ $f)(-1-i)^{12}$

Question 3:

Using De moivre's theorem, express

a) $\cos(2\theta)$ in terms of $\cos\theta$ *and* $\sin\theta$

$b)$ $\sin(3\theta)$ in terms of $\cos\theta$ and $\sin\theta$

c) $\cos(3\theta)$ in terms of $\cos\theta$

Question 4:

The complex number $z=\cos\theta+i\sin\theta\ \ with\,\theta\in\mathbb{R}$.

$a)$*Show that* $z+\frac{1}{z}=2\cos\theta$ *and work out* $z-\frac{1}{z}$ *in terms of* $\sin\theta$.

$b)$ Using the exponential notation, $z=e^{i\theta}$, *show* $z^n+\frac{1}{z^n}=2\cos(n\theta)$ for any integer n.

$c)$ Find a similar identity for $z^n-\frac{1}{z^n}$.

$d)$ *Linearise* $\cos^3\theta$

$e)$ *Linearise* $\sin^4\theta$

$f)$ *Work out* $\int_0^{\frac{\pi}{2}}\sin^4\theta\,d\theta$

Question 5:

Using the De Moivre's theorem, show that

a) $\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$

$b)$ Find a similar expression for $\cos 5\theta$

$c)$ *Hence* or otherwise, show that $\tan 5\theta = \dfrac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$

$d)$ Show that $x = \tan\dfrac{\pi}{20}$ is a solution of the equation $x^5 - 5x^4 - 10x^3 + 10x^2 + 5x + 1 = 0$

$e)$ Find the other 4 solutions.

$f)$ Show that $\tan\dfrac{\pi}{20} + \tan\dfrac{9\pi}{20} - \tan\dfrac{7\pi}{20} - \tan\dfrac{3\pi}{20} = 4$

Question 6:

$a)$ *Solve the complex equation* $z^6 = 1$.

Give your answers in exponential form.

b) If ω is one of the solution (not 1), show that $1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 = 0$

$c)$ Illustrate your solutions in an Argand diagram.

Question 7:

$a)$ *Solve* $z^4 = -i$. Give your answers in the form $e^{i\theta}$ with $-\pi < \theta \le \pi$.

b) Explain why the sum of the solutions is 0.

Question 8:

$a)$ Solve in the set of complex numbers

$$z^5 = 32$$

Give your answers in the exponential form: $e^{i\theta}$ *with* $-\pi < \theta \le \pi$

b) Given that the sum of the root is 0, show that

$$\cos\left(\frac{2\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right) = -\frac{1}{2}$$

Question 9:

$a)$ *Expand* $(z^3 - e^{i\theta})(z^2 - e^{-i\theta})$

$b)$ Hence, *solve* $z^5 + z^3 + z^2 + 1 = 0$

Give your answers in the for $x + iy$

$c)$ Illustrate these solutions on an Argand diagram.

De Moivre’s theorem – exercises - answers

Question 1:

$$a)\left(\cos\frac{3\pi}{7}+i\sin\frac{3\pi}{7}\right)^7=\cos\frac{21\pi}{7}+i\sin\frac{21\pi}{7}=\cos 3\pi+i\sin 3\pi=-1$$

$$b)\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)^5=\left(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right)=-\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}$$

$$c)\frac{e^{i\frac{\pi}{3}}}{e^{i\frac{\pi}{6}}}=e^{i\left(\frac{\pi}{3}-\frac{\pi}{6}\right)}=e^{i\frac{\pi}{3}}=\cos\frac{\pi}{3}+i\frac{\pi}{3}=\frac{1}{2}+i\frac{\sqrt{3}}{2}$$

$$d)\frac{e^{i\frac{\pi}{4}}}{(e^{i\frac{\pi}{2}})^6}=\frac{e^{i\frac{\pi}{4}}}{e^{i\frac{6\pi}{2}}}=e^{i\left(\frac{\pi}{4}-3\pi\right)}=e^{-i\frac{11\pi}{4}}=e^{-i\frac{3\pi}{4}}=\cos\left(-\frac{3\pi}{4}\right)+i\sin\left(-\frac{3\pi}{4}\right)=-\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}$$

Question 2:

$$a)\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)^5=\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)^5=\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}=-\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}$$

$$b)(1+i\sqrt{3})^6=\left(2(\frac{1}{2}+i\frac{\sqrt{3}}{2})\right)^6=\left(2e^{i\frac{\pi}{3}}\right)^6=2^6\times e^{i\frac{6\pi}{3}}=2^6\times e^{i2\pi}=64$$

$$c)(4-4i)^8=\left(4\sqrt{2}\left(\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)\right)^8=\left(4\sqrt{2}e^{-i\frac{\pi}{4}}\right)^8=\left(4\sqrt{2}\right)^8e^{-2\pi i}=1048576$$

$$d)\left(3+i\sqrt{3}\right)^6=\left(2\sqrt{3}\left(\frac{\sqrt{3}}{2}+i\frac{1}{2}\right)\right)^6=\left(2\sqrt{3}e^{i\frac{\pi}{6}}\right)^6=\left(2\sqrt{3}\right)^6\times e^{i\pi}=-1728$$

$$e)\left(-2-2i\sqrt{3}\right)^9=\left(4\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)\right)^9=\left(4e^{-i\frac{2\pi}{3}}\right)^9=4^9\times e^{-i6\pi}=262144$$

$$f)(-1-i)^{12}=\left(\sqrt{2}\left(-\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)\right)^{12}=\left(\sqrt{2}e^{-i\frac{3\pi}{4}}\right)^{12}=\sqrt{2}^{12}\times e^{-9\pi i}=-64$$

Question 3:

$$a)\ \left(\cos\theta+i\sin\theta\right)^2=\cos 2\theta+i\sin 2\theta$$

$$\cos 2\theta+i\sin 2\theta=\cos^2\theta+2i\sin\theta\cos\theta+(i\sin\theta)^2$$

$$=(\cos^2\theta-\sin^2\theta)+i(2\sin\theta\cos\theta)$$

$$\textit{So}\ \cos 2\theta=\cos^2\theta-\sin^2\theta\ \ \textit{and}\ \ \sin 2\theta=2\sin\theta\cos\theta$$

$$b)\cos 3\theta+i\sin 3\theta=\left(\cos\theta+i\sin\theta\right)^3=(c+is)^3$$

$$=c^3+3\times c^2\times(is)+3\times c\times(is)^2+(is)^3$$

$$=(c^3-3cs^2)+i(3c^2s-s^3)$$

Identifying the imaginary parts gives

$$\sin 3\theta=3\cos^2\theta\sin\theta-\sin^3\theta$$

c) *Identifying the real parts gives :*

$$\cos 3\theta\ =\cos^3\theta-3\cos\theta\sin^2\theta$$

$$=\cos^3\theta-3\cos\theta(1-\cos^2\theta)$$

$$=\cos^3\theta+3\cos^2\theta-3\cos\theta$$

Question 4:

$$z=\cos\theta+i\sin\theta \quad and \quad \frac{1}{z}=\left(\cos\theta+i\sin\theta\right)^{-1}=\cos(-\theta)+i\sin(-\theta)$$

$$=\cos\theta-i\sin\theta$$

$$so\, z+\frac{1}{z}=\cos\theta+i\sin\theta+\cos\theta-i\sin\theta=2\cos\theta$$

$$z-\frac{1}{z}=\cos\theta+i\sin\theta-\cos\theta+i\sin\theta=2i\sin\theta$$

$$z=e^{i\theta} \quad so \quad z^n=\left(e^{i\theta}\right)^n=e^{in\theta}=\cos n\theta+i\sin n\theta$$

$$\frac{1}{z^n}=z^{-n}=\left(e^{i\theta}\right)^{-n}=e^{-in\theta}$$

$$=\cos(-n\theta)+i\sin(-n\theta)=\cos n\theta-i\sin n\theta$$

$$so \quad z^n+\frac{1}{z^n}=...=2\cos n\theta \quad and \quad z^n-\frac{1}{z^n}=...=2i\sin n\theta$$

$$d)\cos^3\theta=\left(\frac{1}{2}\left(z+\frac{1}{z}\right)\right)^3=\frac{1}{8}\left(z^3+3\times z^2\times\frac{1}{z}+3\times z\times\frac{1}{z^2}+\frac{1}{z^3}\right)$$

$$=\frac{1}{8}\left(z^3+\frac{1}{z^3}+3(z+\frac{1}{z})\right)$$

$$\cos^3\theta=\frac{1}{4}\cos3\theta+\frac{3}{4}\cos\theta$$

$$e)\sin^4\theta=\left(\frac{1}{2i}\left(z-\frac{1}{z}\right)\right)^4=\frac{1}{16}\left(z^4-4z^3\times\frac{1}{z}+6\times z^2\times\frac{1}{z^2}-4\times z\times\frac{1}{z^3}+\frac{1}{z^4}\right)$$

$$=\frac{1}{16}\left(z^4+\frac{1}{z^4}-4\left(z^2+\frac{1}{z^2}\right)+6\right)$$

$$\sin^4\theta=\frac{1}{8}\cos4\theta-\frac{1}{2}\cos2\theta+\frac{3}{8}$$

$$f)\int_0^{\frac{\pi}{2}}\sin^4\theta d\theta=\int_0^{\frac{\pi}{2}}\frac{1}{8}\cos4\theta-\frac{1}{2}\cos2\theta+\frac{3}{8}d\theta=\left[\frac{1}{32}\sin4\theta-\frac{1}{4}\sin2\theta+\frac{3}{8}\theta\right]_0^{\frac{\pi}{2}}$$

$$\int_0^{\frac{\pi}{2}}\sin^4\theta d\theta=\frac{3\pi}{16}$$

Question 5:

$$a)\cos5\theta+i\sin5\theta=\left(\cos\theta+i\sin\theta\right)^5=(c+is)^5$$

$$=c^5+5ic^4s-10c^3s^2-10ic^2s^3+5cs^4+is^5$$

$$=\left(c^5-10c^3s^2+5cs^4\right)+i\left(5c^4s-10c^2s^3+s^5\right)$$

Equality of the real parts: $\cos5\theta=\cos^5\theta-10\cos^3\theta\sin^2\theta+5\cos\theta\sin^4\theta$

Equality of the imaginary parts:$\sin5\theta=5\cos^4\theta\sin\theta-10\cos^2\theta\sin^3\theta+\sin^5\theta$

$$c)\tan5\theta=\frac{\sin5\theta}{\cos5\theta}=\frac{5\cos^4\theta\sin\theta-10\cos^2\theta\sin^3\theta+\sin^5\theta}{\cos^5\theta-10\cos^3\theta\sin^2\theta+5\cos\theta\sin^4\theta}$$

Tip :divide numerator and denominator by the highest power of $\cos\theta$, *here* $\cos^5\theta$

$$\tan5\theta=\frac{5\frac{\sin\theta}{\cos\theta}-10\frac{\sin^3\theta}{\cos^3\theta}+\frac{\sin^5\theta}{\cos^5\theta}}{1-10\frac{\sin^2\theta}{\cos^2\theta}+5\frac{\sin^4\theta}{\cos^4\theta}}=\frac{5\tan\theta-10\tan^3\theta+\tan^5\theta}{1-10\tan^2\theta+5\tan^4\theta}$$

when $\theta=\frac{\pi}{20}$, *then let's call* $\tan\frac{\pi}{20}=x$. The identity above becomes

$$\tan\frac{5\pi}{20}=1=\frac{5x-10x^3+x^5}{1-10x^2+5x^4} \quad by\ re-arranging$$

$$1-10x^2+5x^4=5x-10x^3+x^5$$

$$x^5-5x^4-10x^3+10x^2+5x-1=0$$

e) The other solutions are given by consider each time that: $\tan5\theta=1$

$$5\theta=\frac{\pi}{4}+k\pi \qquad \theta=\frac{\pi}{20}+k\frac{\pi}{5}$$

$k=0,\ \theta=\frac{\pi}{20}$ $\quad x_1=\tan\frac{\pi}{20}$ $\quad k=1,\theta=\frac{5\pi}{20}=\frac{\pi}{4}$ $\quad x_2=\tan\frac{\pi}{4}=1$

$k=2,\theta=\frac{9\pi}{20}$ $\quad x_3=\tan\frac{9\pi}{20}$ $\quad k=-1,\theta=-\frac{3\pi}{20}$ $\quad x_4=\tan\left(-\frac{3\pi}{20}\right)=-\tan\left(\frac{3\pi}{20}\right)$

$k=-2,\theta=-\frac{7\pi}{20}$ $\quad x_5=\tan\left(-\frac{7\pi}{20}\right)=-\tan\left(\frac{7\pi}{20}\right)$

(You could chose $k=3$ and 4 instead of -1 and -2)

$$f)x^5-5x^4-10x^3+10x^2+5x-1=0$$

The sum of the roots is $5\ (-\frac{b}{a})$

This gives $\quad \tan\frac{\pi}{20}+1+\tan\frac{9\pi}{20}-\tan\frac{3\pi}{20}-\tan\frac{7\pi}{20}=5$

$$\tan\frac{\pi}{20}+\tan\frac{9\pi}{20}-\tan\frac{3\pi}{20}-\tan\frac{7\pi}{20}=4$$

Question 6:

$$z = re^{i\theta} \quad so \quad z^5 = r^5 e^{i5\theta} \quad and \quad 1 = e^{i0}$$

$$z^5 = 1 \quad becomes$$

$$r^5 e^{i5\theta} = 1e^{i0}$$

$$r^5 = 1 \ \ and \ \ 5\theta = 0 + k2\pi$$

$$r = 1 \quad and \quad \theta = k\frac{2\pi}{5} \qquad k = 0,1,2,3,4$$

$$Solutions: z_1 = e^{i0} = 1 \ , \ z_2 = e^{i\frac{2\pi}{5}} \ , \ z_3 = e^{i\frac{4\pi}{5}} \ , \ z_4 = e^{i\frac{6\pi}{5}} \ , \ z_5 = e^{i\frac{8\pi}{5}}$$

$$b) 1 + \omega + \omega^2 + \omega^3 + \omega^4 = \frac{1-\omega^5}{1-\omega} = \frac{1-1}{1-\omega} = 0$$

$$(geometric \ series \ with \ common \ ratio \ \omega)$$

Question 7:

$$a) z^4 = -i$$

$$r^4 e^{i4\theta} = e^{-i\frac{\pi}{2}}$$

$$r^4 = 1 \ \ and \ \ 4\theta = -\frac{\pi}{2} + k2\pi$$

$$r = 1 \ \ and \ \ \theta = -\frac{\pi}{8} + k\frac{\pi}{2} \qquad k = -1,0,1,2$$

$$solutions: z_1 = e^{-i\frac{5\pi}{8}} \ , \ z_2 = e^{-i\frac{\pi}{8}} \ , \ z_3 = e^{i\frac{3\pi}{8}} \ , \ z_4 = e^{i\frac{7\pi}{8}}$$

$$b) z^4 = -i \qquad z^4 + i = 0$$

The coefficient of z^3 *is* 0, hence the sum of the roots is 0.

Question 8:

$$z^5 = 32$$

$$r^5 e^{i5\theta} = 32e^{i0}$$

$$r^5 = 32 \ \ and \ \ 5\theta = 0 + k2\pi$$

$$r = 2 \ \ and \ \ \theta = k\frac{2\pi}{5} \qquad k = -2,-1,0,1,2$$

$$Solutions: z_1 = 2e^{-i\frac{4\pi}{5}} \ , \ z_2 = 2e^{-i\frac{2\pi}{5}} \ , \ z_3 = 2 \ , \ z_4 = 2e^{i\frac{2\pi}{5}} \ , \ z_5 = 2e^{i\frac{4\pi}{5}}$$

$$b) z_1 + z_2 + z_3 + z_4 + z_5 = 0$$

$$\cos\frac{4\pi}{5} - i\sin\frac{4\pi}{5} + \cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5} + 2 + \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5} + \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5} = 0$$

$$2\cos\frac{4\pi}{5} + 2\cos\frac{2\pi}{5} = -2$$

$$\cos\frac{4\pi}{5} + \cos\frac{2\pi}{5} = -\frac{1}{2}$$

Question 9:

$$a) \left(z^3 - e^{i\theta}\right)\left(z^2 - e^{-i\theta}\right) = z^5 - z^3 e^{-i\theta} - z^2 e^{i\theta} + 1$$

$b)$ *when* $\theta = \pi$, the expression becomes $z^5 + z^3 + z^2 + 1$

and can be factorised: $\left(z^3 - e^{i\pi}\right)\left(z^2 - e^{-i\pi}\right)$

$$z^5 + z^3 + z^2 + 1 = 0 \ \ for \ \ z^3 = e^{i\pi} \qquad or \ z^2 = e^{-i\pi}$$

$$z = e^{i\frac{\pi}{3}} \ \ or \ \ z = e^{i\frac{3\pi}{3}} = -1 \ \ or \ \ z = e^{-i\frac{\pi}{3}} \quad OR \ \ z = i \ \ or \ \ z = -i$$

Proof by induction

INDUCTION PRINCIPAL

A proposition P_n, is to be proven true for $n \geq n_0$

- *Basis case*: Show that P_{n_0} is true.
- *Assumption / Hypothesis*

  Suppose that P_k is true.
- *Induction*: Show that P_{k+1} is then also true.
- *Conclusion*:

  "If the proposition P_k is true then P_{k+1} is true.
  Because P_{n_0} is true, according to the induction principal
  we conclude that the proposition is true for all $n \geq n_0$"

Proof by induction - exercises

Question 1:

Prove by induction that

$$\text{for all n} \geq 1, \sum_{r=1}^{n} r^2 - 4r = \frac{1}{6}n(n+1)(2n-11)$$

Question 2:

Prove by induction that

$$\text{for all n} \geq 1,\ \text{n}^2 - 9n + 7 \ \text{ is divisible by 2}$$

Question 3:

A sequence is given by

$$u_1 = 3 \quad and \quad u_{n+1} = 2u_n - 5$$

$a)$ *Work out* u_2, u_3, u_4.

$b)$ Show by induction that for all n ≥ 1, $u_n = 5 - 2^n$

Question 4:

The function f, is defined by

$$f(n) = 3^{2n} - 2^{2n} \quad for\ all\ n \geq 1$$

$a)$ *Work out* $f(1), f(2), f(3)$.

$b)$ *Express* $f(k+1) - 4f(k)$ in terms of k.

$c)$ Prove by induction that for all n $\geq 1, 3^{2n} - 2^{2n}$ is divisible by 5

Question 5:

$a)$ $i)$ Show that $x = 1$ is a root of $x^3 + 5x^2 + 2x - 8 = 0$

$ii)$ Factorise fully $x^3 + 5x^2 + 2x - 8$

$b)$ Prove by induction that

$$\text{for all n} \geq 1, \sum_{r=1}^{n} 4r^3 - 12r = n(n+1)(n+3)(n-2)$$

Question 6:

A sequence is given by

$$u_1 = 6 \quad and \quad u_{n+1} = 3u_n - 6$$

$a)$ *Work out* u_2, u_3, u_4.

$b)$ Show by induction that for all n ≥ 1, $u_n = 3^n + 3$

Question 7:

The function f, is defined by

$$f(n) = 7^{2n} - 2 \times 3^{2n} + 1 \quad for\ all\ n \geq 1$$

$a)$ Work out $f(1), f(2), f(3)$.

$b)$ Show that $f(k+1) - 49f(k) = a \times (b \times 3^{2k} + c)$ where a, b and c are to be found.

$c)$ Prove by induction that for all n ≥ 1, $7^{2n} - 2 \times 3^{2n} + 1$ is divisible by 16

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Proof by induction – exercises - answers

Question 1:

Let's call P_n the proposition: $\sum_{r=1}^{n} r^2 - 4r = \frac{1}{6}n(n+1)(2n-11)$

We have to prove that this proposition is true for all n≥1.

- *Basis case* : $n = 1$

  $Left\ Hand\ Side\,(LHS): \sum_{r=1}^{1} r^2 - 4r = 1^2 - 4\times 1 = -3$

  $Right\ Hand\ \ side\,(RHS): \frac{1}{6}n(n+1)(2n-11) = \frac{1}{6}\times 1\times 2\times -9 = -3$

  The proposition P_1 is true.

- *Assumption* :

  Let's assume that the proposition is true for $n = k$ (P_k *is true*).

  $$i.e: \sum_{r=1}^{k} r^2 - 4r = \frac{1}{6}k(k+1)(2k-11)$$

  Let's show that the proposition is true for $n = k+1$ (*showthat* P_{k+1} *is true*).

  $i.e: let's\ \ show\ \ that \sum_{r=1}^{k+1} r^2 - 4r = \frac{1}{6}(k+1)(k+2)(2k-9)$

- *Induction* :

$$\sum_{r=1}^{k+1} r^2 - 4r = \sum_{r=1}^{k} r^2 - 4r + \left((k+1)^2 - 4(k+1)\right)$$

$$= \frac{1}{6}k(k+1)(2k-11) + (k+1)(k-3)$$

$$= \frac{1}{6}(k+1)\left(k(2k-11) + 6(k-3)\right) = \frac{1}{6}(k+1)(2k^2 - 5k - 18)$$

$$= \frac{1}{6}(k+1)(2k-9)(k+2) \quad Q.E.D$$

- *Conclusion* :

If the proposition is true for $n = k$, then it is also true for $n = k+1$.

Because it is true for $n = 1$, according to the induction principal,

we can conclude that P_n is true for all n ≥ 1.

Question 2:

Let's call P_n the proposition: $n^2 - 9n + 7$ is divisible by 2

We have to prove that this proposition is true for all n≥1.

- *Basis case* : $n = 1$

  $n^2 - 9n + 6 = 1^2 - 9\times 1 + 6 = -2 (= 2\times -1)$

  The proposition P_1 is true.

- *Assumption* :

  Let's assume that the proposition is true for $n = k$ (P_k *is true*).

  $i.e$: $k^2 - 9k + 6$ is divisible by 2

  Let's show that the proposition is true for $n = k+1$ (*showthat* P_{k+1} *is true*).

  $i.e: let's\ \ show\ that\ (k+1)^2 - 9(k+1) + 6$ is divisible by 2.

- *Induction* :

$$(k+1)^2 - 9(k+1) + 6 \ = k^2 + 2k + 1 - 9k - 9 + 6$$

$$= (k^2 - 9k + 6) + 2k - 8$$

$$= (k^2 - 9k + 6) + 2(k-4)$$

$k^2 - 9k + 6$ is divisible by 2 by assumption

2(k - 4) is divisible by 2

so $(k+1)^2 - 9(k+1) + 6$ is divisible by 2   Q.E.D

- *Conclusion* :

If the proposition is true for $n = k$, then it is also true for $n = k+1$.

Because it is true for $n = 1$, according to the induction principal,

we can conclude that P_n is true for all n ≥ 1.

Question 3:

$u_1 = 3 \ \ and \ \ u_{n+1} = 2u_n - 5$

$a) u_2 = 2\times 3 - 5 = 1$

$u_3 = 2\times 1 - 5 = -3$

$u_4 = 2\times -3 - 5 = -11$

$b)$ The proposition P_n is "$u_n = 5 - 2^n$"

●*Basis case*: $n = 1$

$5 - 2^n = 5 - 2^1 = 3 = u_1$

The proposition is true for $n = 1$.(P_1 *is true*)

● *Assumption*:

Let's assume that the proposition is true for $n = k$.

Let's show that it s true for $n = k+1$.

●*Induction*:

$u_{k+1} = 2u_k - 5 = 2(5 - 2^k) - 5 = 10 - 2^{k+1} - 5$

$u_{k+1} = 5 - 2^{k+1} \quad Q.E.D$

●*Conclusion*:

If the proposition is true for $n = k$, then it is also true for $n = k+1$.
Because it is true for $n = 1$, according to the induction principal,
we can conclude that P_n is true for all n ≥ 1.

Question 4:

$a) f(1) = 3^2 - 2^2 = 9 - 4 = 5$

$f(2) = 3^4 - 2^4 = 81 - 16 = 65$

$f(3) = 3^6 - 2^6 = 665$

$$b) f(k+1) - 4f(k) = 3^{2k+2} - 2^{2k+2} - 4\times 3^{2k} + 4\times 2^{2k}$$

$$= 9\times 3^{2k} - 4\times 2^{2k} - 4\times 3^{2k} + 4\times 2^{2k}$$

$$f(k+1) - 4f(k) = 5\times 3^{2k}$$

$c)$ Proposition $\mathrm{P}_n : f(n) = 3^{2n} - 2^{2n}$ is divisible by 5

●*Basis case*: $n = 1$

$f(1) = 5$ is divisible by 5

The proposition is true for $n = 1$ (P_1 *is true*)

●*Assumption*

Let's suppose that P_k true i.e $f(k) = 3^{2k} - 2^{2k}$ is divisible by 5

Let's show that P_{k+1} is true,

i.e. let's show that $f(k+1) = 3^{2k+2} - 2^{2k+2}$ is divisible by 5

•*Induction*

$f(k+1) - 4f(k) = 5\times 3^{2k}$

$f(k+1) = 5\times 3^{2k} - 4f(k)$

5×3^{2k} is a multiple of 5

$4f(k)$ is a multiple of 5 because $f(k)$ is one by supposition.

$f(k+1)$ is therefore a multiple of 5 (as the sum of two multiples of 5)

●*Conclusion*

If P_k is true then P_{k+1} is true.P_1 is true so according to the induction principle, we can say that P_n is true for all n ≥ 1.

Question 5:

$a) i) x^3+5x^2+2x-8=(1)^3+5\times(1)^2+2\times(1)-8=1+5+2-8=0$

1 is a root of x^3+5x^2+2x-8

$ii) x^3+5x^2+2x-8=(x-1)(x^2+6x+8)=(x-1)(x+4)(x+2)$

$b)$ Proposition $P_n : \sum_{r=1}^{n} 4r^3-12r=n(n+1)(n+3)(n-2)$

- *Basis case*: $n=1$

$$LHS: \sum_{r=1}^{1} 4r^3-12r=4\times1^3-12\times1=4-12=-8$$

$$RHS: n(n+1)(n+3)(n-2)=1\times2\times4\times-1=-8$$

P_1 *is true*

- *Assumption*

Let's suppose that P_k is true i.e $\sum_{r=1}^{k} 4r^3-12r=k(k+1)(k+3)(k-2)$

Let's show that P_{k+1} is true , let's show that $\sum_{r=1}^{k+1} 4r^3-12r=(k+1)(k+2)(k+4)(k-1)$

- *Induction*

$$\sum_{r=1}^{k+1} 4r^3-12r=\sum_{r=1}^{k} 4r^3-12r+4(k+1)^3-12(k+1)$$
$$=k(k+1)(k+3)(k-2)+4(k+1)^3-12(k+1)$$
$$=(k+1)\left(k(k+3)(k-2)+4(k+1)^2-12\right)$$
$$=(k+1)\left(k^3+k^2-6k+4k^2+8k+4-12\right)$$
$$=(k+1)\left(k^3+5k^2+2k-8\right)=(k+1)(k-1)(k+4)(k+2)\ Q.E.D$$

- *Conclusion*

If P_k is true then P_{k+1} is true. P_1 is true, so according to the induction principle we conclude that the proposition P_n is true for all n ≥ 1.

Question 6:

$a) u_1=6 \ and \ u_{n+1}=3u_n-6$

$u_2=3\times6-6=12$

$u_3=3\times12-6=30$

$u_4=3\times30-6=84$

$b)$ Proposition $P_n : u_n=3^n+3$

- *Basis case*: $n=1$

$$LHS: u_1=6$$

$$RHS: 3^n+3=3^1+3=6$$

P_1 *is true*

- *Assumption*

Suppose that P_k is true i.e $u_k=3^k+3$

Let's show that P_{k+1} is true, i.e Let's show that $u_{k+1}=3^{k+1}+3$

- *Induction*

$$u_{k+1}=3u_k-6=3(3^k+3)-6=3^{k+1}+9-6$$
$$u_{k+1}=3^{k+1}+3 \qquad Q.E.D$$

- *Conclusion*

If P_k is true the P_{k+1} is true. Because P_1 is true we can conclude that P_n is true for all n ≥ 1.

for all n ≥ 1, $u_n=3^n+3$

Question 7:

$a)\ f(1)=7^2-2\times3^2+1=49-18+1=32$

$f(2)=7^4-2\times3^4+1=2240$

$f(3)=7^6-2\times3^6+1=116192$

$b)\ f(k+1)-49f(k)=7^{2k+2}-2\times3^{2k+2}+1-49(7^{2k}-2\times3^{2k}+1)$

$=49\times7^{2k}-18\times3^{2k}-49\times7^{2k}+98\times3^{2k}-48$

$f(k+1)-49f(k)=80\times3^{2k}-48=16(5\times3^{2k}-3)$

$c)$ Proposition $P_n: f(n)=7^{2n}-2\times3^{2n}+1$ is divisible by 16

- *Basis case*: $n=1$

$f(1)=32=16\times2$ is divisible by 16

The proposition is true for $n=1$ (P_1 *is true*)

- *Assumption*

Let's suppose that P_k true i.e $f(k)=7^{2k}-2\times3^{2k}+1$ is divisible by 16

Let's show that P_{k+1} is true,

i.e. let's show that $f(k+1)=7^{2k+2}-2\times3^{2k+2}+1$ is divisible by 16

- *Induction*

$f(k+1)-49f(k)=16(5\times3^{2k}-3)$

$f(k+1)=16(5\times3^{2k}-3)+49f(k)$

$16(5\times3^{2k}-3)$ is a multiple of 16

$49f(k)$ is a multiple of 16 because $f(k)$ is one by supposition.

$f(k+1)$ is therefore a multiple of 16 (as the sum of two multiples of 16)

- *Conclusion*

If P_k is true then P_{k+1} is true. P_1 is true so according to the induction principle, we can say that P_n is true for all $n\geq1$.

Finite series

$Method\,1:$

You might be asked to prove a result using the induction principal.

e.g.: Show by induction that "$for\ all\ n \geq 1, \sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$"

⚠ In FP2, **do not** use the standard series from the formula book (unless told to do so).

$Method\ \ 2:\ Differencing$

Consider two functions $u(r)$ and $f(r)$

If we can write $u(r) = f(r+1) - f(r)$

$$then\ \boxed{\sum_{r=1}^{n} u(r) = \sum_{r=1}^{n} f(r+1) - f(r) = f(n+1) - f(1)}$$

$e.g: f(r) = 2r^2 - 1$

$a)\,Show\ that\ 4r + 2 = f(r+1) - f(r)$

$b)\,Hence\ work\ out\ \sum_{r=1}^{n} 4r + 2$

$Anwers:$

$$a)\,f(r+1) - f(r) = 2(r+1)^2 - 1 - (2r^2 - 1) = 2r^2 + 4r + 2 - 1 - 2r^2 + 1$$
$$= 4r + 2$$

$$b)\sum_{r=1}^{n} 4r + 2 = \sum_{r=1}^{n} f(r+1) - f(r) = f(n+1) - f(1) = 2(n+1)^2 - 1 - 1$$
$$= 2n^2 + 4n$$

$Partial\ \ fractions$

Property:

$$\boxed{\text{if } F(x) = \frac{ax+b}{(cx+d)(ex+f)} \text{ then it exists two real numbers A and B} \quad so\ that\ \ F(x) = \frac{A}{cx+d} + \frac{B}{ex+f}}$$

$$e.g: \frac{1}{r(r+1)} = \frac{A}{r} + \frac{B}{r+1} \qquad \left(\times r(r+1)\ gives\right)$$

$$1 = A(r+1) + B(r)$$

$r = 0\ \ gives\ \ A = 1$

$r = -1\ gives\ B = -1$

$$\frac{1}{r(r+1)} = \frac{1}{r} - \frac{1}{r+1}$$

Finite series - exercises

Question 1:

(a) Given that $f(r)=\frac{1}{2}r(r+2)$, show that

$$f(r+1)-f(r)=r+\frac{3}{2}$$

(b) Use the method of differences to find the value of

$$\sum_{r=50}^{99}\left(r+\frac{3}{2}\right)$$

Question 2:

(a) *Consider the polynomial* $P(x)=x^3+10x^2+37x+60$

$i)$ *Show that* -5 *is a root of* P

$ii)$ *Show that* $P(x)=\left(x+5\right)\left(x^2+5x+12\right)$

(b) Given that $f(r)=\frac{1}{4}(r+1)^2(r+2)^2$, show that

$$f(r+1)-f(r)=(r+2)^3$$

(c) Use the method of differences to work out in terms of n

$$\sum_{r=1}^{n}(r+2)^3$$

Question 3:

$a)$ *Given that* $\frac{2r^2+2r-2}{r(r+1)}=A+B\left(\frac{1}{r}-\frac{1}{r+1}\right)$

find the value of A *and* B

$b)$ Find the value of

$$\sum_{r=1}^{99}\frac{2r^2+2r-2}{r(r+1)}$$

Question 4:

$a)$ Show that

$$\frac{1}{(n+1)(n+2)}=\frac{1}{n+1}-\frac{1}{n+2}$$

$b)$ Hence find the sum of the first n terms of the series

$$\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\frac{1}{30}+....$$

Question 5:

The sum to r terms, S_r , of a series is given by

$$S_r=r(r+1)(r-1)^2$$

Given that u_r is the r^{th} term of the series whose sum is S_r , show that:

(a) i) $u_1=0$

ii) $u_2=6$ *and* $u_3=42$

(iii) $u_n=n(n-1)(4n-5)$

(b) Show that

$$\sum_{r=n+1}^{2n}u_r=n(15n^3-7n^2-3n+1)$$

Finite series - exercises - Answers

Question 1:

$a)\ f(r)=\frac{1}{2}r(r+2)$

$$f(r+1)-f(r)=\frac{1}{2}(r+1)(r+3)-\frac{1}{2}r(r+2)=\frac{1}{2}\left(r^2+4r+3-r^2-2r\right)$$

$$=\frac{1}{2}(2r+3)=r+\frac{3}{2}$$

$$b)\ \sum_{r=50}^{99}\left(r+\frac{3}{2}\right)=\sum_{r=50}^{99}f(r+1)-f(r)=f(100)-f(50)$$

$$=\frac{1}{2}\times100\times102-\frac{1}{2}\times50\times52=3800$$

Question 2:

$a)$ Is -5 a root of $x^3+10x^2+37x+60$?

$$(-5)^3+10\times(-5)^2+37\times(-5)+60=-125+250-185+60=310-310=0$$

$$x^3+10x^2+37x+60=(x+5)\left(x^2+5x+12\right)$$

$$b)\ f(r+1)-f(r)=\frac{1}{4}(r+2)^2(r+3)^2-\frac{1}{4}(r+1)^2(r+2)^2$$

$$=\frac{1}{4}\left(r^4+10r^3+37r^2+60r+36\right)-\frac{1}{4}\left(r^4+6r^3+13r^2+12r+4\right)$$

$$=\frac{1}{4}\left(4r^3+24r^2+48r+32\right)=r^3+6r^2+12r+8$$

On the other hand $(R.H.S.):(r+2)^3=r^3+3\times r^2\times2+3\times r\times2^2+2^3=r^3+6r^2+12r+8$

$$f(r+1)-f(r)=(r+2)^3$$

$$c)\ \sum_{r=1}^{n}(r+2)^3=\sum_{r=1}^{n}f(r+1)-f(r)=f(n+1)-f(1)=\frac{1}{4}(n+2)^2(n+3)^2-\frac{1}{4}(36)$$

$$=\frac{1}{4}\left(n^4+10n^3+37n^2+60n\right)=\frac{1}{4}n\left(n^3+10n^2+37n+60\right)$$

$$\sum_{r=1}^{n}(r+2)^3=\frac{1}{4}n(n+5)(n^2+5n+12)$$

Question 3:

I am going to express the R.H.S as a single fraction
and identify with the L.H.S

$$A+B\left(\frac{1}{r}-\frac{1}{r+1}\right)=\frac{Ar(r+1)+B(r+1)-Br}{r(r+1)}\quad=\frac{Ar^2+Ar+B}{r(r+1)}$$

This should be equal to $\frac{2r^2+2r-2}{r(r+1)}$ *so* $A=2\ and\ B=-2$

$$b)\ \sum_{r=1}^{99}\frac{2r^2+2r-2}{r(r+1)}=\sum_{r=1}^{99}2-2\left(\frac{1}{r}-\frac{1}{r+1}\right)=\sum_{r=1}^{99}2-2\sum_{r=1}^{99}\frac{1}{r}-\frac{1}{r+1}=2\times99-2\left(1-\frac{1}{100}\right)=196.02$$

Question 4:

$$\frac{1}{(n+1)(n+2)}=\frac{A}{n+1}+\frac{B}{n+2}\qquad(\times(n+1)(n+2)\ gives)$$

$$1=A(n+2)+B(n+1)$$

$for\ n=-1\quad 1=A$

$for\ n=-2\quad 1=-B\quad B=-1$

$$Conclusion:\frac{1}{(n+1)(n+2)}=\frac{1}{n+1}-\frac{1}{n+2}$$

$$b)\ \frac{1}{6}+\frac{1}{12}+\frac{1}{20}+...=\frac{1}{2\times3}+\frac{1}{3\times4}+\frac{1}{4\times5}+...=\sum_{r=1}^{n}\frac{1}{(r+1)(r+2)}=\sum_{r=1}^{n}\frac{1}{r+1}-\frac{1}{r+2}$$

$$=\frac{1}{2}-\frac{1}{n+2}=\frac{n}{2n+4}$$

Question 5:

$$S_r=u_1+u_2+u_3+...+u_r=r(r+1)(r-1)^2$$

$a)\,i)\ S_1=u_1=1\times2\times0^2=0$

$ii)\ S_2=u_1+u_2=2\times3\times1^2=6\qquad u_2=6$

$S_3=u_1+u_2+u_3=0+6+u_3=3\times4\times2^2=48\qquad hence\ \ u_3=48-6=42$

$$iii)\ u_n=S_n-S_{n-1}=n(n+1)(n-1)^2-(n-1)(n)(n-2)^2=n(n-1)\left[(n+1)(n-1)-(n-2)^2\right]$$

$$=n(n-1)\left[n^2-1-n^2+4n-4\right]\qquad u_n=n(n-1)(4n-5)$$

$$b)\ \sum_{r=n+1}^{2n}u_r=\sum_{r=1}^{2n}u_r-\sum_{r=1}^{n}u_r$$

$$=S_{2n}-S_n=2n(2n+1)(2n-1)^2-n(n+1)(n-1)^2=n\left[2(2n+1)(2n-1)^2-(n+1)(n-1)^2\right]$$

$$=n\left[(4n+2)(4n^2-4n+1)-(n+1)(n^2-2n+1)\right]=n\left(16n^3-8n^2-4n+2-n^3+n^2+n-1\right)$$

$$=n\left(15n^3-7n^2-3n+1\right)$$

Techniques of integration

I) The usual functions

1) Power, logarithm, exponential, trigonometric functions

- $\int x^n dx = \frac{1}{n+1}x^{n+1} + c \qquad n \neq -1$
- $\int \frac{1}{x} dx = \ln|x| + c$
- $\int e^x dx = e^x + c$ c is any real number : $c \in \mathbb{R}$
- $\int \cos(x) dx = \sin(x) + c$
- $\int \sin(x) dx = -\cos(x) + c$

2) Usual function composed with a linear function :

The general integral is $\int g(ax+b)dx = \frac{1}{a}G(ax+b) + c$ where G is an integral of g

which means:

- $\int (ax+b)^n dx = \frac{1}{a} \times \frac{1}{n+1}(ax+b)^{n+1} + c \qquad n \neq -1$
- $\int \frac{1}{ax+b} dx = \frac{1}{a}\ln|ax+b| + c$
- $\int e^{ax+b} dx = \frac{1}{a}e^{ax+b} + c$ c is any real number : $c \in \mathbb{R}$
- $\int \cos(ax+b) dx = \frac{1}{a}\sin(ax+b) + c$
- $\int \sin(ax+b) dx = -\frac{1}{a}\cos(ax+b) + c$

II) Recognising expressions

The general integral is $\int f' \times g[f(x)]dx = G[f(x)] + c$ where G is an integral of g

which means:

- $\int f' \times f^n dx = \frac{1}{n+1} f^{n+1} + c \qquad n \neq -1$
- $\int \frac{f'}{f} dx = \ln|f| + c$
- $\int f' \times e^f dx = e^f + c$
- $\int f' \times \cos(f) dx = \sin(f) + c$
- $\int f' \times \sin(f) dx = -\cos(f) + c$

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III) Integration by part

Formula: $\int_a^b u'\times v=[u\times v]_a^b-\int_a^b u\times v'$

This technique is used to integrate functions like:

$x^n e^x,\ x^n\ln(x),\ x^n\cos(x),\ x^n\sin(x)$ where n is an positive integer

IV) Integration by substitution

We use this technique when changing variable make the integral easier to work out

1) Direct substitution

Consider

$$I=\int f(x)dx \quad let\ x=h(u)$$

$$We\ have\ then\ f(x)=f[h(u)]\ and\ dx=h'(u)du$$

$$\text{the integral becomes } I=\int f[h(u)]\times h'(u)du \text{ (and should be easier to integrate)}$$

If the integral is definite (integrate between two values), do not forget to change them according to *u*.

2) Indirect substitution

It is the same principal, but instead, we let a whole expression of *x* be *u*.

V) Partial fractions

An algebraic fraction can be written $\frac{P(x)}{Q(x)}$ where P(x) and Q(x) are polynomials.

Like for example: $\frac{3x+4}{x^2-4x+6}$ or $\frac{4x-5}{x+1}$... How do we integrate those functions?

Case 1: degree of P = degree of Q (=1)

$$If\ \frac{P(x)}{Q(x)}=\frac{ax+b}{cx+d},\ \text{transform into } \frac{a}{c}+\frac{e}{cx+d}\quad where\ e=b-\frac{ad}{c}.$$

Example 1: $\frac{2x+3}{x+2}=\frac{2(x+2)+e}{x+2}=\frac{2(x+2)-1}{x+2}=\frac{2(x+2)}{x+2}-\frac{1}{x+2}=2-\frac{1}{x+2}$

Therefore $\int\frac{2x+3}{x+2}dx=\int 2-\frac{1}{x+2}dx=2x-\ln(x+2)+c$

Example 2: $\frac{3x+2}{2x-1}=\frac{\frac{3}{2}(2x-1)+e}{2x-1}=\frac{\frac{3}{2}(2x-1)+\frac{7}{2}}{2x-1}=\frac{3}{2}+\frac{7/2}{2x-1}=\frac{3}{2}+\frac{7}{2}\times\frac{1}{2x-1}$

Therefore $\int\frac{3x+2}{2x-1}dx=\int\frac{3}{2}+\frac{7}{2}\times\frac{1}{2x-1}dx=\frac{3}{2}x+\frac{7}{2}\times\frac{1}{2}\ln(2x-1)+c=\frac{3}{2}x+\frac{7}{4}\ln(2x-1)+c$

Case 2: degree of P < degree of Q

Theorem:

If an algebraic fraction is $R(x) \;=\; \dfrac{cx+d}{(rx+s)(px+q)}$, then we can find two numbers A and B so that

$$R(x) = \frac{A}{rx+s} + \frac{B}{px+q}.$$

How to find A and B?

Consider the algebraic fraction $\dfrac{5x+4}{x^2+x-2}$.

Factorise the denominator: $\dfrac{5x+4}{(x+2)(x-1)}$, then equal this expression to $\dfrac{A}{x+2} + \dfrac{B}{x-1}$

$\dfrac{5x+4}{(x+2)(x-1)} = \dfrac{A}{(x+2)} + \dfrac{B}{(x-1)}$. Now multiply by $(x+2)(x+1)$ both sides

We have

$5x+4 = A(x-1)+B(x+2)$. Because this identity is true for all values of x,

it is true in particular for $x = 1$ *and* $x = -2$.

If $x = 1$, $\quad 5\times 1+4 = A(1-1)+B(1+2)$

$\qquad 9 = 3B \qquad\qquad B = 3$

If $x = -2$, $\quad 5\times -2+4 = A(-2-1)+B(-2+2)$

$\qquad -6 = -3A \qquad\qquad A = 2$

$\dfrac{5x+4}{(x+2)(x-1)} = \dfrac{2}{x+2} + \dfrac{3}{x-1}$. We can now integrate this function.

$$\int \frac{5x+4}{(x+2)(x-1)}dx = \int \frac{2}{x+2} + \frac{3}{x-1}dx = 2\ln|x+2| + 3\ln|x-1| + c = \ln\left((x+2)^2|x-1|^3\right) + c$$

VI) Trigonometric functions

1) tan(x)

Remember: $\dfrac{d}{dx}\tan(x) = \dfrac{1}{\cos^2(x)} = 1+\tan^2(x) = \sec^2(x)$

Therefore also remember that:

- $\int \sec^2(x)dx = \int \dfrac{1}{\cos^2(x)}dx = \tan(x)+c$
- $\int \tan^2(x) = \tan(x) - x + c$

$Explanation: \int \tan^2(x)dx = \int 1+\tan^2(x) - 1\,dx = \tan(x) - x + c$

2) Recognising expressions

Instead of learning yet another formula, it is often useful to recognise familiar expressions (see chapter II)

$\int \tan(x)dx = \int \dfrac{\sin(x)}{\cos(x)} = -\int \dfrac{-\sin(x)}{\cos(x)}dx$. This is the form $\dfrac{f\,'}{f}$ which integrate into $\ln|f| + c$

$$\int \tan(x)dx = -\ln|\cos(x)| + c = \ln\left|\frac{1}{\cos(x)}\right| + c = \ln|\sec(x)| + c$$

$\int \sec(x)tan(x)dx = \int \frac{1}{\cos(x)} \times \frac{\sin(x)}{\cos(x)} dx = -\int \frac{-\sin(x)}{\cos^2(x)} dx$. This is the form $f' \times f^n$ with $n = -2$, which integrate into $\frac{1}{n+1} f^{n+1} + c$.

- $\int \sec(x)\tan(x)dx = -\frac{1}{-1}\cos^{-1}(x) + c = \sec(x) + c$

3) Linearisation of cos² and sin²

How to integrate $\cos^2(x)$ *and* $\sin^2(x)$?
We use the following formulae ("double angle" formulae re-arranged)

$$\cos^2(x) = \frac{1}{2} + \frac{1}{2}\cos(2x)$$

$$\sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x)$$

VII) *Useful substitution

If the integral includes	try
$(ax+b)^n$	$ax+b=u$
$\sqrt[n]{ax+b}$	$ax+b=u^n$
$a-bx^2$	$x=\sqrt{\frac{b}{a}}\sin(u)$
$a+bx^2$	$x=\sqrt{\frac{b}{a}}\tan(u)$
bx^2-a	$x=\sqrt{\frac{b}{a}}\sec(u)$
e^x	$e^x=u$
$\ln(ax+b)$	$ax+b=e^u$

VIII) Useful trigonometric substitution $t = Tan\left(\frac{1}{2}x\right)$

When using this substitution, Tan(x), Cos(x) and Sin(x) become rational functions.

$$Tan(x) = \frac{2t}{1-t^2}$$

$$Cos(x) = \frac{1-t^2}{1+t^2}$$

$$Sin(x) = \frac{2t}{1+t^2}$$

$$dx = \frac{2}{1+t^2}dt$$

Example:

$$\int \frac{1}{Sinx}dx = \int \frac{1}{\frac{2t}{1+t^2}} \times \frac{2}{1+t^2}dt = \int \frac{2}{2t}dt = \int \frac{1}{t}dt = \ln|t| + c$$

$$\int \frac{1}{Sinx}dx = \ln\left|Tan\left(\frac{1}{2}x\right)\right| + c$$

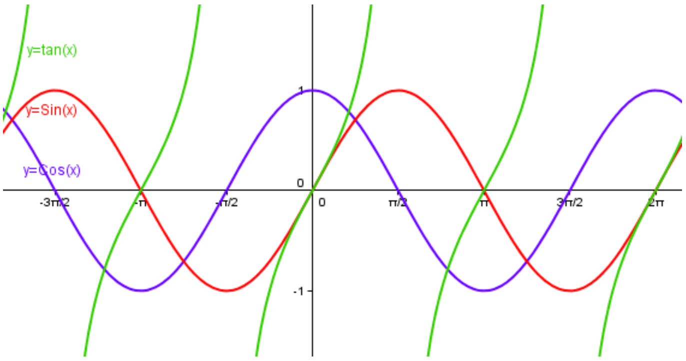
Calculus of inverse trig functions

Reminder

$$y=\cos x \qquad \frac{dy}{dx}=-\sin x$$

$$y=\sin x \qquad \frac{dy}{dx}=\cos x$$

$$y=\tan x \qquad \frac{dy}{dx}=1+\tan^2 x=\frac{1}{\cos^2 x}=\sec^2 x$$



Inverse trig functions

$\arccos(x)=\cos^{-1}(x)$

$[-1,1]\to[0,\pi]$

$a\to\theta \qquad so\ that\ \cos(\theta)=a$

$\arcsin(x)=\sin^{-1}(x)$

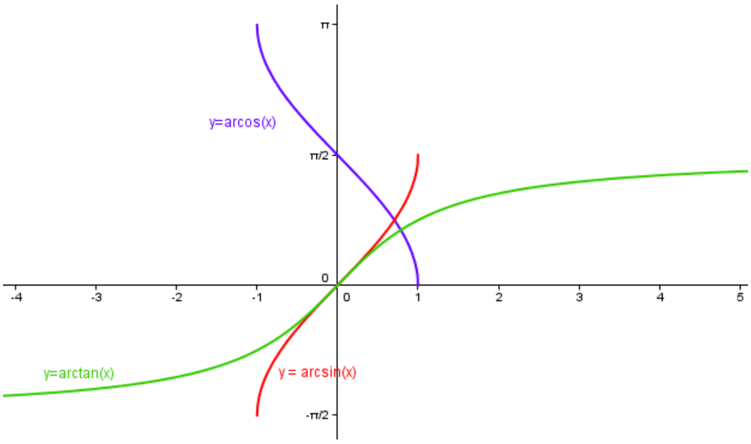
$[-1,1]\to[-\frac{\pi}{2},\frac{\pi}{2}]$

$a\to\theta \qquad so\ that\ \sin(\theta)=a$

$\arctan(x)=\tan^{-1}(x)$

$\mathbb{R}\to[-\frac{\pi}{2},\frac{\pi}{2}]$

$a\to\theta \qquad so\ that\ \tan(\theta)=a$



Calculus with inverse trig functions

Differentiation

$$y=\cos^{-1}x \qquad \frac{dy}{dx}=-\frac{1}{\sqrt{1-x^2}} \qquad for\ \ -1<x<1$$

$$y=\sin^{-1}x \qquad \frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}} \qquad for -1<x<1$$

$$y=\tan^{-1}x \qquad \frac{dy}{dx}=\frac{1}{1+x^2} \qquad for\ all\ x$$

Integration

If $a>0$

$$\int\frac{dx}{a^2+x^2}=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+k \qquad \int\frac{dx}{\sqrt{a^2-x^2}}=\sin^{-1}\left(\frac{x}{a}\right)+k$$

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Inverse trig functions and calculus - exercises

Question 1: Differentiation

Differentiate the following functions

a) $y = \sin^{-1} 4x$ $\quad$ b) $y = \sin^{-1} \frac{1}{2}x$ $\quad$ c) $y = 3\cos^{-1} \frac{x}{3}$ $\quad$ d) $y = x\tan^{-1}(x)$

e) $y = \cos^{-1}(\sqrt{x})$ $\quad$ f) $y = 3\tan^{-1}(4x^2)$

Question 2: Integration

Find the value of the integrals

a) $\int_0^6 \frac{dx}{\sqrt{36 - x^2}}$ $\quad$ b) $\int_0^1 \frac{dx}{1 + x^2}$ $\quad$ c) $\int_{-\frac{3}{2}}^{\frac{3}{2}} \frac{dx}{4x^2 + 9}$ $\quad$ d) $\int \frac{dx}{x^2 + 6x + 18}$

e) $\int \frac{4x + 5}{\sqrt{4 - 6x - x^2}} dx$ $\quad$ f) $\int \frac{6x + 3}{x^2 + 6x + 13} dx$

Question 3:

Use the substitution $x\sqrt{3} = 2Tan\theta$ to show that $\int_0^2 \frac{1}{(3x^2 + 4)^{\frac{3}{2}}} dx = \frac{1}{8}$

Question 4:

(a) Differentiate $x\tan^{-1}x$ with respect to x. *(2 marks)*

(b) Show that

$$\int_0^1 \tan^{-1} x \, \mathrm{d}x = \frac{\pi}{4} - \ln\sqrt{2}$$

(5 marks)

Question 5:

By using the substitution $u = x - 2$, or otherwise, find the exact value of

$$\int_{-1}^{5} \frac{\mathrm{d}x}{\sqrt{32 + 4x - x^2}}$$

(5 marks)

Question 6:

a) Use the substitution $t = \tan\theta$ to transforms the integral $\int \frac{d\theta}{9Cos^2\theta + Sin^2\theta}$ into $\int \frac{dt}{9 + t^2}$

b) *Hence show that* $\int_0^{\frac{\pi}{3}} \frac{d\theta}{9Cos^2\theta + Sin^2\theta} = \frac{\pi}{18}$

Inverse trig functions and calculus - exercises - answers

Question 1: Differentiation

$a)\ \frac{dy}{dx}=\frac{4}{\sqrt{1-16x^2}}$ $\quad b)\ \frac{dy}{dx}=\frac{1}{\sqrt{4-x^2}}$ $\quad c)\ \frac{dy}{dx}=\frac{-3}{\sqrt{9-x^2}}$

$d)\ \frac{dy}{dx}=\tan^{-1}(x)+\frac{x}{1+x^2}$ $\quad e)\ \frac{dy}{dx}=\frac{-1}{2\sqrt{x}\sqrt{1-x}}$ $\quad f)\ \frac{dy}{dx}=\frac{24x}{16x^4+1}$

Question 2: Integration

$$a)\int_0^6\frac{dx}{\sqrt{36-x^2}}=\left[\sin^{-1}\left(\frac{x}{6}\right)\right]_0^6=\sin^{-1}1-\sin^{-1}0=\frac{\pi}{2}$$

$$b)\int_0^1\frac{dx}{1+x^2}=\left[\tan^{-1}(x)\right]_0^1=\tan^{-1}1-\tan^{-1}0=\frac{\pi}{4}$$

$$c)\int_{-\frac{3}{2}}^{\frac{3}{2}}\frac{dx}{4x^2+9}=\frac{1}{4}\int_{-\frac{3}{2}}^{\frac{3}{2}}\frac{dx}{x^2+\frac{9}{4}}=\left[\frac{1}{4}\times\frac{2}{3}\tan^{-1}\left(\frac{2x}{3}\right)\right]_{-\frac{3}{2}}^{\frac{3}{2}}=\frac{1}{6}\tan^{-1}(1)-\frac{1}{6}\tan^{-1}(-1)=\frac{\pi}{12}$$

$$d)\int\frac{dx}{x^2+6x+18}=\int\frac{dx}{(x+3)^2+9}=\frac{1}{3}\tan^{-1}\left(\frac{x+3}{9}\right)+c$$

$$e)\int\frac{4x+5}{\sqrt{4-6x-x^2}}dx=\int\frac{4x+12}{\sqrt{4-6x-x^2}}-\frac{7}{\sqrt{4-6x-x^2}}dx=-2\int\frac{-2x-6}{\sqrt{4-6x-x^2}}-\frac{7}{\sqrt{4-6x-x^2}}dx$$

$$=-2\left(2\sqrt{4-6x-x^2}\right)-7\int\frac{dx}{\sqrt{4-6x-x^2}}=-4\sqrt{4-6x-x^2}-7\int\frac{dx}{\sqrt{13-(x+3)^2}}$$

$$=-4\sqrt{4-6x-x^2}-7\sin^{-1}\left(\frac{x+3}{\sqrt{13}}\right)+c$$

$$f)\int\frac{6x+3}{x^2+6x+13}dx=\int\frac{6x+18}{x^2+6x+13}-\frac{15}{x^2+6x+13}dx=3\int\frac{2x+6}{x^2+6x+13}dx-15\int\frac{1}{x^2+6x+13}dx$$

$$=3\ln\left(x^2+6x+13\right)-15\int\frac{dx}{(x+3)^2+4}=3\ln\left(x^2+6x+13\right)-\frac{15}{2}\tan^{-1}\left(\frac{x+3}{2}\right)+c$$

Question 3:

$x\sqrt{3}=2\tan\theta \qquad x=\frac{2}{\sqrt{3}}\tan\theta \qquad \frac{dx}{d\theta}=\frac{2}{\sqrt{3}}(1+\tan^2\theta)$

$x=0\quad \theta=0 \quad$ *and* $\quad x=2\ \ \tan\theta=\sqrt{3}\quad \theta=\frac{\pi}{3}$

$$I=\int_0^2\frac{1}{\left(3x^2+4\right)^{\frac{3}{2}}}dx=\int_0^{\frac{\pi}{3}}\frac{2}{\sqrt{3}}\frac{1+\tan^2\theta}{\left(4\tan^2\theta+4\right)^{\frac{3}{2}}}d\theta=\int_0^{\frac{\pi}{3}}\frac{1}{4\sqrt{3}}\frac{1}{\sqrt{1+\tan^2\theta}}d\theta$$

$$=\frac{\sqrt{3}}{12}\int_0^{\frac{\pi}{3}}\cos\theta d\theta=\frac{\sqrt{3}}{12}[\sin\theta]_0^{\frac{\pi}{3}}=\frac{\sqrt{3}}{12}\left(\sin\frac{\pi}{3}-\sin 0\right)=\frac{\sqrt{3}}{12}\times\frac{\sqrt{3}}{2}=\frac{3}{24}=\frac{1}{8}$$

Question 4:

$a)\ y=x\tan^{-1}x \qquad \frac{dy}{dx}=\tan^{-1}x+\frac{x}{1+x^2}$

$$b)\int_0^1\tan^{-1}(x)dx=\int_0^1\frac{dy}{dx}-\frac{x}{1+x^2}dx=\left[x\tan^{-1}x\right]_0^1-\left[\frac{1}{2}\ln(1+x^2)\right]_0^1$$

$$=\tan^{-1}1-\frac{1}{2}\ln(2)=\frac{\pi}{4}-\ln\sqrt{2}$$

Question 5:

$$\int_{-1}^5\frac{dx}{\sqrt{32+4x-x^2}}=\int_{-1}^5\frac{dx}{\sqrt{36-(x-2)^2}}=\int_{-3}^3\frac{du}{\sqrt{36-u^2}}=\left[\sin^{-1}\left(\frac{u}{6}\right)\right]_{-3}^3$$

$$=\sin^{-1}\frac{1}{2}-\sin^{-1}-\frac{1}{2}=\frac{\pi}{6}+\frac{\pi}{6}=\frac{\pi}{3}$$

Question 6:

$t=\tan\theta \qquad \frac{dt}{d\theta}=1+\tan^2\theta=1+t^2\ d\theta=\frac{dt}{1+t^2}$

$\cos^2\theta=\frac{1}{1+\tan^2\theta}=\frac{1}{1+t^2}$

$\sin^2\theta=1-\cos^2\theta=1-\frac{1}{1+t^2}=\frac{t^2}{1+t^2}$

$$\int\frac{d\theta}{9\cos^2\theta+\sin^2\theta}=\int\frac{dt}{1+t^2}\times\frac{1}{\frac{9}{1+t^2}+\frac{t^2}{1+t^2}}=\int\frac{dt}{9+t^2}$$

$\theta=\frac{\pi}{3}\quad t=\tan\frac{\pi}{3}=\sqrt{3}$

$\theta=0\quad t=\tan 0=0$

$$\int_0^{\frac{\pi}{3}}\frac{d\theta}{9\cos^2\theta+\sin^2\theta}=\int_0^{\sqrt{3}}\frac{dt}{9+t^2}=\left[\frac{1}{3}\tan^{-1}\left(\frac{t}{3}\right)\right]_0^{\sqrt{3}}=\frac{1}{3}\tan^{-1}\frac{\sqrt{3}}{3}=\frac{1}{3}\times\frac{\pi}{6}=\frac{\pi}{18}$$

Formulae | Roots | Complex numbers | De Moivre theorem | Proof by induction | Finite series | Inverse trig functions | Hyperbolic functions | Arc length | Past Papers | Jan 2006 | Jun 2006 | Jan 2007 | Jun 2007 | Jan 2008 | Jun 2008 | Jan 2009 | Jun 2009 | Jan 2010 | Jun 2010

Hyperbolic functions

Property:Any function can be written as the some of an even and odd function

In particular, $y = e^x$ can be written $e^x = \cosh x + \sinh x$.

$\mathbb{R} \to [1, +\infty[$ *and* $\mathbb{R} \to \mathbb{R}$

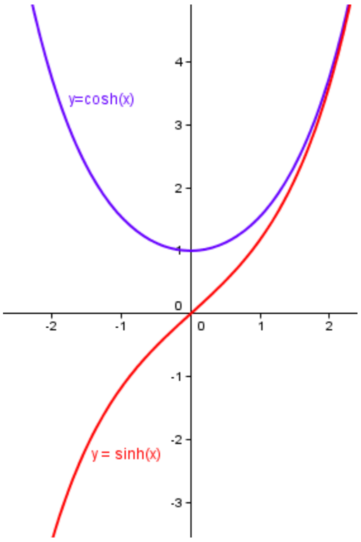
$x \to \cosh(x) = \frac{1}{2}\left(e^x + e^{-x}\right)$ $\qquad$ $x \to \sinh(x) = \frac{1}{2}\left(e^x - e^{-x}\right)$

- cosh is an even function because $\cosh(-x) = \cosh(x)$

  The graph of cosh is symmetrical around the y-axis
- sinh is an odd function because $\sinh(-x) = -\sinh(x)$

  The graph of sinh is symmetrical around the point O

  (or rotational symmetry order 2)



Properties and calculus

In a certain respect, the properties of cosh and sinh are very similar to those of cos and sin. You need to know and to be able to prove that:

- $\cosh^2 x - \sinh^2 x \equiv 1$ $\qquad$ ("$\equiv$"indicates an identity and means "= for all x")
- $\cosh(A \pm B) \equiv \cosh A \cosh B \pm \sinh A \sinh B$

  $\sinh(A \pm B) \equiv \sinh A \cosh B \pm \cosh A \sinh B$
- $\cosh(2A) \equiv \cosh^2 A + \sinh^2 A \equiv 1 + 2\sinh^2 A \equiv 2\cosh^2 A - 1$

  $\sinh 2A \equiv 2 \sinh A \cosh A$

Calculus

$y = \cosh x \qquad \frac{dy}{dx} = \sinh x$ $\qquad$ *and* $\qquad \int \sinh x \, dx = \cosh x + k$

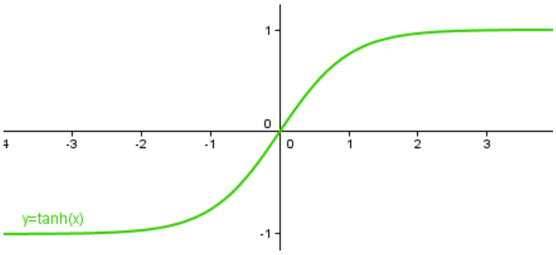
$y = \sinh x \qquad \frac{dy}{dx} = \cosh x$ $\qquad$ *and* $\qquad \int \cosh x \, dx = \sinh x + k$

Other hyperbolic functions

Tanh

- $\mathbb{R} \to]-1, 1[$

  $x \to \tanh(x) = \frac{\sinh x}{\cosh x}$
- $y = \tanh x \qquad \frac{dy}{dx} = 1 - \tanh^2 x = \frac{1}{\cosh^2 x} = \sec h^2 x$



- $\frac{1}{\cosh x} = \operatorname{sech} x$ $\qquad$ $\frac{1}{\sinh x} = \operatorname{cosech}(x)$

  $\frac{1}{\tanh x} = \coth x$

Hyperbolic functions - exercises

Question 1:

Express, in terms of exponentials:

a) $\operatorname{sech} x$ b) $\coth x$ c) $\tanh\left(\frac{1}{2}x\right)$ (d) $\operatorname{cosech}(3x)$.

Question 2:

Show that

$a)\cosh^2 x-\sinh^2 x=1$

$b)\ \sinh(x-y)=\sinh x\cosh y\ -\ \cosh x\sinh y$

$c)\ \cosh(x\pm y)=\cosh x\cosh y\pm\sinh x\sinh y$

$d)\tanh 2x=\dfrac{2\tanh x}{1+\tanh^2 x}$

Question 3:

Given that $u=\tanh x$, use the definitions of $\sinh x$ *and* $\cosh x$ in terms of e^x *and* e^{-x}

to show that $x=\frac{1}{2}\ln\left(\frac{1+u}{1-u}\right)$

Question 4:

Differentiate the following expressions:

$a)\ \cosh 3x$ $b)\ \cosh^2(3x)$ $c)\ x^2\cosh x,$

$d)\ \dfrac{\cosh 2x}{x}$ $e)\ x\tanh x$ $f)\ \operatorname{sech} x$ $g)\ \operatorname{cosech} x.$

Question 5:

It is given that $x=\frac{1}{2}\cosh 2t$ *and* $y=2\sinh t$

Express $\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2$ in terms of $\cosh t$.

Question 6:

Given that $y=\ln\left(\tanh\frac{x}{2}\right)$, *where* $x>0$, *show that* $\frac{dy}{dx}=\operatorname{cosech} x$ (6 *marks*)

Question 7:

Evaluate the following integrals:

(a) $\int\cosh 3x\,dx$ b) $\int\cosh^2 x\,dx$ c) $\int x\sinh 2x\,dx$ d) $\int\tanh^2 x\,dx$

Question 8:

$a)$ Given that $u=\cosh^2 x$, *show that* $\frac{du}{dx}=\sinh 2x.$

$b)$ *Hence show that* $\int_0^1\frac{\sinh 2x}{1+\cosh^4 x}dx=\tan^{-1}(\cosh^2 1)-\frac{\pi}{4}$ (5 *marks*)

Question 9:

Use the substitution $x=4\sinh^2\theta$ to show that $\int\sqrt{\frac{x+4}{x}}dx=2\sinh 2\theta+4\theta+c$ (5 *marks*)

Hyperbolic functions – exercises - answers

Question 1:

$a)\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$ $\qquad b)\coth x = \frac{1}{\tanh x} = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{e^{2x}+1}{e^{2x}-1}$ $(after \times \frac{e^x}{e^x})$

$c)\tanh\left(\frac{1}{2}x\right) = \frac{e^{\frac{x}{2}} - e^{-\frac{x}{2}}}{e^{\frac{x}{2}} + e^{-\frac{x}{2}}} = \frac{e^x - 1}{e^x + 1}$ $\quad (after \times \frac{e^{\frac{x}{2}}}{e^{\frac{x}{2}}})$ $\qquad d)\operatorname{cosech} 3x = \frac{1}{\sinh 3x} = \frac{2}{e^{3x} - e^{-3x}}$

Question 2:

$a)\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{1}{4}\left(e^{2x} + e^{-2x} + 2 - e^{2x} - e^{-2x} + 2\right) = \frac{1}{4} \times 4 = 1$

$b)\sinh x \cosh y - \cosh x \sinh y = \frac{e^x - e^{-x}}{2} \times \frac{e^y + e^{-y}}{2} - \frac{e^x + e^{-x}}{2} \times \frac{e^y - e^{-y}}{2}$

$= \frac{e^{x+y} + e^{x-y} - e^{-x+y} - e^{-x-y}}{4} - \frac{e^{x+y} - e^{x-y} + e^{-x+y} - e^{-x-y}}{4} = \frac{2e^{x-y} - 2e^{-x+y}}{4} = \frac{e^{x-y} - e^{-x+y}}{2} = \sinh(x-y)$

$c)$ *same technique*

$d)1 + \tanh^2 x = 1 + \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)^2 = \frac{(e^x + e^{-x})^2 + (e^x - e^{-x})^2}{(e^x + e^{-x})^2} = \frac{2(e^{2x} + e^{-2x})}{(e^x + e^{-x})^2}$

$\frac{2\tanh x}{1 + \tanh^2 x} = \frac{2(e^x - e^{-x})}{(e^x + e^{-x})} \times \frac{(e^x + e^{-x})^2}{2(e^{2x} + e^{-2x})} = \frac{(e^x - e^{-x})(e^x + e^{-x})}{(e^{2x} + e^{-2x})} = \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}} = \tanh 2x$

Question 3:

$u = \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1}$

$u(e^{2x} + 1) = e^{2x} - 1$ $\qquad make(e^{2x}\ the\ subject)$ $\qquad e^{2x}(u-1) = -1 - u$

$e^{2x} = \frac{-1-u}{u-1} = \frac{1+u}{1-u}$ $\quad so \quad 2x = \ln\left(\frac{1+u}{1-u}\right)$ $\qquad and \qquad x = \frac{1}{2}\ln\left(\frac{1+u}{1-u}\right)$

Question 4:

$a)3\sinh 3x$ $\qquad b)2 \times 3\sinh(3x) \times \cosh(3x) = 3\sinh 6x$

$c)2x\cosh x + x^2 \sinh x$ $\qquad d)\frac{2\sinh 2x \times x - \cosh 2x}{x^2} = \frac{2x\sinh 2x - \cosh 2x}{x^2}$

$e)\tanh x + \frac{x}{\cosh^2 x}$ $\qquad f)\ y = \operatorname{sech} x = \frac{1}{\cosh x}, \ \frac{dy}{dx} = -\frac{\sinh x}{\cosh^2 x} = -\tanh x \operatorname{sech} x$

$g)\ y = \operatorname{cosech} x = \frac{1}{\sinh x}, \ \frac{dy}{dx} = -\frac{\cosh x}{\sinh^2 x} = -\coth x \operatorname{cosech} x$

Question 5:

$x = \frac{1}{2}\cosh 2t \qquad \frac{dx}{dt} = \frac{1}{2} \times 2\sinh 2t = \sinh 2t \ \ and \ \ y = 2\sinh t \qquad \frac{dy}{dx} = 2\cosh t$

$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \sinh^2 2t + 4\cosh^2 t$

$\sinh^2 2t = (2\sinh t \cosh t)^2 = 4\sinh^2 t \cosh^2 t = 4(\cosh^2 t - 1)\cosh^2 t = 4\cosh^4 t - 4\cosh^2 t$

$Hence \ \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 4\cosh^4 t$

Question 6:

$y = \ln\left(\tanh\frac{x}{2}\right) \qquad First\ let's\ work\ out\ \frac{d}{dx}\left(tanh\frac{x}{2}\right)$

$\frac{d}{dx}\left(tanh\frac{x}{2}\right) = \frac{1}{2\cosh^2\frac{x}{2}}$

$Now \ \frac{dy}{dx} = \frac{\frac{d}{dx}\left(tanh\frac{x}{2}\right)}{\tanh\frac{x}{2}} = \frac{1}{2\cosh^2\frac{x}{2}} \times \frac{\cosh\frac{x}{2}}{\sinh\frac{x}{2}} = \frac{1}{2\sinh\frac{x}{2}\cosh\frac{x}{2}} = \frac{1}{\sinh x} = \operatorname{cosech} x$

Question 7:

$a)\int \cosh 3x\, dx = \frac{1}{3}\sinh 3x + c \qquad b)\int \cosh^2 x\, dx = \int \frac{1}{2}\cosh 2x + \frac{1}{2}dx = \frac{1}{4}\sinh 2x + \frac{1}{2}x + c$

$c)\int x\sinh 2x\, dx = \int uv' = [uv] - \int u'v = \frac{1}{2}x\cosh 2x - \int\frac{1}{2}\cosh 2x\, dx$

$= \frac{1}{2}x\cosh 2x - \frac{1}{4}\sinh 2x + c$

$d)\int \tanh^2 x\, dx = -\int 1 - \tanh^2 x - 1\, dx = -\tanh x + x + c$

Question 8:

$a)u = \cosh^2 x \qquad \frac{du}{dx} = 2 \times \sinh x \times \cosh x = \sinh 2x \qquad (du = \sinh 2x dx)$

$b)\ when\ \ x = 0,\ u = \cosh^2 0 = 1 \quad and \quad when\ \ x = 1,\ \ u = \cosh^2 1$

$I = \int_0^1 \frac{\sinh 2x\, dx}{1 + \cosh^4 x} = \int_0^{\cosh^2 1} \frac{du}{1+u^2} = \left[\tan^{-1} u\right]_0^{\cosh^2 1} = \tan^{-1}\left(\cosh^2 1\right) - \tan^{-1} 0 = \tan^{-1}\left(\cosh^2 1\right) - \frac{\pi}{4}$

Question 9:

$x = 4\sinh^2\theta \qquad \frac{dx}{d\theta} = 4 \times 2\cosh\theta\sinh\theta = 4\sinh 2\theta$

$\int\sqrt{\frac{x+4}{x}}dx = \int\sqrt{\frac{4\sinh^2\theta + 4}{4\sinh^2\theta}} \times 4\sinh 2\theta\, d\theta = \int 4\sqrt{\frac{\cosh^2\theta}{\sinh^2\theta}} \times 2\sinh\theta\cosh\theta\, d\theta = \int 8\cosh^2\theta\, d\theta$

$\int 8\cosh^2\theta\, d\theta = \int 8\left(\frac{1}{2}\cosh 2\theta + \frac{1}{2}\right)d\theta = \int 4\cosh 2\theta + 4\, d\theta = 2\sinh 2\theta + 4\theta + c$

Inverse hyperbolic functions

Definition and graphs

- $[1,+\infty[\to[0,+\infty[$

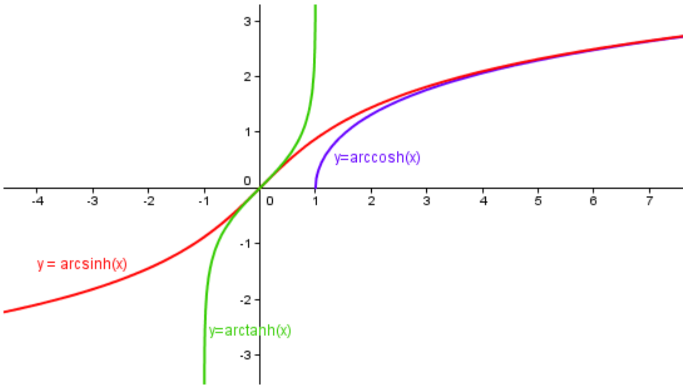
  $x \quad \to \cosh^{-1}(x)=\ln\left(x+\sqrt{x^2-1}\right)$

- $\mathbb{R} \quad \to \quad \mathbb{R}$

  $x \quad \to \sinh^{-1}(x)=\ln\left(x+\sqrt{x^2+1}\right)$

- $]-1,1[\ \to\mathbb{R}$

  $x \quad \to \tanh^{-1}(x)=\dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$



CALCULUS

Differentiation

$$y=\cosh^{-1}x \qquad \frac{dy}{dx}=\frac{1}{\sqrt{x^2-1}}$$

$$y=\sinh^{-1}x \qquad \frac{dy}{dx}=\frac{1}{\sqrt{1+x^2}}$$

$$y=\tanh^{-1} \qquad \frac{dy}{dx}=\frac{1}{1-x^2}$$

Integration

$$if\ a>0, \quad \int\frac{1}{\sqrt{x^2-a^2}}dx=\cosh^{-1}\left(\frac{x}{a}\right)+k=\ln\left(x+\sqrt{x^2-a^2}\right)+k$$

$$\int\frac{1}{\sqrt{a^2+x^2}}dx=\sinh^{-1}\left(\frac{x}{a}\right)+k=\ln\left(x+\sqrt{x^2+a^2}\right)+k$$

$$\int\frac{1}{a^2-x^2}dx=\frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right)+k=\frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right)+k$$

Inverse hyperbolic functions - exercises

Question 1:

Show that

$a) Sinh^{-1}x = \ln\left(x + \sqrt{x^2+1}\right)$

$b) Tanh^{-1}x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$

Question 2:

Differentiate the following:

$a)\ \tanh^{-1}\frac{x}{3}$ $\quad b)\ \sinh^{-1}\frac{x}{4}$ $\quad c)\ \cosh^{-1}2x$

$d)\ \mathrm{e}^{x}\sinh^{-1}x$ $\quad e)\ \frac{1}{x}\cosh^{-1}x^2$ $\quad f)\cosh^{-1}\frac{1}{x}$

Question 3:

a) Show that the equation $14\sinh x - 10\cosh x = 5$ can be expressed as $2e^{2x} - 5e^{x} - 12 = 0.$

b) Hence solve the equation $14\sinh x - 10\cosh x = 5$, giving your answer as a natural logarithm.

Question 4:

Solve the equations:

$a)\ 4\sinh x + 3e^{x} = 9$ $\quad b)\ 3\sinh x + 4\cosh x = 4$ $\quad c)\ \cosh 2x - 3\sinh x = 5$

$d)\ \cosh 2x - 3\cosh x = 4$ $\quad e)\ 8\sinh x = 3\,\mathrm{sech}\,x$ $\quad f) 3\,\mathrm{sech}^2 x + 7\tanh x = 5$

Question 5:

Evaluate the following integrals

a) $\int \frac{dx}{\sqrt{x^2+9}}$ $\quad b) \int \frac{dx}{\sqrt{x^2-16}}$ $\quad c) \int \frac{dx}{\sqrt{4x^2+25}}$

$d) \int \frac{dx}{\sqrt{9x^2+49}}$ $\quad e) \int \frac{dx}{\sqrt{(x+1)^2+4}}$ $\quad f) \int \frac{dx}{\sqrt{(x-2)^2-16}}$

$g) \int \frac{dx}{\sqrt{x^2+4x+5}}dx$ $\quad h) \int \frac{dx}{\sqrt{x^2-2x-2}}$

Inverse hyperbolic functions – exercises - answers

Question 1:

$$a)\; y=\sinh^{-1}x \quad so \quad \sinh y = x$$

$$e^{y}=\sinh y+\cosh y=\sinh y+\sqrt{1+\sinh^{2}y}$$

$$e^{y}=x+\sqrt{1+x^{2}}$$

$$y=\ln\left(x+\sqrt{1+x^{2}}\right)$$

$$b)\; y=\tanh^{-1}x \quad so \quad \tanh y = x$$

$$\tanh y=\frac{e^{y}-e^{-y}}{e^{y}+e^{-y}}=\frac{e^{2y}-1}{e^{2y}+1}=x \quad making\ e^{2y}\ the\, subject:$$

$$e^{2y}-1=e^{2y}x+x \qquad e^{2y}=\frac{1+x}{1-x} \qquad y=\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

Question 2:

$$a)\frac{dy}{dx}=\frac{1}{3}\times\frac{1}{1-\left(\frac{x}{3}\right)^{2}}=\frac{3}{9-x^{2}} \qquad b)\frac{dy}{dx}=\frac{1}{4}\times\frac{1}{\sqrt{1+\left(\frac{x}{4}\right)^{2}}}=\frac{1}{\sqrt{16+x^{2}}}$$

$$c)\frac{dy}{dx}=2\times\frac{1}{\sqrt{(2x)^{2}-1}}=\frac{2}{\sqrt{4x^{2}-1}} \qquad d)\frac{dy}{dx}=e^{x}\sinh^{-1}x+\frac{e^{x}}{\sqrt{1+x^{2}}}$$

$$e)-\frac{1}{x^{2}}\cosh^{-1}x^{2}+\frac{1}{x}\times 2x\times\frac{1}{\sqrt{(x^{2})^{2}-1}}=-\frac{1}{x^{2}}\cosh^{-1}x^{2}+\frac{2}{\sqrt{x^{4}-1}}$$

Question 3:

$$a)14\sinh x-10\cosh x=14\times\frac{e^{x}-e^{-x}}{2}-10\times\frac{e^{x}+e^{-x}}{2}=2e^{x}-12e^{-x}$$

$$so\,14\sinh x-10\cosh x=5 \;\; becomes \;\; 2e^{x}-12e^{-x}=5$$

$$(\times e^{x}) \;\; 2e^{2x}-12-5e^{x}=0$$

$$b)\,2e^{2x}-5e^{x}-12=0$$

$$(2e^{x}+3)(e^{x}-4)=0$$

$$e^{x}=-\frac{3}{2}\;(impossible)\;\; or\;\; e^{x}=4$$

$$x=\ln 4=2\ln 2$$

Question 4:

$$a)\,4\sinh x+3e^{x}=9$$

$$4\times\frac{e^{x}-e^{-x}}{2}+3e^{x}-9=0$$

$$5e^{x}-2e^{-x}-9=0 \quad (\times e^{x})$$

$$5e^{2x}-9e^{x}-2=0$$

$$(5e^{x}+1)(e^{x}-2)=0$$

$$e^{x}=-\frac{1}{2}(impossible)\,or\,e^{x}=2$$

$$x=\ln 2$$

$$b)3\sinh x+4\cosh x=4$$

$$3\times\frac{e^{x}-e^{-x}}{2}+4\times\frac{e^{x}+e^{-x}}{2}=4 \quad (\times 2e^{x})$$

$$7e^{2x}+1-8e^{x}=0$$

$$\left(7e^{x}-1\right)\left(e^{x}-1\right)=0$$

$$e^{x}=\frac{1}{7}\;\; or\;\; e^{x}=1$$

$$x=-\ln(7)\;\; or\;\; x=0$$

$$c)\cosh 2x-3\sinh x=6$$

$$2\sinh^{2}x+1-3\sinh x-6=0$$

$$2\sinh^{2}x-3\sinh x-5=0$$

$$(2\sinh x-5)(\sinh x+1)=0$$

$$\sinh x=\frac{5}{2}\;\; or\;\; \sinh x=-1$$

$$x=\ln\left(\frac{5}{2}+\sqrt{\left(\frac{5}{2}\right)^{2}+1}\right)$$

$$or\;\; x=\ln\left(-1+\sqrt{(-1)^{2}+1}\right)$$

$$x=\ln\left(\frac{5+\sqrt{29}}{2}\right)\;\; or\;\; x=\ln\left(\sqrt{2}-1\right)$$

$$d)\ \cosh 2x - 3\cosh x = 4$$
$$2\cosh^2 x - 1 - 3\cosh x - 4 = 0$$
$$2\cosh^2 x - 3\cosh x - 5 = 0$$
$$(2\cosh x - 5)(\cosh x + 1) = 0$$
$$\cosh x = \frac{5}{2} \quad or \quad \cosh x = -1 \ \ (impossible)$$
$$x = \pm\cosh^{-1}\left(\frac{5}{2}\right) = \pm\ln\left(\frac{5}{2} + \sqrt{\left(\frac{5}{2}\right)^2 - 1}\right)$$
$$x = \pm\ln\left(\frac{5+\sqrt{21}}{2}\right)$$

$$e)\ 8\sinh x = 3\operatorname{sech} x$$
$$8\sinh x = \frac{3}{\sinh x}$$
$$\sinh^2 x = \frac{3}{8}$$
$$\sinh x = \pm\sqrt{\frac{3}{8}}$$
$$x = \ln\left(\sqrt{\frac{3}{8}} + \sqrt{\frac{3}{8}+1}\right) \quad or \quad x = \ln\left(-\sqrt{\frac{3}{8}} + \sqrt{\frac{3}{8}+1}\right)$$
$$x = \ln\left(\frac{\sqrt{3}+\sqrt{11}}{2\sqrt{2}}\right) \quad or \quad x = \ln\left(\frac{\sqrt{11}-\sqrt{3}}{2\sqrt{2}}\right)$$
$$x = \ln\left(\frac{\sqrt{22}+\sqrt{6}}{4}\right) \quad or \quad x = \ln\left(\frac{\sqrt{22}-\sqrt{6}}{4}\right)$$

$$f)\ 3\operatorname{sech}^2 x + 7\tanh x = 5$$
$$3(1-\tanh^2 x) + 7\tanh x - 5 = 0$$
$$3\tanh^2 x - 7\tanh x + 2 = 0$$
$$(3\tanh x + 1)(\tanh x + 2) = 0$$
$$\tanh x = -\frac{1}{3} \quad or \quad \tanh x = -2\,(impossible)$$
$$x = \tanh^{-1}\left(-\frac{1}{3}\right) = \frac{1}{2}\ln\left(\frac{1-\frac{1}{3}}{1+\frac{1}{3}}\right) = \frac{1}{2}\ln\left(\frac{1}{2}\right)$$

Question 5:

$$a)\int \frac{dx}{\sqrt{x^2+9}} = \sinh^{-1}\left(\frac{x}{3}\right) + c$$

$$b)\int \frac{dx}{\sqrt{x^2-16}} = \cosh^{-1}\left(\frac{x}{4}\right) + c$$

$$c)\int \frac{dx}{\sqrt{4x^2+25}} = \int \frac{1}{2} \times \frac{dx}{\sqrt{x^2+\left(\frac{25}{4}\right)}} = \frac{1}{2}\sinh^{-1}\left(\frac{2x}{5}\right) + c$$

$$d)\int \frac{dx}{\sqrt{9x^2+49}} = \int \frac{1}{3} \times \frac{dx}{\sqrt{x^2+\left(\frac{49}{9}\right)}} = \frac{1}{3}\cosh^{-1}\left(\frac{3x}{7}\right) + c$$

$$e)\int \frac{dx}{\sqrt{(x+1)^2+4}} = \sinh^{-1}\left(\frac{x+1}{2}\right) + c$$

$$f)\int \frac{dx}{\sqrt{(x-2)^2-16}} = \cosh^{-1}\left(\frac{x-2}{4}\right) + c$$

$$g)\int \frac{dx}{\sqrt{x^2+4x+5}} = \int \frac{dx}{\sqrt{(x+2)^2+1}} = \sinh^{-1}(x+2) + c$$

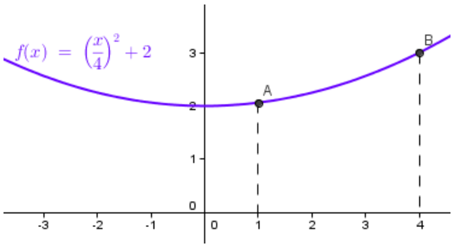
$$h)\int \frac{dx}{\sqrt{x^2-2x-2}} - \int \frac{dx}{\sqrt{(x-1)^2-3}} = \sinh^{-1}\left(\frac{x-1}{\sqrt{3}}\right) + c$$

Arc length and area of surface of revolution

Arc of length

If $y = f(x)$, the length of the arc of the curve from the point where x=a to the point where x=b is given by

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$



Parametric curves

The length of arc of a curve in terms of a parameter t is given by

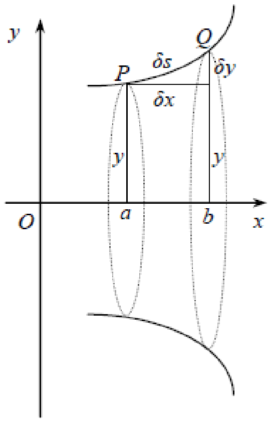
$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dx$$

where t_1 and t_2 are the values of the parameter at each end of the arc

Area of surface of revolution

The area of surface of revolution obtained by rotating an arc of the curve $y = f(x)$ through 2π radians about the x-axis between the points where $x = a$ and $x = b$ is given by

$$S_x = 2\pi \int_a^b y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$



Parametric curves

The area of surface of revolution obtained by rotating an arc of a parametric curve through 2π radians about the x-axis is given by

$$S_x = 2\pi \int_{t_1}^{t_2} y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dx$$

where t_1 and t_2 are the values of the parameter at each end of the arc

Arc length and area of surface of revolution - exercises

Question 1:

Find the length of the arc of the curve with equation $y=\frac{1}{3}x^{\frac{3}{2}}$, from the origin to the point with x-coordinate 12. (Hint: *substitute* $\left(1+\frac{1}{4}x\right)$ *by* "u" *to* integrate)

Question 2:

The curve C has equation $y=\ln\left(\cos x\right)$. Find the length of the arc of C between the points with x-coordinate 0 and $\frac{\pi}{3}$.

Question 3:

The parabola P has the following parametric definition: $x=4t^2$ *and* $y=4t$.

Work out the length of the arc between the point A($t=0$) and B($t=3$).

(Integration hint: *Let* $\frac{1}{2}\sinh u=t$)

Question 4:

(a) Use the definitions

$$\sinh\theta=\tfrac{1}{2}(e^{\theta}-e^{-\theta}) \qquad \text{and} \qquad \cosh\theta=\tfrac{1}{2}(e^{\theta}+e^{-\theta})$$

to show that:

(i) $2\sinh\theta\cosh\theta=\sinh 2\theta$; *(2 marks)*

(ii) $\cosh^2\theta+\sinh^2\theta=\cosh 2\theta$. *(3 marks)*

(b) A curve is given parametrically by

$$x=\cosh^3\theta,\quad y=\sinh^3\theta$$

(i) Show that

$$\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2=\frac{9}{4}\sinh^2 2\theta\cosh 2\theta$$ *(6 marks)*

(ii) Show that the length of the arc of the curve from the point where $\theta=0$ to the point where $\theta=1$ is

$$\frac{1}{2}\left[(\cosh 2)^{\frac{3}{2}}-1\right]$$ *(6 marks)*

Question 5:

(a) Given that $y=\ln\tanh\frac{x}{2}$, where $x>0$, show that

$$\frac{dy}{dx}=\operatorname{cosech} x$$ *(6 marks)*

(b) A curve has equation $y=\ln\tanh\frac{x}{2}$, where $x>0$. The length of the arc of the curve between the points where $x=1$ and $x=2$ is denoted by s.

(i) Show that

$$s=\int_1^2\coth x\,dx$$ *(2 marks)*

(ii) Hence show that $s=\ln(2\cosh 1)$. *(4 marks)*

Question 6:

The arc of the curve $y = x^3$, between the origin and the point (1,1), is rotated through 4 right-angles about the x-axis. Find the area of the surface generated.

Question 7:

The arc, in the first quadrant, of the curve with parametric equations

$$x = \operatorname{sech} t \quad \text{and} \quad y = \tanh t$$

between the points where $t = 0$ *and* $t = \ln 2$, is rotated completely about the x-axis.

Show that the area of the surface generated is $\frac{2\pi}{5}$.

Question 8:

A curve has parametric equations

$$x = t - \frac{1}{3}t^3, \quad y = t^2$$

(a) Show that

$$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = (1 + t^2)^2$$

(3 marks)

(b) The arc of the curve between $t = 1$ and $t = 2$ is rotated through 2π radians about the x-axis.

Show that S, the surface area generated, is given by $S = k\pi$, where k is a rational number to be found. *(5 marks)*

Question 9:

(a) Use the definition $\cosh x = \frac{1}{2}(\mathrm{e}^x + \mathrm{e}^{-x})$ to show that $\cosh 2x = 2\cosh^2 x - 1$. *(2 marks)*

(b) (i) The arc of the curve $y = \cosh x$ between $x = 0$ and $x = \ln a$ is rotated through 2π radians about the x-axis. Show that S, the surface area generated, is given by

$$S = 2\pi \int_0^{\ln a} \cosh^2 x \, \mathrm{d}x$$

(3 marks)

(ii) Hence show that

$$S = \pi\left(\ln a + \frac{a^4 - 1}{4a^2}\right)$$

(5 marks)

Question 10:

A curve has parametric equations

$x = a(1 - \cos^3\theta)$ and $y = a\sin^3\theta$, for $0 \le \theta \le \frac{\pi}{2}$, where a is a positive constant.

Show that, when this curve is rotated through 2π radians about the x-axis,

the curved surface area generated is $\frac{6}{5}\pi a^2$.

Formulae
Roots
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Arc length and area of surface of revolution - exercises - answers

Question 1:

$$y=\frac{1}{3}x^{\frac{3}{2}} \qquad \frac{dy}{dx}=\frac{1}{3}\times\frac{3}{2}x^{\frac{1}{2}}=\frac{1}{2}\sqrt{x}$$

The length of the arc is $s=\int_0^{12}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\int_0^{12}\sqrt{1+\frac{1}{4}x}\,dx$

Using the substitution $u=1+\frac{1}{4}x$:

$$\frac{du}{dx}=\frac{1}{4} \quad and \quad when\ x=0,u=1\ when\,x=12,u=4$$

$$s\,becomes\int_1^4\sqrt{u}\times 4du=\left[4\times\frac{2}{3}u^{\frac{3}{2}}\right]_1^4=\frac{64}{3}-\frac{8}{3}=\frac{56}{3}$$

Question 2:

$$y=\ln(\cos x) \qquad \frac{dy}{dx}=\frac{-\sin x}{\cos x}=-\tan x$$

$$s=\int_0^{\frac{\pi}{3}}\sqrt{1+(-\tan x)^2}\,dx=\int_0^{\frac{\pi}{3}}\sec x\,dx \qquad (formula\ book)$$

$$s=\left[\ln(\sec x+\tan x)\right]_0^{\frac{\pi}{3}}=\ln\left(2+\sqrt{3}\right)-\ln(1)=\ln\left(2+\sqrt{3}\right)$$

Question 3:

$$x=4t^2 \quad and \quad y=4t \qquad \frac{dx}{dt}=8t \quad and \quad \frac{dy}{dt}=4$$

$$s=\int_0^3\sqrt{(8t)^2+(4)^2}\,dt=\int_0^3\sqrt{64t^2+16}\,dt$$

$$Let\ \frac{1}{2}\sinh u=t \qquad \frac{dt}{du}=\frac{1}{2}\cosh u$$

$$when\ t=0,\ u=0 \quad when\,t=3,u=\sinh^{-1}6$$

$$s=\int_0^{\sinh^{-1}6}\sqrt{64\times\frac{1}{4}\sinh^2 u+16}\times\frac{1}{2}\cosh u\,du=\int_0^{\sinh^{-1}6}4\cosh u\times\frac{1}{2}\cosh u\,du$$

$$s=\int_0^{\sinh^{-1}6}2\cosh^2 u\,du=\int_0^{\sinh^{-1}6}\cosh 2u+1\,du=\left[\frac{1}{2}\sinh 2u+u\right]_0^{\sinh^{-1}6}$$

$$s=\left[\sinh u\cos u+u\right]_0^{\sinh^{-1}6}=\left[\sinh u\sqrt{1+\sinh^2 u}+u\right]_0^{\sinh^{-1}6}$$

$$s=6\times\sqrt{1+6^2}+\sinh^{-1}6=6\sqrt{37}+\ln\left(6+\sqrt{37}\right)$$

Question 4:

$$a)\,i)\,2\sinh\theta\cosh\theta=2\times\frac{e^{\theta}-e^{-\theta}}{2}\times\frac{e^{\theta}+e^{-\theta}}{2}=\frac{e^{2\theta}-e^{2\theta}}{2}=\sinh 2\theta$$

$$ii)\cosh^2\theta+\sinh^2\theta=\left(\frac{e^{\theta}+e^{-\theta}}{2}\right)^2+\left(\frac{e^{\theta}-e^{-\theta}}{2}\right)^2=\frac{e^{2\theta}+e^{-2\theta}+2}{4}+\frac{e^{2\theta}+e^{-2\theta}-2}{4}$$

$$=\frac{e^{2\theta}+e^{-2\theta}}{2}=\cosh 2\theta$$

$$b)\,x=\cosh^3\theta \quad and \quad y=\sinh^3\theta$$

$$\frac{dx}{d\theta}=3\sinh\theta\cosh^2\theta \quad and \quad \frac{dy}{dx}=3\cosh\theta\sinh^2\theta$$

$$i)\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2=9\sinh^2\theta\cosh^4\theta+9\cosh^2\theta\sinh^4\theta$$

$$=9\sinh^2\theta\cosh^2\theta\left(\cosh^2\theta+\sinh^2\theta\right)$$

$$=9\left(\frac{\sinh^2 2\theta}{4}\right)(\cosh 2\theta)=\frac{9}{4}\sinh^2 2\theta\cosh 2\theta$$

$$ii)\,s=\int_0^1\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta=\int_0^1\frac{3}{2}\sinh 2\theta\sqrt{\cosh 2\theta}\,d\theta$$

$$s=\frac{3}{4}\int_0^1 2\sinh 2\theta(\cosh 2\theta)^{\frac{1}{2}}\,d\theta=\int f'\times f^n=\frac{1}{n+1}f^{n+1}$$

$$s=\frac{3}{4}\left[\frac{2}{3}(\cosh 2\theta)^{\frac{3}{2}}\right]_0^1=\frac{1}{2}\left[(\cosh 2)^{\frac{3}{2}}-(\cosh 0)^{\frac{3}{2}}\right]=\frac{1}{2}\left[(\cosh 2)^{\frac{3}{2}}-1\right]$$

Question 5:

$$a)\,y=\ln\left(\tanh\frac{x}{2}\right)$$

$$\frac{dy}{dx}=\frac{\frac{1}{2}\left(\operatorname{sech}^2\frac{x}{2}\right)}{\tanh\frac{x}{2}}=\frac{1}{2}\times\frac{1}{\cosh^2\frac{x}{2}}\times\frac{\cosh\frac{x}{2}}{\sinh\frac{x}{2}}=\frac{1}{2\sinh\frac{x}{2}\cosh\frac{x}{2}}=\frac{1}{\sinh x}=\operatorname{cosech} x$$

$$b)\,i)\,s=\int_1^2\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\int_1^2\sqrt{1+\operatorname{cosech}^2 x}\,dx=\int_1^2\sqrt{\frac{\sinh^2 x+1}{\sinh^2 x}}dx=\int_1^2\frac{\cosh x}{\sinh x}dx$$

$$s=\left[\ln(\sinh x)\right]_1^2=\ln(\sinh 2)-\ln(\sinh 1)=\ln\left(\frac{\sinh 2}{\sinh 1}\right)=\ln\left(\frac{2\sinh 1\cosh 1}{\sinh 1}\right)$$

$$s=\ln(2\cosh 1)$$

Question 6:

$$y = x^3 \qquad \frac{dy}{dx} = 3x^2$$

$$S = 2\pi\int_0^1 x^3\sqrt{1+(3x^2)^2}\,dx = \frac{2\pi}{36}\int_0^1 36x^3\sqrt{1+9x^4}\,dx$$

$$S = \frac{2\pi}{36}\left[\frac{2}{3}(1+9x^4)^{\frac{3}{2}}\right]_0^1 = \frac{\pi}{27}\left(10\sqrt{10}-1\right)$$

Question 7:

$$x = \operatorname{sech} t \quad and \quad y = \tanh t$$

$$\frac{dx}{dt} = -\frac{\sinh t}{\cosh^2 t} \quad and \quad y = \frac{1}{\cosh^2 t}$$

$$S = 2\pi\int_0^{\ln 2} \tanh t \times \sqrt{\left(-\frac{\sinh t}{\cosh^2 t}\right)^2 + \left(\frac{1}{\cosh^2 t}\right)^2}\,dt$$

$$S = 2\pi\int_0^{\ln 2}\frac{\sinh t}{\cosh t}\times\frac{\cosh t}{\cosh^2 t}\,dt = 2\pi\int_0^{\ln 2}\sinh t \times \cosh^{-2} t\,dt$$

$$S = 2\pi\left[-\frac{1}{\cosh t}\right]_0^{\ln 2} = 2\pi\left(1-\frac{1}{\cosh(\ln 2)}\right)$$

$$\cosh(\ln 2) = \frac{e^{\ln 2}+e^{-\ln 2}}{2} = \frac{e^{\ln 2}+e^{\ln\frac{1}{2}}}{2} = \frac{2+\frac{1}{2}}{2} = \frac{5}{4}$$

$$so \quad S = 2\pi\left(1-\frac{4}{5}\right) = \frac{2\pi}{5}$$

Question 8:

$$x = t - \frac{1}{3}t^3 \quad and \quad y = t^2 \qquad\qquad \frac{dx}{dt} = 1-t^2 \quad and \quad \frac{dy}{dt} = 2t$$

$$a)\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (1-t^2)^2 + (2t)^2 = 1+t^4-2t^2+4t^2 = 1+t^4+2t^2 = (1+t^2)^2$$

$$b)\,S = 2\pi\int_1^2 t^2\sqrt{(1+t^2)^2}\,dt = 2\pi\int_1^2 t^2(1+t^2)\,dt$$

$$S = 2\pi\int_1^2 t^2+t^4\,dt = 2\pi\left[\frac{1}{3}t^3+\frac{1}{5}t^5\right]_1^2 = 2\pi\left(\frac{8}{3}+\frac{32}{5}-\frac{1}{3}-\frac{1}{5}\right)$$

$$S = \frac{256}{15}\pi$$

Question 9:

$$a)\,2\cosh^2 x - 1 = 2\left(\frac{e^x+e^{-x}}{2}\right)^2 - 1 = \frac{e^{2x}+e^{-2x}+2}{2} - \frac{2}{2} = \frac{e^{2x}+e^{-2x}}{2} = \cosh 2x$$

$$b)\,i)\; y = \cosh x \qquad \frac{dy}{dx} = \sinh x$$

$$S = 2\pi\int_0^{\ln a}\cosh x\sqrt{1+\sinh^2 x}\,dx = 2\pi\int_0^{\ln a}\cosh^2 x\,dx$$

$$S = \pi\int_0^{\ln a}\cosh 2x + 1\,dx = \pi\left[\frac{1}{2}\sinh 2x + x\right]_0^{\ln a} = \pi\left(\frac{1}{2}\sinh(2\ln a) + \ln a\right)$$

$$\bullet\,\sinh(2\ln a) = \sinh(\ln a^2) = \frac{e^{\ln a^2} - e^{-\ln a^2}}{2} = \frac{e^{\ln a^2} - e^{\ln\frac{1}{a^2}}}{2} = \frac{a^2 - \frac{1}{a^2}}{2} = \frac{a^4-1}{2a^2}$$

$$so \quad S = \pi\left(\frac{a^4-1}{4a^2} + \ln a\right)$$

Question 10

$$x = a(1-\cos^3\theta) \quad and \quad y = a\sin^3\theta$$

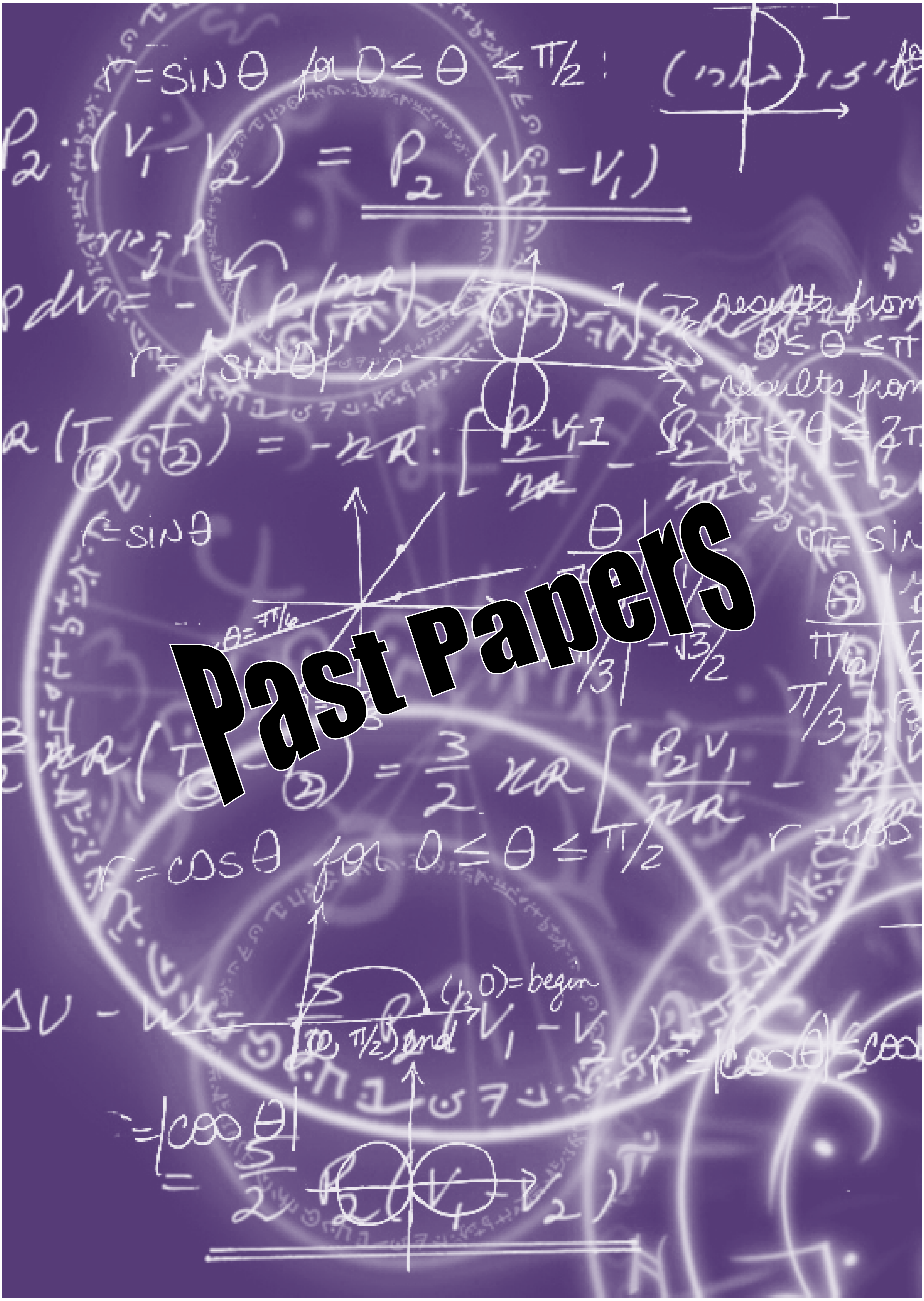
$$\frac{dx}{d\theta} = 3a\sin\theta\cos^2\theta \quad and \quad \frac{dy}{d\theta} = 3a\cos\theta\sin^2\theta$$

$$S = 2\pi\int_0^{\frac{\pi}{2}} a\sin^3\theta\sqrt{(3a\sin\theta\cos^2\theta)^2 + \left(3a\cos\theta\sin^2\theta\right)^2}\,d\theta$$

$$S = 6\pi a^2\int_0^{\frac{\pi}{2}}\sin^3\theta\sqrt{\sin^2\theta\cos^4\theta + \cos^2\theta\sin^4\theta}\,d\theta$$

$$S = 6\pi a^2\int_0^{\frac{\pi}{2}}\sin^3\theta\sin\theta\cos\theta\sqrt{\cos^2\theta + \sin^2\theta}\,d\theta$$

$$S = 6\pi a^2\int_0^{\frac{\pi}{2}}\cos\theta\sin^4\theta\,d\theta = 6\pi a^2\left[\frac{1}{5}\sin^5\theta\right]_0^{\frac{\pi}{2}} = 6\pi a^2\left(\frac{1}{5}-0\right) = \frac{6}{5}\pi a^2$$


r=sinθ for 0≤θ≤π/2
r=|sinθ| is
r=sinθ
r=cosθ for 0≤θ≤π/2
(1,0)=begin
(0,π/2)end
=|cosθ|

Past Papers

General Certificate of Education
January 2006
Advanced Level Examination

MATHEMATICS **MFP2**
Unit Further Pure 2

Friday 27 January 2006 1.30 pm to 3.00 pm

For this paper you must have:
- an 8-page answer book
- the **blue** AQA booklet of formulae and statistical tables

You may use a graphics calculator.

Time allowed: 1 hour 30 minutes

Instructions
- Use blue or black ink or ball-point pen. Pencil should only be used for drawing.
- Write the information required on the front of your answer book. The *Examining Body* for this paper is AQA. The *Paper Reference* is MFP2.
- Answer **all** questions.
- All necessary working should be shown; otherwise marks for method may be lost.

Information
- The maximum mark for this paper is 75.
- The marks for questions are shown in brackets.

Advice
- Unless stated otherwise, formulae may be quoted, without proof, from the booklet.

Answer **all** questions.

---

1 (a) Show that

$$\frac{1}{r^2}-\frac{1}{(r+1)^2}=\frac{2r+1}{r^2(r+1)^2}$$ *(2 marks)*

(b) Hence find the sum of the first n terms of the series

$$\frac{3}{1^2\times 2^2}+\frac{5}{2^2\times 3^2}+\frac{7}{3^2\times 4^2}+\ldots$$ *(4 marks)*

2 The cubic equation

$$x^3+px^2+qx+r=0$$

where p, q and r are real, has roots α, β and γ.

(a) Given that

$$\alpha+\beta+\gamma=4 \qquad \text{and} \qquad \alpha^2+\beta^2+\gamma^2=20$$

find the values of p and q. *(5 marks)*

(b) Given further that one root is $3+\mathrm{i}$, find the value of r. *(5 marks)*

3 The complex numbers z_1 and z_2 are given by

$$z_1=\frac{1+\mathrm{i}}{1-\mathrm{i}} \qquad \text{and} \qquad z_2=\frac{1}{2}+\frac{\sqrt{3}}{2}\mathrm{i}$$

(a) Show that $z_1=\mathrm{i}$. *(2 marks)*

(b) Show that $|z_1|=|z_2|$. *(2 marks)*

(c) Express both z_1 and z_2 in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\leqslant\pi$. *(3 marks)*

(d) Draw an Argand diagram to show the points representing z_1, z_2 and z_1+z_2. *(2 marks)*

(e) Use your Argand diagram to show that

$$\tan\frac{5}{12}\pi=2+\sqrt{3}$$ *(3 marks)*

Formulae | Roots | Complex numbers | De Moivre theorem | Proof by induction | Finite series | Inverse trig functions | Hyperbolic functions | Arc length | Past Papers | Jan 2006 | Jun 2006 | Jan 2007 | Jun 2007 | Jan 2008 | Jun 2008 | Jan 2009 | Jun 2009 | Jan 2010 | Jun 2010

4 (a) Prove by induction that

$$2 + (3 \times 2) + (4 \times 2^2) + \ldots + (n+1)\, 2^{n-1} = n\, 2^n$$

for all integers $n \geqslant 1$. *(6 marks)*

(b) Show that

$$\sum_{r=n+1}^{2n} (r+1)\, 2^{r-1} = n\, 2^n (2^{n+1} - 1)$$ *(3 marks)*

5 The complex number z satisfies the relation

$$|z + 4 - 4\mathrm{i}| = 4$$

(a) Sketch, on an Argand diagram, the locus of z. *(3 marks)*

(b) Show that the greatest value of $|z|$ is $4(\sqrt{2} + 1)$. *(3 marks)*

(c) Find the value of z for which

$$\arg(z + 4 - 4\mathrm{i}) = \tfrac{1}{6}\pi$$

Give your answer in the form $a + \mathrm{i}b$. *(3 marks)*

6 It is given that $z = \mathrm{e}^{\mathrm{i}\theta}$.

(a) (i) Show that

$$z + \frac{1}{z} = 2\cos\theta$$ *(2 marks)*

(ii) Find a similar expression for

$$z^2 + \frac{1}{z^2}$$ *(2 marks)*

(iii) Hence show that

$$z^2 - z + 2 - \frac{1}{z} + \frac{1}{z^2} = 4\cos^2\theta - 2\cos\theta$$ *(3 marks)*

(b) Hence solve the quartic equation

$$z^4 - z^3 + 2z^2 - z + 1 = 0$$

giving the roots in the form $a + \mathrm{i}b$. *(5 marks)*

7 (a) Use the definitions

$$\sinh\theta = \tfrac{1}{2}(\mathrm{e}^{\theta} - \mathrm{e}^{-\theta}) \qquad \text{and} \qquad \cosh\theta = \tfrac{1}{2}(\mathrm{e}^{\theta} + \mathrm{e}^{-\theta})$$

to show that:

(i) $2\sinh\theta\cosh\theta = \sinh 2\theta$; *(2 marks)*

(ii) $\cosh^2\theta + \sinh^2\theta = \cosh 2\theta$. *(3 marks)*

(b) A curve is given parametrically by

$$x = \cosh^3\theta, \quad y = \sinh^3\theta$$

(i) Show that

$$\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2 = \frac{9}{4}\sinh^2 2\theta\cosh 2\theta$$ *(6 marks)*

(ii) Show that the length of the arc of the curve from the point where $\theta = 0$ to the point where $\theta = 1$ is

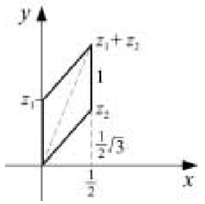
$$\frac{1}{2}\left[(\cosh 2)^{\frac{3}{2}} - 1\right]$$ *(6 marks)*

END OF QUESTIONS

MFP2

Q	Solution	Marks	Total	Comments
1(a)	$\frac{1}{r^2}-\frac{1}{(r+1)^2}=\frac{(r+1)^2-r^2}{r^2(r+1)^2}$	M1		
	$=\frac{2r+1}{r^2(r+1)^2}$	A1	2	AG
(b)	$\frac{3}{1^2\times 2^2}=\frac{1}{1^2}-\frac{1}{2^2}$			
	$\frac{5}{2^2\times 3^2}=\frac{1}{2^2}-\frac{1}{3^2}$			
	$\frac{7}{3^2\times 4^2}=\frac{1}{3^2}-\frac{1}{4^2}$	M1A1		A1 for at least 3 lines
	$\frac{2n+1}{n^2(n+1)^2}=\frac{1}{n^2}-\frac{1}{(n+1)^2}$			
	Clear cancellation	M1		
	$1-\frac{1}{(n+1)^2}$	A1F	4	
	Total		**6**	
2(a)	$p=-4$	B1		
	$(\alpha+\beta+\gamma)^2=\Sigma\alpha^2+2\Sigma\alpha\beta$	M1		
	$16=20+2\Sigma\alpha\beta$	A1		
	$\Sigma\alpha\beta=-2$	A1F		
	$q=-2$	A1F	5	
(b)	$3-\mathrm{i}$ is a root	B1		
	Third root is -2	B1F		
	$\alpha\beta\gamma=(3+\mathrm{i})(3-\mathrm{i})(-2)$	M1		
	$=-20$	A1F		Real $\alpha\beta\gamma$
	$r=+20$	A1F	5	Real r
	Alternative to (b) Substitute 3 + i into equation	M1		
	$(3+\mathrm{i})^2=8+6\mathrm{i}$	B1		
	$(3+\mathrm{i})^3=18+26\mathrm{i}$	B1		
	$r=20$	A2,1,0		Provided r is real
	Total		**10**	

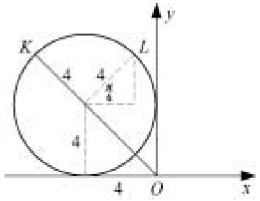
MFP2 (cont)

Q	Solution	Marks	Total	Comments
3(a)	$\frac{1+\mathrm{i}}{1-\mathrm{i}}=\frac{(1+\mathrm{i})^2}{1-\mathrm{i}^2}=\mathrm{i}$	M1A1	2	AG
(b)	$\|z_2\|=\sqrt{\frac{1}{4}+\frac{3}{4}}=1=\|z_1\|$	M1A1	2	
(c)	$r=1$	B1		PI
	$\theta=\frac{1}{2}\pi,\ \frac{1}{3}\pi$	B1B1	3	Deduct 1 mark if extra solutions
(d)		B2,1F	2	Positions of the 3 points relative to each other, must be approximately correct
(e)	$\mathrm{Arg}(z_1+z_2)=\frac{5}{12}\pi$	B1		**Clearly** shown
	$\tan\frac{5}{12}\pi=\frac{1+\frac{1}{2}\sqrt{3}}{\frac{1}{2}}$	M1		Allow if BO earned
	$=2+\sqrt{3}$	A1	3	AG must earn BO for this
	Total		**12**	

MFP2 (cont)

Q	Solution	Marks	Total	Comments
4(a)	Assume result true for $n=k$			
	$\sum_{r=1}^{k}(r+1)2^{r-1}=k2^{k}$			
	$\sum_{r=1}^{k+1}(r+1)2^{r-1}=k2^{k}+(k+2)2^{k}$	M1A1		
	$=2^{k}(k+k+2)$	m1		
	$=2^{k}(2k+2)$			
	$=2^{k+1}(k+1)$	A1		
	$n=1 \quad 2\times2^{0}=2=1\times2^{1}$	B1		
	$P_k \Rightarrow P_{k+1}$ and P_1 is true	E1	6	Provided previous 5 marks earned
(b)	$\sum_{r=1}^{2n}(r+1)2^{r-1}-\sum_{r=1}^{n}(r+1)2^{r-1}$	M1		Sensible attempt at the difference between 2 series
	$=2n\,2^{2n}-n2^{n}$	A1		
	$=n\left(2^{n+1}-1\right)2^{n}$	A1	3	AG
	Total		**9**	

MFP2 (cont)

Q	Solution	Marks	Total	Comments
5(a)		B1 B1 B1	3	Circle Correct centre Touching both axes
(b)	$\lvert z\rvert \max = OK$	M1		Accept $\sqrt{4^2+4^2}+4$ as a method
	$=\sqrt{4^2+4^2}+4$	A1F		Follow through circle in incorrect position
	$=4\left(\sqrt{2}+1\right)$	A1F	3	AG
(c)	Correct position of z, ie L	M1		
	$a=-\left(4-4\cos\frac{1}{6}\pi\right)$			
	$=-\left(4-2\sqrt{3}\right)$	A1F		Follow through circle in incorrect position
	$b=4+4\sin\frac{1}{6}\pi=6$	A1F	3	
	Total		**9**	

MFP2 (cont)

Q	Solution	Marks	Total	Comments
6(a)(i)	$z+\frac{1}{z}=\cos\theta+\mathrm{i}\sin\theta+$			Or $z+\frac{1}{z}=\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{-\mathrm{i}\theta}$
	$\cos(-\theta)+\mathrm{i}\sin(-\theta)$	M1		
	$=2\cos\theta$	A1	2	AG
(ii)	$z^2+\frac{1}{z^2}=\cos 2\theta+i\sin 2\theta$			
	$+\cos(-2\theta)+\mathrm{i}\sin(-2\theta)$	M1		
	$=2\cos 2\theta$	A1	2	OE
(iii)	$z^2-z+2-\frac{1}{z}+\frac{1}{z^2}$			
	$=2\cos 2\theta-2\cos\theta+2$	M1		
	Use of $\cos 2\theta=2\cos^2\theta-1$	m1		
	$=4\cos^2\theta-2\cos\theta$	A1	3	AG
(b)	$z+\frac{1}{z}=0 \qquad z=\pm\mathrm{i}$	M1A1		
				Alternative:
	$z+\frac{1}{z}=1 \qquad z^2-z+1=0$	M1A1		$\cos\theta=0 \qquad \theta=\pm\frac{1}{2}\pi$ M1
				$z=\pm\mathrm{i}$ A1
	$z=\frac{1\pm\mathrm{i}\sqrt{3}}{2}$	A1F	5	$\cos\theta=\frac{1}{2} \qquad \theta=\pm\frac{1}{3}\pi$ M1
	Accept solution to (b) if done otherwise			$z=\mathrm{e}^{\pm\frac{1}{3}\pi\mathrm{i}}=\frac{1}{2}(1\pm\mathrm{i}\sqrt{3})$ A1 A1
	Alternative			
	If $\theta=+\frac{1}{2}\pi \quad \theta=\frac{1}{3}\pi$	M1		
	$z=\mathrm{i} \quad z=\frac{1+\sqrt{3}\,\mathrm{i}}{2}$	A1		
	Or any correct z values of θ	M1		
	Any 2 correct answers	A1		
	One correct answer only	B1		
	Total		**12**	

MFP2 (cont)

Q	Solution	Marks	Total	Comments
7(a)(i)	$2\left(\frac{\mathrm{e}^{\theta}-\mathrm{e}^{-\theta}}{2}\right)\left(\frac{\mathrm{e}^{\theta}+\mathrm{e}^{-\theta}}{2}\right)$			
	$=\frac{\mathrm{e}^{2\theta}-\mathrm{e}^{-2\theta}}{2}=\sinh 2\theta$	M1A1	2	AG
(ii)	$\left(\frac{\mathrm{e}^{\theta}-\mathrm{e}^{-\theta}}{2}\right)^2+\left(\frac{\mathrm{e}^{\theta}+\mathrm{e}^{-\theta}}{2}\right)^2$	M1		
	$=\frac{\mathrm{e}^{2\theta}-2+\mathrm{e}^{-2\theta}+\mathrm{e}^{2\theta}+2+\mathrm{e}^{-2\theta}}{4}$	A1		
	$=\cosh 2\theta$	A1	3	AG
(b)(i)	$\dot{x}=3\cosh^2\theta\sinh\theta''$	M1A1		Allow M1 for **reasonable** attempt at differentiation, but M0 for putting in terms of $\mathrm{e}^{k\theta}$ or $\sinh 3\theta$ unless real progress made towards $\dot{x}^2+\dot{y}^2$
	$\dot{y}=3\sinh^2\theta\cosh\theta$	A1		Allow this M1 if not squared out, must be clear sum in question is $\dot{x}^2+\dot{y}^2$
	$\dot{x}^2+\dot{y}^2=9\cosh^4\theta\sinh^2\theta$			
	$+9\sinh^4\theta\cosh^2\theta$	M1		AG
	$=9\sinh^2\theta\cosh^2\theta\left(\cosh^2\theta+\sinh^2\theta\right)$	A1		Accept $\int_0^1\sqrt{\frac{9}{4}\sinh^2 2\theta\cosh 2\theta}\ \mathrm{d}\theta$ but limits must appear somewhere
	$=\frac{9}{4}\sinh^2 2\theta\cosh 2\theta$	A1	6	AG
(ii)	$S=\int_0^1\frac{3}{2}\sinh 2\theta\sqrt{\cosh 2\theta}\,\mathrm{d}\theta$	M1		
	$u=\cosh 2\theta \qquad \mathrm{d}u=2\sinh 2\theta\,\mathrm{d}\theta$	M1A1		
	$I=\int\frac{3}{4}u^{\frac{1}{2}}\,\mathrm{d}u=\frac{3}{4}\times\frac{2}{3}u^{\frac{3}{2}}$	A1F		
	$S=\left\{\frac{1}{2}(\cosh 2\theta)^{\frac{3}{2}}\right\}_0^1$	A1F		
	$=\frac{1}{2}\left\{(\cosh 2)^{\frac{3}{2}}-1\right\}$	A1	**6**	AG
	Total		**17**	
	TOTAL		75	

AQA – Further pure 2 – Jan 2006 – Answers

Question 1	**Exam report**
$a)\ \frac{1}{r^2}-\frac{1}{(r+1)^2}=\frac{(r+1)^2}{r^2(r+1)^2}-\frac{r^2}{r^2(r+1)^2}=\frac{r^2+2r+1-r^2}{r^2(r+1)^2}=\frac{2r+1}{r^2(r+1)^2}$ $b)\ \frac{3}{1^2\times 2^2}+\frac{5}{2^2\times 3^2}+\frac{7}{3^2\times 4^2}+\ldots+\frac{2n+1}{n^2(n+1)^2}=\sum_{r=1}^{n}\frac{2r+1}{r^2(r+1)^2}\ \ and$ $\sum_{r=1}^{n}\frac{2r+1}{r^2(r+1)^2}=\sum_{r=1}^{n}\frac{1}{r^2}-\frac{1}{(r+1)^2}$ $=1-\frac{1}{4}+\frac{1}{4}-\frac{1}{9}+\frac{1}{9}-\frac{1}{16}+\ldots.+\frac{1}{(n-1)^2}-\frac{1}{n^2}+\frac{1}{n^2}-\frac{1}{(n+1)^2}$ *All the terms cancel out except the first and the last one:* $\sum_{r=1}^{n}\frac{2r+1}{r^2(r+1)^2}=1-\frac{1}{(n+1)^2}$	There were many fully correct answers to this question. A few candidates did not spot the connection between the parts (a) and (b). Otherwise, the only errors in part (b) were errors of sign leading to the answer $1-\frac{1}{(n+1)^2}$ or the summation of n +1 terms rather than n terms of the given series.

Question 2:	**Exam report**
$a)\ x^3+px^2+qx+r=0$ *has three roots* α,β *and* γ. $\alpha+\beta+\gamma=4\quad so\ \ p=-4$ $\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)$ $20=(4)^2-2q$ $q=-2$ $b)$ *p, q and r are REAL numbers so* *if* $3+i$ *is a root, then its conjugate is also a root* $\alpha=3+i,\ \beta=3-i$ $\alpha+\beta+\gamma=4\ \ gives\ \ 6+\gamma=4\quad and\ \ \gamma=-2$ $r=-\alpha\beta\gamma=-(3+i)(3-i)(-2)=(9-i^2)\times 2=20$ $r=20$	Candidates were also able to achieve good results on this question. In part (a), where errors occurred they were almost always errors of signs. For instance the value of *p* was given as 2 instead of .2 or the formula for $\left(\sum\alpha\right)^2$ was incorrectly quoted as $\sum\alpha^2-2\sum\alpha\beta$. There were fewer completely correct solutions to part (b) often due to inelegant methods of solution. The most successful candidates obtained the values of the other two roots and then worked out the product $\alpha\beta\gamma$. The main loss of marks using this method was to equate *r* to $\alpha\beta\gamma$ instead of - $\alpha\beta\gamma$. The other main method of approach to this part of the question was to substitute 3 + i into the cubic equation with the values of *p* and *q* already found in part (a). However any error in the values of *p* and *q* or in the substitution inevitably led to *r* having a complex value. Surprisingly this did not seem to worry the candidates in spite of the fact that the question stated that *r* was real.

Question 3:	**Exam report**
$z_1=\frac{1+i}{1-i}\quad and\quad z_2=\frac{1}{2}+\frac{\sqrt{3}}{2}i$ $a)\ z_1=\frac{1+i}{1-i}\times\frac{1+i}{1+i}=\frac{1+2i+i^2}{1-i^2}=\frac{2i}{2}=i$ $b)\ \lvert z_1\rvert=\lvert i\rvert=\lvert 0+1i\rvert=\sqrt{0^2+1^2}=1$ $\lvert z_2\rvert=\left\lvert\frac{1}{2}+\frac{\sqrt{3}}{2}i\right\rvert=\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}=\sqrt{\frac{1}{4}+\frac{3}{4}}=\sqrt{1}=1$ $\lvert z_1\rvert=\lvert z_2\rvert$ $c)\ z_1=i=e^{i\frac{\pi}{2}}$ $z_2=\frac{1}{2}+\frac{\sqrt{3}}{2}i=Cos(\frac{\pi}{3})+iSin(\frac{\pi}{3})=e^{i\frac{\pi}{3}}$	Part (a) was well done, as was part (b). In part (c) it was surprising to note how many candidates could not express the complex number in the form $re^{i\theta}$, although z₂ was almost invariably correctly written as $e^{i\frac{\pi}{3}}$ Errors in part (c) however did not deter candidates from drawing a correct Argand diagram as they usually used the form ***a*** + i***b*** when plotting their points. Although the vast majority of scripts ended with $Tan\frac{5\pi}{12}=2+\sqrt{3}$, very, very few candidates gave convincing proof that $\arg(z_1+z_2)$ was $\frac{5\pi}{12}$, but rather seemed to take it for granted.

Question 3: (continues)

$d)$

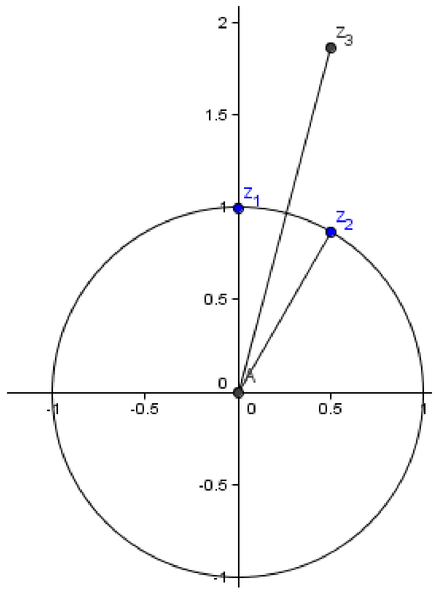
$e)$ The argument of z_3 *is* $\arg(z_2)+\dfrac{\arg(z_1)-\arg(z_2)}{2}$

$$\frac{\pi}{2}-\frac{\pi}{3}=\frac{\pi}{6} \quad and \quad \frac{1}{2}\, of\, \frac{\pi}{6}=\frac{\pi}{12}$$

$$\arg(z_3)=\frac{\pi}{3}+\frac{\pi}{12}=\frac{5\pi}{12}.$$

$$and \quad z_3=z_1+z_2=i+\frac{1}{2}+\frac{\sqrt{3}}{2}i=\frac{1}{2}+i(1+\frac{\sqrt{3}}{2})$$

$$So \; Tan\frac{5\pi}{12}=\frac{1+\frac{\sqrt{3}}{2}}{\frac{1}{2}}=2+\sqrt{3}$$



Question 4:

$a)$ *Notation*: $2+(3\times 2)+(4+2^2)+...+(n+1)2^{n-1}=\sum_{r=1}^{n}(r+1)2^{r-1}.$

Proposition $\mathrm{P}_n : \sum_{r=1}^{n}(r+1)2^{r-1}=n2^n$ is to be proven by induction.

•Base case: $n=1$

$LHS: 2 \quad and \quad RHS: (1+1)\times 2^{1-1}=2$ The proposition is true for $n=1$.

•Propostion P_k :Let's suppose that for $n=k$, the proposition is true,

meaning we suppose that $\sum_{r=1}^{k}(r+1)2^{r-1}=k2^k$.

Let's show that the proposition is then true for $n=k+1$,

meaning let's show that $\sum_{r=1}^{k+1}(r+1)2^{r-1}=(k+1)2^{k+1}$.

$$\bullet\sum_{r=1}^{k+1}(r+1)2^{r-1}=\sum_{r=1}^{k}(r+1)2^{r-1}+(k+2)2^k$$

$$=k2^k+(k+2)2^k=k2^k+k2^k+2^{k+1}=2k2^k+2^{k+1}$$

$$=k2^{k+1}+2^{k+1}=(k+1)2^{k+1}.$$

•Conclusion:

If the propostion is true for $n=k$ then it is true for $n=k+1$,

because it is true for n=1,

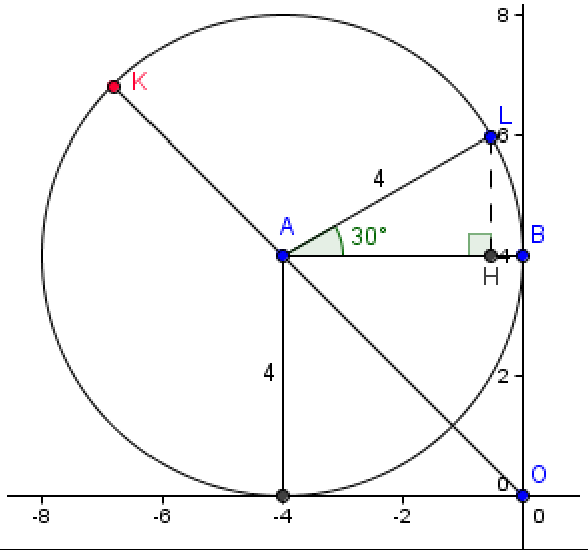
we can conclude, according to the induction principal,

that it is true for all $n\geq 1$: $for \; all \; n\geq 1, \sum_{r=1}^{n}(r+1)2^{r-1}=n2^n$

Exam report

Responses to this question were only fair. Although candidates had some idea of what was required for the inductive process, in part (a) they appeared to be easily confused. Common statements were for instance $(k+2)2^k=k2^k$ or $(k+1)2^{k-1}+(k+2)2^k$ or to even write down correctly $k2^k+(k+2)2^k$ but without any reference whatsoever as to what the expression represented.

Question 4:	Exam report
b) $\sum_{r=n+1}^{2n}(r+1)2^{r-1} = \sum_{r=1}^{2n}(r+1)2^{r-1} - \sum_{r=1}^{n}(r+1)2^{r-1}$ $= 2n\times 2^{2n} - n2^n = n2^n(2\times 2^n - 1)$ $\sum_{r=n+1}^{2n}(r+1)2^{r-1} = n2^n(2^{n+1}-1)$	In part (b), unless candidates realised that the given series was the difference of two other series no progress was made and only a few realised the connection with part (a). A common approach was to try and prove this result by induction also.

Question 5:	Exam report
Call the complex number $-4+4i$, z_A *and* A$(-4,4)$ the corresponding point in the Argand diagram. M is the point corresponding to the complex number z. $a)\ \lvert z+4-4i\rvert = 4$ $\lvert z-(-4+4i)\rvert = 4$ *corresponds to the locus* $AM = 4$ This is the circle centre A radius $r=4$. b) The furthest point away from O on the circle is K. $\lvert z\rvert_{\max} = OK = OA + r = \lvert z_A\rvert + 4 = \sqrt{(-4)^2+4^2}+4$ $\lvert z\rvert_{\max} = \sqrt{32}+4 = 4\sqrt{2}+4 = 4(\sqrt{2}+1)$ $c)\ \arg(z+4-4i) = \frac{\pi}{6}$ *means* $angle(AM,ox) = \frac{\pi}{6}$. *We* show the position with the point L. Using trigonometry in ALH, we have $x_L = a = -4 + AH = -4+4Cos\frac{\pi}{6} = -4+2\sqrt{3}$ *and* $y_L = b = 4 + LH = 4+4Sin\frac{\pi}{6} = 6$ $z_L = (-4+2\sqrt{3})+6i$ 	Most candidates realised that the locus in part (a) was a circle although it was frequently drawn in an incorrect quadrant, and occasionally with a radius of 2 rather than a radius of 4. The correct answer to part (b) was usually obtained although sometimes with a less than convincing argument. There were relatively fewer correct solutions to part (c). Those candidates who addressed the geometry of the figure were the most successful, but those who converted the equations of the circle and line into the Cartesian form and then attempted to solve a pair of simultaneous equations usually abandoned their solution after making algebraic errors.

Question 6:	**Exam report**
$z=e^{i\theta}$ a) i) $z+\frac{1}{z}=e^{i\theta}+\frac{1}{e^{i\theta}}=e^{i\theta}+e^{-i\theta}$ $=Cos\theta+iSin\theta+Cos\theta-iSin\theta$ $z+\frac{1}{z}=2Cos\theta$ ii) $z^2+\frac{1}{z^2}=e^{i2\theta}+\frac{1}{e^{i2\theta}}=e^{i2\theta}+e^{-i2\theta}$ $=Cos2\theta+iSin2\theta+Cos2\theta-iSin2\theta$ $z^2+\frac{1}{z^2}=2Cos2\theta$ iii) $z^2-z+2-\frac{1}{z}+\frac{1}{z^2}=2Cos2\theta-2Cos\theta+2$ we know that $Cos2\theta=2Cos^2\theta-1$ so $z^2-z+2-\frac{1}{z}+\frac{1}{z^2}=2(2Cos^2\theta-1)-2Cos\theta+2$ $z^2-z+2-\frac{1}{z}+\frac{1}{z^2}=4Cos^2\theta-2Cos\theta$ b) $z^4-z^3+2z^2-z+1=0$ factorise by z^2 ($z=0$ is not a solution) $z^2(z^2-z+2-\frac{1}{z}+\frac{1}{z^2})=0$ This gives $z^2-z+2-\frac{1}{z}+\frac{1}{z^2}=0$ $4Cos^2\theta-2Cos\theta=0$ $2Cos\theta(2Cos\theta-1)=0$ $Cos\theta=0$ or $Cos\theta=\frac{1}{2}$ $\theta=\frac{\pi}{2}$ or $\theta=-\frac{\pi}{2}$ or $\theta=\frac{\pi}{3}$ or $\theta=-\frac{\pi}{3}$ $z=e^{\pm i\frac{\pi}{2}}=\pm i$ or $z=e^{\pm i\frac{\pi}{3}}=\frac{1}{2}\pm i\frac{\sqrt{3}}{2}$	Parts (a) was quite well done and many candidates scored the available seven marks. There were however some serious algebraic errors, the commonest of which was to equate $z^2+\frac{1}{z^2}$ to $\left(z+\frac{1}{z}\right)^2$ in part (a)(ii) with some consequent faking to arrive at the printed result in part (a)(iii). Part (b) however was very poorly attempted. Candidates did not seem to realise that z could be equal to zero and consequently multiplied $(Cos\theta+iSin\theta)^2$ by $4Cos^2\theta-2Cos\theta$. Of those candidates who realised that $4Cos^2-2Cos\theta$ was equal to zero, the factorisation of this quadratic in $Cos\theta$ evaded most and, even when attempts were made to solve $4Cos^2\theta-2Cos\theta=0$, the factor $Cos\theta$ disappeared and the other solution $Cos\theta=\frac{1}{2}$ usually produced just one root from $\theta=\frac{\pi}{3}$. Candidates who were able to obtain both $Cos\theta=0$ and $Cos\theta=\frac{1}{2}$ usually produced only two solutions and subsequently two roots. It did not seem to occur to candidates that a quartic equation would have four roots.

Question 7:	**Exam report**
$a)i)\ Sinh\theta=\frac{1}{2}(e^{\theta}-e^{-\theta})\ \ and\ \ Cosh\theta=\frac{1}{2}(e^{\theta}+e^{-\theta})$ $2\,Sinh\theta\,Cosh\theta=2\times\frac{1}{2}(e^{\theta}-e^{-\theta})\times\frac{1}{2}(e^{\theta}+e^{-\theta})$ $=\frac{1}{2}\left(e^{2\theta}+e^{0}-e^{0}-e^{-2\theta}\right)$ $=\frac{1}{2}(e^{2\theta}-e^{-2\theta})=Sinh(2\theta)$ $ii)\ Cosh^{2}\theta+Sinh^{2}\theta=\left(\frac{1}{2}(e^{\theta}-e^{-\theta})\right)^{2}+\left(\frac{1}{2}(e^{\theta}+e^{-\theta})\right)^{2}$ $=\frac{1}{4}\left(e^{2\theta}-2e^{0}+e^{-2\theta}\right)+\frac{1}{4}\left(e^{2\theta}+2e^{0}+e^{-2\theta}\right)$ $=\frac{1}{4}\left(2e^{2\theta}+2e^{2\theta}\right)=\frac{1}{2}(e^{2\theta}+e^{-2\theta})=Cosh(2\theta)$ $b)\ x=Cosh^{3}\theta\quad,\quad y=Sinh^{3}\theta$ $i)\left(\frac{dx}{d\theta}\right)^{2}+\left(\frac{dy}{d\theta}\right)^{2}=\left(3Sinh\theta Cosh^{2}\theta\right)^{2}+\left(3Cosh\theta Sinh^{2}\theta\right)^{2}$ $=9Sinh^{2}\theta Cosh^{4}\theta+9Cosh^{2}\theta Sinh^{4}\theta$ $=9Sinh^{2}\theta Cosh^{2}\theta\left(Cosh^{2}\theta+Sinh^{2}\theta\right)$ $=9\left(\frac{1}{2}Sinh2\theta\right)^{2}\ (Cosh2\theta)$ $\left(\frac{dx}{d\theta}\right)^{2}+\left(\frac{dy}{d\theta}\right)^{2}=\frac{9}{4}Sinh^{2}2\theta Cosh2\theta$ $ii)\ S=\int_{0}^{1}\sqrt{\left(\frac{dx}{d\theta}\right)^{2}+\left(\frac{dy}{d\theta}\right)^{2}}\ d\theta=\int_{0}^{1}\sqrt{\frac{9}{4}Sinh^{2}2\theta Cosh2\theta}\ d\theta$ $S=\int_{0}^{1}\frac{3}{2}Sinh2\theta\sqrt{Cosh2\theta}\ d\theta=\frac{3}{2}\int_{0}^{1}Sinh2\theta\sqrt{Cosh2\theta}\ d\theta$ $S=\frac{3}{2}\int_{0}^{1}Sinh2\theta\times Cosh^{\frac{1}{2}}2\theta\,d\theta.$ This is an integral of the form $\int\ f'\times f^{n}=\frac{1}{n+1}f^{n+1}$ $S=\frac{3}{2}\left[\frac{1}{2}\times\frac{2}{3}\times Cosh^{\frac{3}{2}}2\theta\right]_{0}^{1}=\frac{1}{2}\left(Cosh^{\frac{3}{2}}2-Cosh^{\frac{3}{2}}0\right)$ $S=\frac{1}{2}\left((Cosh2)^{\frac{3}{2}}-1\right)$	Candidates were generally well drilled in proving the identities of parts (a)(i) and (a)(ii) although in (a)(ii) sometimes $\left(e^{\theta}-e^{-\theta}\right)^{2}$ was written as $e^{2\theta}+e^{-2\theta}$ with the same result from $\left(e^{\theta}+e^{-\theta}\right)^{2}$ thus obtaining the correct answer from incorrect algebra. Part (b)(i) was usually quite well done so long as candidates did not write $Cosh^{3}\theta\ \ as\ \ \frac{1}{8}\left(e^{\theta}+e^{-\theta}\right)^{3}$. Those who worked in powers of $e^{\theta}\ \ and\ \ e^{-\theta}$ found it impossible to reconcile their formula with the printed result and so make little meaningful progress. Part (b)(ii) proved to be beyond all but the most able candidates. The required integral, $\int\ \frac{3}{2}Sinh2\theta\sqrt{Cosh2\theta}d\theta$, was tackled successfully by these candidates by a variety of methods. Some spotted the integral, others used the substitution $u=Cosh2\theta$ whilst others again integrated by parts.

Grade Boundaries

Comp. Code	**Component Title**	**Maximum Scaled Mark**	**Scaled Mark Grade Boundaries**				
			A	**B**	**C**	**D**	**E**
MFP2	GCE MATHEMATICS UNIT FP2	75	58	51	44	38	32

June 2006

Answer **all** questions.

1 (a) Given that

$$\frac{r^2+r-1}{r(r+1)} = A + B\left(\frac{1}{r} - \frac{1}{r+1}\right)$$

find the values of A and B. *(3 marks)*

(b) Hence find the value of

$$\sum_{r=1}^{99} \frac{r^2+r-1}{r(r+1)}$$

(4 marks)

2 A curve has parametric equations

$$x = t - \frac{1}{3}t^3, \quad y = t^2$$

(a) Show that

$$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = (1+t^2)^2$$

(3 marks)

(b) The arc of the curve between $t = 1$ and $t = 2$ is rotated through 2π radians about the x-axis.

Show that S, the surface area generated, is given by $S = k\pi$, where k is a rational number to be found. *(5 marks)*

3 The curve C has equation

$$y = \cosh x - 3\sinh x$$

(a) (i) The line $y = -1$ meets C at the point $(k, -1)$.

Show that

$$\mathrm{e}^{2k} - \mathrm{e}^k - 2 = 0$$

(3 marks)

(ii) Hence find k, giving your answer in the form $\ln a$. *(4 marks)*

(b) (i) Find the x-coordinate of the point where the curve C intersects the x-axis, giving your answer in the form $p\ln a$. *(4 marks)*

(ii) Show that C has no stationary points. *(3 marks)*

(iii) Show that there is exactly one point on C for which $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$. *(1 mark)*

4 (a) On one Argand diagram, sketch the locus of points satisfying:

(i) $|z - 3 + 2\mathrm{i}| = 4$; *(3 marks)*

(ii) $\arg(z-1) = -\frac{1}{4}\pi$. *(3 marks)*

(b) Indicate on your sketch the set of points satisfying both

$$|z - 3 + 2\mathrm{i}| \leqslant 4$$

and $$\arg(z-1) = -\frac{1}{4}\pi$$

(1 mark)

5 The cubic equation

$$z^3 - 4\mathrm{i}z^2 + qz - (4 - 2\mathrm{i}) = 0$$

where q is a complex number, has roots α, β and γ .

(a) Write down the value of:

(i) $\alpha + \beta + \gamma$; *(1 mark)*

(ii) $\alpha\beta\gamma$. *(1 mark)*

(b) Given that $\alpha = \beta + \gamma$, show that:

(i) $\alpha = 2\mathrm{i}$; *(1 mark)*

(ii) $\beta\gamma = -(1 + 2\mathrm{i})$; *(2 marks)*

(iii) $q = -(5 + 2\mathrm{i})$. *(3 marks)*

(c) Show that β and γ are the roots of the equation

$$z^2 - 2\mathrm{i}z - (1 + 2\mathrm{i}) = 0$$

(2 marks)

(d) Given that β is real, find β and γ . *(3 marks)*

6 (a) The function f is given by

$$\mathrm{f}(n) = 15^n - 8^{n-2}$$

Express

$$\mathrm{f}(n+1) - 8\mathrm{f}(n)$$

in the form $k \times 15^n$. *(4 marks)*

(b) Prove by induction that $15^n - 8^{n-2}$ is a multiple of 7 for all integers $n \geqslant 2$. *(4 marks)*

7 (a) Find the six roots of the equation $z^6 = 1$, giving your answers in the form $\mathrm{e}^{\mathrm{i}\phi}$, where $-\pi < \phi \leqslant \pi$. *(3 marks)*

(b) It is given that $w = \mathrm{e}^{\mathrm{i}\theta}$, where $\theta \neq n\pi$.

(i) Show that $\dfrac{w^2 - 1}{w} = 2\mathrm{i}\sin\theta$. *(2 marks)*

(ii) Show that $\dfrac{w}{w^2 - 1} = -\dfrac{\mathrm{i}}{2\sin\theta}$. *(2 marks)*

(iii) Show that $\dfrac{2\mathrm{i}}{w^2 - 1} = \cot\theta - \mathrm{i}$. *(3 marks)*

(iv) Given that $z = \cot\theta - \mathrm{i}$, show that $z + 2\mathrm{i} = zw^2$. *(2 marks)*

(c) (i) Explain why the equation

$$(z + 2\mathrm{i})^6 = z^6$$

has five roots. *(1 mark)*

(ii) Find the five roots of the equation

$$(z + 2\mathrm{i})^6 = z^6$$

giving your answers in the form $a + \mathrm{i}b$. *(4 marks)*

END OF QUESTIONS

Key To Mark Scheme And Abbreviations Used In Marking

M	mark is for method		
m or dM	mark is dependent on one or more M marks and is for method		
A	mark is dependent on M or m marks and is for accuracy		
B	mark is independent of M or m marks and is for method and accuracy		
E	mark is for explanation		
√or ft or F	follow through from previous incorrect result	MC	mis-copy
CAO	correct answer only	MR	mis-read
CSO	correct solution only	RA	required accuracy
AWFW	anything which falls within	FW	further work
AWRT	anything which rounds to	ISW	ignore subsequent work
ACF	any correct form	FIW	from incorrect work
AG	answer given	BOD	given benefit of doubt
SC	special case	WR	work replaced by candidate
OE	or equivalent	FB	formulae book
A2,1	2 or 1 (or 0) accuracy marks	NOS	not on scheme
$-x$ EE	deduct x marks for each error	G	graph
NMS	no method shown	c	candidate
PI	possibly implied	sf	significant figure(s)
SCA	substantially correct approach	dp	decimal place(s)

No Method Shown

Where the question specifically requires a particular method to be used, we must usually see evidence of use of this method for any marks to be awarded. However, there are situations in some units where part marks would be appropriate, particularly when similar techniques are involved. Your Principal Examiner will alert you to these and details will be provided on the mark scheme.

Where the answer can be reasonably obtained without showing working and it is very unlikely that the correct answer can be obtained by using an incorrect method, we must award **full marks**. However, the obvious penalty to candidates showing no working is that incorrect answers, however close, earn **no marks.**

Where a question asks the candidate to state or write down a result, no method need be shown for full marks.

Where the permitted calculator has functions which reasonably allow the solution of the question directly, the correct answer without working earns **full marks**, unless it is given to less than the degree of accuracy accepted in the mark scheme, when it gains **no marks.**

Otherwise we require evidence of a correct method for any marks to be awarded.

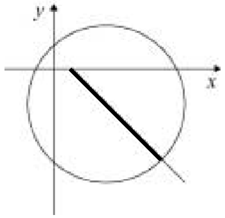
MFP2

Q	Solution	Marks	Total	Comments
1(a)	$r^2+r-1=A\left(r^2+r\right)+B$	M1		Any correct method
	$A=1, B=-1$	A1		
		A1F	3	ft B if incorrect A and vice versa
				Or $\frac{r^2+r-1}{r^2+r}=1-\frac{1}{r(r+1)}$ B1
				$=1-\left(\frac{1}{r}-\frac{1}{r+1}\right)$ M1A1
(b)	$r=1 \quad 1-\frac{1}{1}+\cancel{\frac{1}{2}}$			
	$r=2 \quad 1-\cancel{\frac{1}{2}}+\cancel{\frac{1}{3}}$	M1		Do not allow M1 if merely $\sum\frac{1}{r}-\sum\frac{1}{r+1}$ is summed
	$r=99 \quad 1-\cancel{\frac{1}{99}}+\frac{1}{100}$	A1F		A1 for suitable (3 at least) number of rows
	Sum $=98+\frac{1}{100}$	m1		Must have 98 or 99
	$=98.01$	A1F	4	OE Allow correct answer with no working 4 marks
	Total		**7**	
2(a)	$\dot{x}=1-t^2, \dot{y}=2t$	B1		
	$\dot{x}^2+\dot{y}^2=\left(1-t^2\right)^2+4t^2$	M1		
	$=\left(1+t^2\right)^2$	A1	3	AG; must be intermediate line
(b)	$S=2\pi\int_1^2\left(1+t^2\right)t^2\,\mathrm{d}t$	M1A1		Must be correct substitutions for M1
	$=2\pi\left[\frac{t^3}{3}+\frac{t^5}{5}\right]_1^2$	m1		Allow if one term integrated correctly
	$=2\pi\left[\frac{8}{3}+\frac{32}{5}-\frac{1}{3}-\frac{1}{5}\right]$	A1F		Any form
	$=\frac{256\pi}{15}$	A1F	5	
	Total		**8**	

MFP2 (cont)

Q	Solution	Marks	Total	Comments
3(a)(i)	$\dfrac{e^k + e^{-k}}{2} - \dfrac{3\left(e^k - e^{-k}\right)}{2} = -1$	M1		Allow if 2's are missing or if cosh*x* and sinh*x* interchanged
	$-2e^k + 4e^{-k} = -2$	A1		
	$e^{2k} - e^k - 2 = 0$	A1	3	AG Condone *x* instead of *k*
(ii)	$\left(e^k + 1\right)\left(e^k - 2\right) = 0$	M1		
	$e^k \neq -1$	E1		Must state something to earn E1. Do not accept ignoring or crossing out.
	$e^k = 2$	A1		
	$k = \ln 2$	A1F	4	
(b)(i)	$\cosh x = 3\sinh x$ or in terms of e^x	M1		
	$\tanh x = \dfrac{1}{3}$ or $2e^x = 4e^{\ x}$	A1		
	$x = \dfrac{1}{2}\ln\left(\dfrac{1+\frac{1}{3}}{1-\frac{1}{3}}\right)$ or $e^{2x} = 2$	A1F		
	$x = \dfrac{1}{2}\ln 2$	A1	4	CAO
(ii)	$\dfrac{dy}{dx} = \sinh x - 3\cosh x$ or $-e^x - 2e^{-x}$	M1		
	$= 0$ when $\tanh x = 3$ or $e^{2x} = -2$	A1		
	Correct reason	E1	3	Must give a reason
(iii)	$\dfrac{d^2 y}{dx^2} = y = 0$ at $\left(\frac{1}{2}\ln 2, 0\right)$ ie one point	B1F	1	
	Total		**15**	

MFP2 (cont)

Q	Solution	Marks	Total	Comments
4				
(a)(i)	Circle	B1		
	Correct centre	B1		
	Enclosing the origin	B1	3	
(ii)	Half line	B1		
	Correct starting point	B1		
	Correct angle	B1	3	
(b)	Correct part of the line **indicated**	B1F	1	
	Total		**7**	
5(a)(i)	$\alpha + \beta + \gamma = 4i$	B1	1	
(ii)	$\alpha\beta\gamma = 4 - 2i$	B1	1	
(b)(i)	$\alpha + \alpha = 4i, \ \alpha = 2i$	B1	1	AG
(ii)	$\beta\gamma = \dfrac{4-2i}{2i} = -2i - 1$	M1		Some method must be shown, eg $\dfrac{2}{i} - 1$
		A1	2	AG
(iii)	$q = \alpha\beta + \beta\gamma + \gamma\alpha$	M1		
	$= \alpha(\beta + \gamma) + \beta\gamma$	M1		Or $\alpha^2 + \beta\gamma$, ie suitable grouping
	$= 2i.2i - 2i - 1 = -2i - 5$	A1	3	AG
(c)	Use of $\beta + \gamma = 2i$ and $\beta\gamma = -2i - 1$	M1		Elimination of say γ to arrive at
	$z^2 - 2iz - (1 + 2i) = 0$	A1	2	$\beta^2 - 2i\beta - (1 + 2i) = 0$ M1A0 unless also some reference to γ being a root AG
(d)	$f(-1) = 1 + 2i - 1 - 2i = 0$	M1		For any correct method
	$\beta = -1, \ \gamma = 1 + 2i$	A1A1	3	A1 for each answer
	Total		**13**	

MFP2 (cont)

Q	Solution	Marks	Total	Comments
6(a)	$f(n+1) - 8f(n) = 15^{n+1} - 8^{n-1}$			
	$- 8(15^n - 8^{n-2})$	M1A1		
	$= 15^{n+1} - 8.15^n$			
	$= 15^n(15-8)$	M1		For multiples of powers of 15 only
	$= 7.15^n$	A1	4	For valid method ie not using 120^n etc
(b)	Assume $f(n)$ is M(7)			
	Then $f(n+1) - 8f(n) = 7 \times 15^n$	M1		Or considering $f(n+1) - f(n)$
	$f(n+1) = M(7) + M(7)$			
	$= M(7)$	A1		
	$n = 2$: $f(n) = 15^2 - 8^0 = 224$			
	$= 7 \times 32$	B1		$n = 1$ B0
	$P(n) \Rightarrow P(n+1)$ and $P(2)$ true	E1	4	Must score previous 3 marks to be awarded E1
	Total		**8**	

MFP2 (cont)

Q	Solution	Marks	Total	Comments
7(a)	$z = e^{\frac{2k\pi i}{6}}$, $k = 0, \pm 1, \pm 2, 3$	M1 A2,1,0	3	OE M1A1 only if: (1) range for k is incorrect eg 0,1,2,3,4,5 (2) i is missing
(b)(i)	$\frac{w^2-1}{w} = w - \frac{1}{w} = 2i\sin\theta$	M1A1	2	AG
(ii)	$\frac{w}{w^2-1} = \frac{1}{2i\sin\theta}$	M1		
	$= -\frac{i}{2\sin\theta}$	A1	2	AG
(iii)	$\frac{2i}{w^2-1} = \frac{-2iw^{-1}i}{2\sin\theta}$	M1		Or for $\frac{1}{\sin\theta\, e^{i\theta}}$
	$= \frac{1}{\sin\theta}(\cos\theta - i\sin\theta)$	A1		
	$= \cot\theta - i$	A1	3	AG
(iv)	$z = \frac{2i}{w^2-1}$ Or $z + 2i = \frac{2i}{w^2-1} + 2i$	M1		ie any correct method
	$z + 2i = zw^2$	A1	2	AG
(c)(i)	No coefficient of z^6	E1	1	
(ii)	$(w^2)^6 = 1 \quad w^2 = e^{\frac{k\pi i}{3}}$	B1		**Alternatively:**
	$z = \cot\frac{k\pi}{6} - i$, $k = \pm 1, \pm 2, 3$	M1 A2,1,0	4	$z + 2i = e^{\frac{k\pi i}{3}} z$ B1 $z = \frac{2i}{e^{\frac{k\pi i}{3}} - 1}$ M1 roots A2,1,0 (NB roots are $\pm\sqrt{3} - i$; $\pm\frac{1}{\sqrt{3}} - i$; $-i$)
	Total		**17**	
	TOTAL		**75**	

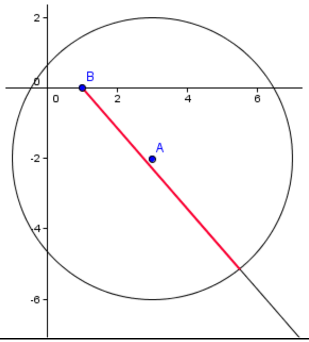
AQA – Further pure 2 – Jun 2006 – Answers

Question 1:	**Exam report**
1)a) $\frac{r^2+r-1}{r(r+1)} = A+B\left(\frac{1}{r}-\frac{1}{r+1}\right) = A+B\left(\frac{r+1-r}{r(r+1)}\right)$ $= \frac{Ar(r+1)+B}{r(r+1)} = \frac{Ar^2+Ar+B}{r(r+1)}$ *It is now clear that* $A=1$ *and* $B=-1$ b) $\sum_{r=1}^{99}\frac{r^2+r-1}{r(r+1)} = \sum_{r=1}^{99}1-\left(\frac{1}{r}-\frac{1}{r+1}\right) = \sum_{r=1}^{99}1-\sum_{r=1}^{99}\frac{1}{r}-\frac{1}{r+1}$ $\sum_{r=1}^{99}\frac{r^2+r-1}{r(r+1)} = 99-\left(1-\cancel{\frac{1}{2}}+\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}+\ldots+\cancel{\frac{1}{98}}-\cancel{\frac{1}{99}}+\cancel{\frac{1}{99}}-\frac{1}{100}\right)$ $= 99-1+\frac{1}{100} = 98.01$	Many candidates experienced difficulty in finding the values of A and B in part (a). They seemed to want to equate the left hand side of the identity to $\frac{C}{r}+\frac{D}{r+1}$, thus ignoring the fact that the powers of *r* in the numerator and denominator were equal. Generally the most successful candidates were those who rewrote the left hand side of the equation as $1-\frac{1}{r(r+1)}$ with the subsequent expressing of $\frac{1}{r(r+1)}$ in partial fractions. If candidates were successful in finding the values of *A* and *B,* they usually went on to complete part (b) correctly. The main source of error in this part, if mistakes were made, was to overlook the fact that that the constant term, as well as the variable terms, had to be summed from 1 to 99. The constant term was often left as 1.

Question 2:	**Exam report**
a) $\left(\frac{dx}{dt}\right)^2+\left(\frac{dx}{dt}\right)^2 = \left(1-t^2\right)^2+\left(2t\right)^2 = 1-2t^2+t^4+4t^2$ $= 1+2t^2+t^4 = \left(1+t^2\right)^2$ b) *From the formula book* : $S = 2\pi\int_1^2 y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$ $S = 2\pi\int_1^2 y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt = 2\pi\int_1^2 t^2\times(1+t^2)dt$ $S = 2\pi\int_1^2 t^2+t^4dt = 2\pi\left[\frac{1}{3}t^3+\frac{1}{5}t^5\right]_1^2 = 2\pi\left(\frac{8}{3}+\frac{32}{5}-\frac{1}{3}-\frac{1}{5}\right)$ $S = \frac{256}{15}\pi$	This was a very well-answered question with the vast majority of candidates either gaining full marks or losing one mark through faulty arithmetic. Very occasionally a candidate differentiated t^2+t^4 or integrated t^2+t^4, but wrote down $\frac{t^3}{3}+\frac{t^4}{4}$

Question 3:	**Exam report**
a)i) $y = Cosh\,x-3Sinh\,x$ *meets* $y=-1$ *at* $(k,-1)$ *This gives* $-1 = Cosh\,k-3Sinh\,k$ $-1 = \frac{1}{2}\left(e^k+e^{-k}\right)-\frac{3}{2}\left(e^k-e^{-k}\right)$ $-1 = \frac{1}{2}e^k+\frac{1}{2}e^{-k}-\frac{3}{2}e^k+\frac{3}{2}e^{-k}$ $-1 = -e^k+2e^{-k} \qquad (\times e^k)$ $-e^k = -e^{2k}+2$ $e^{2k}-e^k-2=0$ ii) $(e^k)^2-e^k-2=0$ $(e^k-2)(e^k+1)=0$ $e^k=2$ *or* $e^k=-1(No\,solution)$ $k=\ln(2) \qquad (e^x$ *is positive for all* $x)$	Again, this question proved to be a good source of marks for many candidates. Some candidates in part (a)(i) mixed the exponential forms for cosh *x* and sinh *x*,whilst others, having expressed cosh *x* and sinh *x* in exponential form correctly and having arrived at $-e^k+2e^{-k}=-1$ were unable to take the final step which led to the printed result. There was, also, not always a very convincing reason for the rejection of $e^k=-1$ in this part of the question. A very common error in part (a)(ii) was to write 2*k* – *k* – ln 2 = 0 after the printed answer, leading to the correct answer by totally incorrect mathematics.

Question 3: (Continues)	**Exam report**
$b)$ C intersect the x-$axis$ when $y=0$, we have to solve: $Cosh\,x - 3Sinh\,x = 0$ $\frac{1}{2}(e^x+e^{-x}) - \frac{3}{2}(e^x - e^{-x}) = 0$ $e^x + e^{-x} - 3e^x + 3e^{-x} = 0$ $4e^{-x} - 2e^x = 0 \qquad (\times e^x / 2)$ $2 - e^{2x} = 0$ $e^{2x} = 2 \qquad 2x = \ln(2)$ $x = \frac{1}{2}\ln(2)$ $ii)\frac{dy}{dx} = Sinh\,x - 3Cosh\,x = 0$ $Sinh\,x = 3Cosh\,x$ $Tanh\,x = 3$ No solution as : for all x, $-1 < Tanh\,x < 1$ C has no stationary point. $iii)\frac{d^2y}{dx^2} = Cosh\,x - 3Sinh\,x = y$ $and\ \ y = 0\ only\ \ for\ \ x = \frac{1}{2}\ln(2)$	Part (b)(i) was usually answered well, as was part (b)(ii), apart from those candidates who thought that the derivatives of hyperbolic functions followed the pattern of the derivatives of trigonometrical functions and so incurred sign errors. The explanation for the rejection of solutions to $e^{2x} = -2$ was more convincing in this part of the question than was the rejection of $e^k = -1$ in part (a)(ii). Part (b)(ii) was usually correct.

Question 4:	**Exam report**
$a)i)\,Let\ z_A\ be\,3-2i \quad and\ \ A(3,-2)$ $and\ \ M(z)$ $Then\ \lvert z-3+2i \rvert = 4$ $is\,equivalent\,to\ \ AM = 4$ This is a circle, centre A, radius $r = 4$. $ii)\,Let\ z_B = 1\ \ and\ \ B(1,0)$ $andf\ \ M(z)$ $Then\ \arg(z-1) = -\frac{\pi}{4}$ $is\ equivalent\ to\ \ angle\ (BM, ox) = -\frac{\pi}{4}$ This is a half-line, $y = Tan(-\frac{\pi}{4})x = -x \ \ , \ x > 1$ 	Responses to this question were usually quite good and it was pleasing to note some quite accurate neat diagrams using a ruler and compasses. Errors in parts (a)(i) and part (a)(ii) were usually errors of sign. For instance in part (a)(i) the centre of the circle was sometimes taken to be the point (–3,2)or even (3,2) and in part (a)(ii) the half line would be drawn from either (0,1)or (–1,0) . Just occasionally the radius of the circle was taken to be 2, or the direction of the line was taken to be $+\frac{\pi}{4}\,or + \frac{3\pi}{4}$. In part (b), a substantial number of candidates thought that the set of points must involve an area and consequently shaded some region in their sketch.

Question 5:	**Exam report**
$z^3-4iz^2+qz-(4-2i)=0\ \ has\ \ roots\ \ \alpha,\beta,\gamma$ $a)\,i)\ \alpha+\beta+\gamma=4i \qquad ii)\,\alpha\beta\gamma=4-2i$ $b)\,\alpha=\beta+\gamma$ $i)\,\alpha+\beta+\gamma=4i\ \ becomes$ $\alpha+\alpha=4i\quad so\quad \alpha=2i$ $ii)\,\alpha\beta\gamma=4-2i$ $\beta\gamma=\frac{4-2i}{\alpha}=\frac{4-2i}{2i}\times\frac{i}{i}=\frac{4i+2}{-2}$ $\beta\gamma=-2i-1=-(1+2i)$ $iii)\,q=\alpha\beta+\alpha\gamma+\beta\gamma$ $=\alpha(\beta+\gamma)+\beta\gamma$ $=\alpha^2+\beta\gamma=(2i)^2-(1+2i)$ $q=-4-1-2i$ $q=-5-2i$ $c)\,\beta+\gamma=2i\quad and\quad \beta\gamma=-(1+2i)$ $so\ \beta\ and\ \gamma$ are roots of the equations $z^2-2iz-(1+2i)=0$ $d)\,\beta=1\ is\ an\ "obvious"\ root\ \ (1^2-2i-(1+2i)=0)$ $z^2-2iz-(1+2i)=(z-1)(z^2+(1+2i))=0$ $roots\ are\ \beta=1\ \ and\ \ \gamma=1+2i$	Apart from the occasional sign errors, part (a) was answered well. Where sign errors did occur there was some faking to establish the printed answers in part (b). Part (b) is an example of what was mentioned at the beginning of this report in that with all three answers being printed, sufficient working needed to be shown in order to obtain full credit. Whilst most candidates knew roughly what was required for part (c), few candidates could express their argument succinctly. A number of candidates attempted to divide the cubic equation by z–2*i* with varying success. Probably the commonest method of approach in part (d) was to substitute β for z in the quadratic equation in z and then to equate real parts. Equating real parts led to $\beta^2=1$ from which a substantial number of candidates assumed that $\beta=1$ instead of considering the imaginary parts of the equation as well.

Question 6:	**Exam report**
$a)\,f(n+1)-8f(n)=(15^{n+1}-8^{n-1})-8(15^n-8^{n-2})$ $=15^{n+1}-8^{n-1}-8\times15^n+8^{n-1}$ $=15^n\left(15-8\right)$ $f(n+1)-8f(n)=7\times15^n$ $b)$ Propostion, P_n :For all $n\geq2,\ 15^n-8^{n-2}$ is a multiple of 7. Base case: for $n=2,\quad 15^2-8^{2-2}=225-1=224=7\times32$ the proposition is true for $n=2$. We suppose that P_k *is true : for* $n=k,\ \ f(k)=15^k-8^{k-2}$ is a multiple of 7. Let's show that P_{k+1} is the true:let's show that $f(k+1)=15^{k+1}-8^{k-1}$ is a multiple of 7. – According to question a) $f(k+1)=7\times15^k+8f(k)$ 7×15^k is a multiple of 7 $8f(k)$ is a multiple of 7, because $f(k)$ is (hypothesis) therefore $f(k+1)$ is a multiple of 7 – *Conclusion* :If P_k is true then P_{k+1} is also true, because P_2 is true, according to the induction principle, we can conclude that for all $n\geq2,\mathrm{P}_n$ *is true.*	Although there were some good solutions to part(a) of this question it did show in many cases a lack of understanding of the theory of indices. It was quite common to see 8×15^n written as a 120^n and $8\times8^{n-2}$ as 64^{n-2} . There was also a lack of clarity in part (b). It was not unusual to see the first line of the inductive proof to state “Assume result true for *n* = *k* i.e. that f (*k*) $=15^k-8^k-2$ ” to be followed by “ f (*k* +1) – 8f (*k*) is a multiple of 7 ”, showing a lack of understanding of the proof by induction in the case of multiples of integers. Some candidates tried to establish the result for *n* = 1 in spite of being told that *n* was greater than or equal to 2. A substantial minority of candidates ignored the hint in part (a) and in part (b) considered f (*k* +1) – f (*k*) with a measure of success

Question 7:	**Exam report**
a) Let note $z = re^{i\phi}$ then $z^6 = r^6 \times e^{i6\phi}$ and $1 = 1e^{i0}$ The equation $z^6 = 1$ is equivalent to $r^6 \times e^{i6\phi} = 1 \times e^{i0}$ This gives $r^6 = 1 \qquad r = 1$ $6\phi = 0 + k2\pi \qquad -3 < k \le 3$ $\phi = k\frac{\pi}{3} \qquad -2 \le k \le 3$	
so $z = e^{-i\frac{2\pi}{3}}$ or $e^{-i\frac{\pi}{3}}$ or e^{i0} or $e^{i\frac{\pi}{3}}$ or $e^{i\frac{2\pi}{3}}$ or $e^{i\pi}$ b)i) $\frac{w^2-1}{w} = \frac{e^{i2\theta}-1}{e^{i\theta}} = \frac{e^{i\theta}(e^{i\theta}-e^{-i\theta})}{e^{i\theta}} = e^{i\theta}-e^{-i\theta} = 2iSin\theta$ ii) $\frac{w}{w^2-1} = \frac{1}{2iSin\theta} = \frac{1}{2iSin\theta} \times \frac{i}{i} = -\frac{i}{2Sin\theta}$ iii) $\frac{2i}{w^2-1} = \frac{2i}{2iwSin\theta} = \frac{1}{e^{i\theta} \times Sin\theta} = \frac{e^{-i\theta}}{Sin\theta} = \frac{Cos\theta - iSin\theta}{Sin\theta}$ $= \frac{Cos\theta}{Sin\theta} - i\frac{Sin\theta}{Sin\theta} = Cot\theta - i$	Although part (a) of this question was standard work it was surprising to see many candidates fail to obtain full marks. The commonest errors were either to express the six roots of $z^6 = 1$ in the form a + ib , or to give the roots in the range 0 to 2π . A few candidates wrote down the 6 roots as $e^{i\frac{k\pi}{3}}$ with k = ± 1,± 2,±3. In part (b), parts (i) and (iv) were often well done, but relatively few candidates spotted part (b)(ii) as the reciprocal of part (b)(i), and it was not unusual to see $\frac{w}{w^2-1}$ rewritten as $w^{-1} - w$.
iv) $z = Cot\theta - i$ so $\frac{2i}{w^2-1} = z$ $2i = zw^2 - z$ $z + 2i = zw^2$ c)i) $(z+2i)^6 = z^6$ is equivalent to order 5 polynomial=0 (the terms in z^6 cancel out)	Part (b)(iii) was beyond all but the most able candidates although quite a number arrived at $\frac{1}{Sin\theta\, e^{i\theta}}$ at which point their solutions usually petered out. There was a wide variety of reasons why the equation $(z + 2i)^6 = z^6$ had only 5 roots with about 50% of them spurious.
ii) $(z+2i)^6 = z^6$ $\left(\frac{z+2i}{z}\right)^6 = 1$ $(w^2)^6 = 1$ So $w^2 = e^{i\frac{\pi k}{3}}$ (question (a)) $w = e^{i\frac{\pi k}{6}}$ This gives $z = \cot 0 - i$, $\cot\frac{\pi}{6} - i$, $\cot\frac{\pi}{3} - i$ $\cot\frac{2\pi}{3} - i$, $\cot\frac{5\pi}{6} - i$ $z = -i$, $\sqrt{3} - i$, $\frac{\sqrt{3}}{3} - i$, $-\frac{\sqrt{3}}{3} - i, -\sqrt{3} - i$	In part (c)(ii) only one or two candidates used the hints given in the earlier parts of the question, but instead, solved the equation $(z + 2i)^6 = z^6$ from first principles by writing $z + 2i = z\,e^{i\frac{k\pi}{3}}$ followed by $z = \frac{2i}{e^{i\frac{\pi}{3}} - 1}$. Of the few serious attempts made by candidates at this part of the question, most solutions ended at the point indicated and only the most able candidates found the five roots of the equation in the required form.

January 2007

Answer **all** questions.

1 (a) Given that

$$4\cosh^2 x = 7\sinh x + 1$$

find the two possible values of $\sinh x$. *(4 marks)*

(b) Hence obtain the two possible values of x, giving your answers in the form $\ln p$. *(3 marks)*

2 (a) Sketch on one diagram:

(i) the locus of points satisfying $|z - 4 + 2\mathrm{i}| = 2$; *(3 marks)*

(ii) the locus of points satisfying $|z| = |z - 3 - 2\mathrm{i}|$. *(3 marks)*

(b) Shade on your sketch the region in which

both $|z - 4 + 2\mathrm{i}| \leqslant 2$

and $|z| \leqslant |z - 3 - 2\mathrm{i}|$ *(2 marks)*

3 The cubic equation

$$z^3 + 2(1 - \mathrm{i})z^2 + 32(1 + \mathrm{i}) = 0$$

has roots α, β and γ.

(a) It is given that α is of the form $k\mathrm{i}$, where k is real. By substituting $z = k\mathrm{i}$ into the equation, show that $k = 4$. *(5 marks)*

(b) Given that $\beta = -4$, find the value of γ. *(2 marks)*

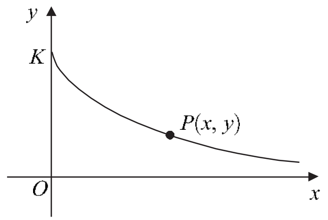
4 (a) Given that $y = \operatorname{sech} t$, show that:

(i) $\dfrac{\mathrm{d}y}{\mathrm{d}t} = -\operatorname{sech} t \tanh t$; *(3 marks)*

(ii) $\left(\dfrac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = \operatorname{sech}^2 t - \operatorname{sech}^4 t$. *(2 marks)*

(b) The diagram shows a sketch of part of the curve given parametrically by

$$x = t - \tanh t \qquad y = \operatorname{sech} t$$



The curve meets the y-axis at the point K, and $P(x, y)$ is a general point on the curve. The arc length KP is denoted by s. Show that:

(i) $\left(\dfrac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\dfrac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = \tanh^2 t$; *(4 marks)*

(ii) $s = \ln\cosh t$; *(3 marks)*

(iii) $y = \mathrm{e}^{-s}$. *(2 marks)*

(c) The arc KP is rotated through 2π radians about the x-axis. Show that the surface area generated is

$$2\pi(1 - \mathrm{e}^{-s})$$ *(4 marks)*

5 (a) Prove by induction that, if n is a positive integer,

$$(\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta$$ *(5 marks)*

(b) Find the value of $\left(\cos\frac{\pi}{6} + \mathrm{i}\sin\frac{\pi}{6}\right)^6$. *(2 marks)*

(c) Show that

$$(\cos\theta + \mathrm{i}\sin\theta)(1 + \cos\theta - \mathrm{i}\sin\theta) = 1 + \cos\theta + \mathrm{i}\sin\theta$$ *(3 marks)*

(d) Hence show that

$$\left(1 + \cos\frac{\pi}{6} + \mathrm{i}\sin\frac{\pi}{6}\right)^6 + \left(1 + \cos\frac{\pi}{6} - \mathrm{i}\sin\frac{\pi}{6}\right)^6 = 0$$ *(4 marks)*

6 (a) Find the three roots of $z^3 = 1$, giving the non-real roots in the form $\mathrm{e}^{\mathrm{i}\theta}$, where $-\pi < \theta \leqslant \pi$. *(2 marks)*

(b) Given that ω is one of the non-real roots of $z^3 = 1$, show that

$$1 + \omega + \omega^2 = 0$$ *(2 marks)*

(c) By using the result in part (b), or otherwise, show that:

(i) $\dfrac{\omega}{\omega + 1} = -\dfrac{1}{\omega}$; *(2 marks)*

(ii) $\dfrac{\omega^2}{\omega^2 + 1} = -\omega$; *(1 mark)*

(iii) $\left(\dfrac{\omega}{\omega + 1}\right)^k + \left(\dfrac{\omega^2}{\omega^2 + 1}\right)^k = (-1)^k\, 2\cos\frac{2}{3}k\pi$, where k is an integer. *(5 marks)*

7 (a) Use the identity $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$ with $A = (r + 1)x$ and $B = rx$ to show that

$$\tan rx \tan(r + 1)x = \frac{\tan(r + 1)x}{\tan x} - \frac{\tan rx}{\tan x} - 1$$ *(4 marks)*

(b) Use the method of differences to show that

$$\tan\frac{\pi}{50}\tan\frac{2\pi}{50} + \tan\frac{2\pi}{50}\tan\frac{3\pi}{50} + \ldots + \tan\frac{19\pi}{50}\tan\frac{20\pi}{50} = \frac{\tan\frac{2\pi}{5}}{\tan\frac{\pi}{50}} - 20$$ *(5 marks)*

END OF QUESTIONS

Key to mark scheme and abbreviations used in marking

M	mark is for method
m or dM	mark is dependent on one or more M marks and is for method
A	mark is dependent on M or m marks and is for accuracy
B	mark is independent of M or m marks and is for method and accuracy
E	mark is for explanation

√or ft or F	follow through from previous incorrect result	MC	mis-copy
CAO	correct answer only	MR	mis-read
CSO	correct solution only	RA	required accuracy
AWFW	anything which falls within	FW	further work
AWRT	anything which rounds to	ISW	ignore subsequent work
ACF	any correct form	FIW	from incorrect work
AG	answer given	BOD	given benefit of doubt
SC	special case	WR	work replaced by candidate
OE	or equivalent	FB	formulae book
A2,1	2 or 1 (or 0) accuracy marks	NOS	not on scheme
$-x$ EE	deduct x marks for each error	G	graph
NMS	no method shown	c	candidate
PI	possibly implied	sf	significant figure(s)
SCA	substantially correct approach	dp	decimal place(s)

No Method Shown

Where the question specifically requires a particular method to be used, we must usually see evidence of use of this method for any marks to be awarded. However, there are situations in some units where part marks would be appropriate, particularly when similar techniques are involved. Your Principal Examiner will alert you to these and details will be provided on the mark scheme.

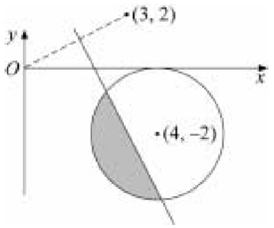
Where the answer can be reasonably obtained without showing working and it is very unlikely that the correct answer can be obtained by using an incorrect method, we must award **full marks**. However, the obvious penalty to candidates showing no working is that incorrect answers, however close, earn **no marks**.

Where a question asks the candidate to state or write down a result, no method need be shown for full marks.

Where the permitted calculator has functions which reasonably allow the solution of the question directly, the correct answer without working earns **full marks**, unless it is given to less than the degree of accuracy accepted in the mark scheme, when it gains **no marks**.

Otherwise we require evidence of a correct method for any marks to be awarded.

MFP2

Q	Solution	Marks	Total	Comments
1(a)	Use of $\cosh^2 x = 1 + \sinh^2 x$	M1		Must be correct for M1
	$4\sinh^2 x - 7\sinh x + 3 = 0$	A1		
	$(4\sinh x - 3)(\sinh x - 1) = 0$	A1√		Provided quadratic factorizes
	$\sinh x = \frac{3}{4}$ or 1	A1√	4	
(b)	Use of formula for $\sinh^{-1}$	M1		
	$x = \ln 2$ or $\ln\left(1+\sqrt{2}\right)$	A1√		
		A1√	3	
	Total		7	
2(a)				
(i)	**Circle**			
	Correct centre	B1		
	Correct radius	B1		
	Touching x-axis	B1	3	
(ii)	**Line**			
	Point $(3,2)$ indicated	B1		
	Line through $\left(1\frac{1}{2}, 1\right)$	B1√		
	Perpendicular to $(0,0) \to (3,2)$	B1	3	
(b)	Correct shaded area	B1	2	For shading inside the circle provided no other area is shaded
		B1√		Must be a circle and a straight line for second B1
	Total		8	

MFP2 (cont)

Q	Solution	Marks	Total	Comments
3(a)	$-k^3\mathrm{i}+2(1-\mathrm{i})\left(-k^2\right)+32(1+\mathrm{i})=0$	M1		Any form
	Equate real and imaginary parts: $-k^3+2k^2+32=0$	A1		
	$-2k^2+32=0$	A1		
	$k=\pm 4$	A1		
	$k=+4$	E1	5	AG
(b)	Sum of roots is $-2(1-\mathrm{i})$	M1		Or $\alpha\beta\gamma=-(32+32\mathrm{i})$ Must be correct for M1
	Third root $2-2\mathrm{i}$	A1√	2	
	Total		**7**	
4(a)(i)	$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\cosh t}\right)=-1(\cosh t)^{-2}\sinh t$	M1A1		Or $\frac{-2\left(\mathrm{e}^t-\mathrm{e}^{-t}\right)}{\left(\mathrm{e}^t+\mathrm{e}^{-t}\right)^2}$
	$=-\operatorname{sech} t\tanh t$	A1	3	AG
(ii)	Use of $\tanh^2 t=1-\operatorname{sech}^2 t$	M1		
	Printed result	A1	2	
(b)(i)	$\dot{x}=1-\operatorname{sech}^2 t \quad (\dot{y}=-\operatorname{sech} t\tanh t)$	B1		
	$\dot{x}^2+\dot{y}^2=\left(1-\operatorname{sech}^2 t\right)^2+\operatorname{sech}^2 t-\operatorname{sech}^4 t$	M1A1		Any form
	$=1-\operatorname{sech}^2 t=\tanh^2 t$	A1	4	AG
(ii)	$s=\int_0^t \tanh t\,\mathrm{d}t$	M1		Ignore limits for M1 and first A1
	$=\left[\ln\cosh t\right]_0^t$	A1		
	$=\ln\cosh t$	A1	3	AG
(iii)	$\mathrm{e}^s=\cosh t$	M1		
	$y=\mathrm{e}^{-s}$	A1	2	AG
(c)	$S=2\pi\int_0^t \operatorname{sech} t\tanh t\,\mathrm{d}t$	M1		Ignore limits for M1 and first A1
	$=2\pi\left[-\operatorname{sech} t\right]_0^t$	A1		
	$=2\pi(1-\operatorname{sech} t)$	A1		
	$=2\pi\left(1-\mathrm{e}^{-s}\right)$	A1	4	AG
	Total		**18**	

MFP2 (cont)

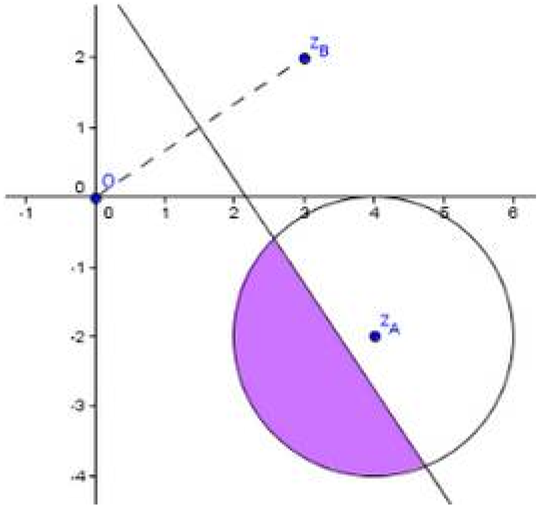
Q	Solution	Marks	Total	Comments
5(a)	Assume true for $n=k$ $(\cos\theta+\mathrm{i}\sin\theta)^{k+1}$			
	$=(\cos k\theta+\mathrm{i}\sin k\theta)(\cos\theta+\mathrm{i}\sin\theta)$	M1		
	Multiply out	A1		Any form
	$=\cos(k+1)\theta+\mathrm{i}\sin(k+1)\theta$	A1		
	True for $n=1$ shown	B1		
	$P(k)\Rightarrow P(k+1)$ and $P(1)$ true	E1	5	Allow E1 only if previous 4 marks earned
(b)	$\left(\cos\frac{\pi}{6}+\mathrm{i}\sin\frac{\pi}{6}\right)^6=\cos\frac{6\pi}{6}+\mathrm{i}\sin\frac{6\pi}{6}$	M1		
	$=-1$	A1	2	
(c)	$(\cos\theta+\mathrm{i}\sin\theta)(1+\cos\theta-\mathrm{i}\sin\theta)$	M1		
	$=\cos\theta+\cos^2\theta-\mathrm{i}\sin\theta\cos\theta$ $+\mathrm{i}\sin\theta+\mathrm{i}\sin\theta\cos\theta+\sin^2\theta$	A1		(Accept $-\mathrm{i}^2\sin^2\theta$) Or $\mathrm{e}^{\mathrm{i}\theta}\left(1+\mathrm{e}^{-\mathrm{i}\theta}\right)$
	$=1+\cos\theta+\mathrm{i}\sin\theta$	A1	3	AG
(d)	$\theta=\frac{\pi}{6}$ used	M1		In the context of part (c)
	Part (c) raised to power 6	M1		
	Use of result in part (b)	A1		
	$\left(1+\cos\frac{\pi}{6}+\mathrm{i}\sin\frac{\pi}{6}\right)^6+$ $\left(1+\cos\frac{\pi}{6}-\mathrm{i}\sin\frac{\pi}{6}\right)^6=0$	A1	4	AG
	Total		**14**	

Q	Solution	Marks	Total	Comments
6(a)	$1, \mathrm{e}^{\pm\frac{2\pi\mathrm{i}}{3}}$	M1A1	2	M1 for any method which would lead to the correct answers Accept e^0 or $\mathrm{e}^{0\mathrm{i}}$ Also accept answers written down correctly
(b)	Any correct method Shown for one root	M1 A1	2	AG
(c)(i)	$\frac{\omega}{\omega+1}=\frac{\omega}{-\omega^2}$	M1		ie use of result in (b)
	$=-\frac{1}{\omega}$	A1	2	AG
(ii)	$\frac{\omega^2}{\omega^2+1}=-\omega$	A1	1	AG
(iii)	$\left(\frac{\omega}{\omega+1}\right)^k+\left(\frac{\omega^2}{\omega^2+1}\right)^k=\left(-\frac{1}{\omega}\right)^k+(-\omega)^k$	M1A1		
	Use of $\omega=\mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}$	m1		
	$=(-1)^k\left(\mathrm{e}^{\frac{-2k\pi\mathrm{i}}{3}}+\mathrm{e}^{\frac{2k\pi\mathrm{i}}{3}}\right)$	A1		
	$=(-1)^k 2\cos\frac{2k\pi}{3}$	A1	5	AG
	Total		**12**	

Q	Solution	Marks	Total	Comments
7(a)	$\tan\left((r+1)x-rx\right)$			
	$=\frac{\tan(r+1)x-\tan rx}{1+\tan(r+1)x\tan rx}$	M1A1		
	Multiplying up Printed result	A1 A1	4	AG
(b)	$x=\frac{\pi}{50}$			
	$\tan\frac{\pi}{50}\tan\frac{2\pi}{50}=\frac{\tan\frac{2\pi}{50}}{\tan\frac{\pi}{50}}-\frac{\tan\frac{\pi}{50}}{\tan\frac{\pi}{50}}-1$			
	$\tan\frac{2\pi}{50}\tan\frac{3\pi}{50}=\frac{\tan\frac{3\pi}{50}}{\tan\frac{\pi}{50}}-\frac{\tan\frac{2\pi}{50}}{\tan\frac{\pi}{50}}-1$	M1A1		At least three lines to be shown Accept if x's used
	………………..			
	$\tan\frac{19\pi}{50}\tan\frac{20\pi}{50}=\frac{\tan\frac{20\pi}{50}}{\tan\frac{\pi}{50}}-\frac{\tan\frac{19\pi}{50}}{\tan\frac{\pi}{50}}-1$			
	Clear cancellation	m1		
	$\text{Sum }=\frac{\tan\frac{20\pi}{50}}{\tan\frac{\pi}{50}}-\frac{\tan\frac{\pi}{50}}{\tan\frac{\pi}{50}}-19$	A1		
	$=\frac{\tan\frac{2\pi}{5}}{\tan\frac{\pi}{50}}-20$	A1	5	AG
	Total		**9**	
	TOTAL		**75**	

AQA – Further pure 2 – Jan 2007 – Answers

Question 1:	**Exam report**
$a)\ 4Cosh^2x = 7Sinh\,x+1$ Using $\text{Cosh}^2x - Sinh^2x = 1$, The equation becomes $4(\text{Sinh}^2x+1) = 7\,Sinh\,x+1$ $4\,Sinh^2x - 7\,Sinh\,x+3=0$ $(4Sinh\,x-3)(Sinh\,x-1)=0$ $Sinh\,x = \frac{3}{4}\quad or\quad Sinh\,x=1$ $b)\ x = Sinh^{-1}(\frac{3}{4}) = \ln\left(\frac{3}{4}+\sqrt{\left(\frac{3}{4}\right)^2+1}\right)$ $= \ln\left(\frac{3+\sqrt{25}}{4}\right) = \ln(2)$ $or\ \ x = Sinh^{-1}(1) = \ln\left(1+\sqrt{1^2+1}\right) = \ln\left(1+\sqrt{2}\right)$	Apart from a few candidates who factorised the quadratic in sinh *x* incorrectly, most candidates worked part (a) correctly. However, many candidates spent more time on part (b) than was necessary. They expressed sinh *x* in exponential form and solved the ensuing quadratic equations rather than quote the formula for $\sinh^{-1} x$ given in the formulae booklet which they were entitled to do. This method also led to superfluous incorrect solutions which candidates needed to reject

Question 2:	**Exam report**
$a)\ i)\ Let\ \ z_A = 4-2i\ \ and\ \ A(4,-2)$ The point M represents z in the Argand diagram. $\lvert z-4+2i\rvert = 2$ $\lvert z - z_A\rvert = 2\ \ is\ equivalent\ to\ \ AM=2$ The locus of M is the circle centre $A(4,-2)$ radius $r=2$ $ii)\ Let\ z_B = 3+2i\ \ and\ \ B(3,2)$ $\lvert z\rvert = \lvert z-3-2i\rvert$ $\lvert z - z_o\rvert = \lvert z-z_B\rvert$ is equivalent to OM= BM The locus of M is the prependicular bisector of OB. b) $\lvert z-4+2i\rvert \le 2$ *is "inside" the circle* $\lvert z\rvert \le \lvert z-3-2i\rvert$ *is* the "half-plane" containing O. 	A few candidates misplotted the centre of the circle, usually at (.4, 2). Apart from this most drew the circle correctly. Not all recognised the line as the perpendicular bisector of the line joining the origin to the point (3, 2) in part (b), but rather thought that this equation represented another circle. This in turn had an effect on the shading in part (c) although the interior of the circle was usually shaded. It should be said that the diagrams were neat and in the main well labelled and in proportion; a great improvement on sketches submitted in previous years.

Question 3:	**Exam report**
$a)\ z^3+2(1-i)z^2+32(1+i)=0\ \ has\ roots\ \alpha,\beta,\gamma$ $\alpha=ki\ \ so$ $(ki)^3+2(1-i)(ki)^2+32+32i=0$ $-ik^3-2k^2+2ik^2+32+32i=0$ $(-2k^2+32)+i(-k^3+2k^2+32)=0$ *This gives* $-2k^2+32=0$ $and\ \ -k^3+2k^2+32=0$ The first equation gives $k=4\ or\ k=-4$ $-(-4)^3+2\times(-4)^2+32=64+32+32=128\quad k\neq-4$ $-(4)^3+2\times(4)^2+32=-64+32+32=0\quad k=4$ $b)\ \alpha=4i,\beta=-4\ and\ we\ know\ that$ $\alpha+\beta+\gamma=-2(1-i)$ $4i-4+\gamma=-2+2i$ $\gamma=2-2i$	Although this question was attempted by almost every candidate, there were few whose solutions presented the rigour required. Most substituted *ki* for *z* in the cubic equation, equated real parts and subsequently wrote z^2 = 16 so *z* = 4, not realising that *z* =- 4 was also a root of z^2 = 16 and that imaginary parts had to be equated in order to reject the solution *z* = 4. Part (b) was not particularly well answered either. The most common errors were errors of sign in the use of α +β +γ or αβγ , and those candidates using the product of the roots made extra work for themselves as they obtained a rational expression for γ which needed to be simplified. Some candidates thought that γ equalled -4i , the complex conjugate of α .

Question 4:	**Exam report**
$a)\ y=Sech\,t=\dfrac{1}{Cosh\,t}$ $i)\dfrac{dy}{dx}=-\dfrac{Sech\,t}{Cosh^2\,t}\qquad (if\ \ y=\dfrac{1}{f}\ then\ \dfrac{dy}{dx}=-\dfrac{f\,'}{f^2})$ $\dfrac{dy}{dx}=-\dfrac{Sech\,t}{Cosh\,t}\times\dfrac{1}{Cosh\,t}=-Sech\,t\times Tanh\,t$ $ii)\left(\dfrac{dy}{dx}\right)^2=(-Sech\,t\times Tanh\,t)^2=Sech^2t\times Tanh^2t$ Using $\boxed{Tanh^2t=1-Sech^2t}$ $\left(\dfrac{dy}{dx}\right)^2=Sech^2t(1-Sech^2t)=Sech^2t-Sech^4t$ $b)\ x=t-Tanh\,t\quad and\quad y=Sech\,t$ $i)\dfrac{dx}{dt}=1-Sech^2t\ \ and\ \ \left(\dfrac{dx}{dt}\right)^2=1-2Sech^2t+Sech^4t$ $\left(\dfrac{dx}{dt}\right)^2+\left(\dfrac{dy}{dt}\right)^2=1-2Sech^2t+Sech^4t+Sech^2t-Sech^4t$ $=1-Sech^2t=Tanh^2t$ $ii)\ s=\int_0^t\sqrt{\left(\dfrac{dx}{dt}\right)^2+\left(\dfrac{dy}{dt}\right)^2}\,dt=\int_0^t Tanh(t)dt=\left[\ln(Cosh\,t)\right]_0^t=\ln(Cosh\,t)$ $iii)\ e^s=Cosh\,t\qquad so\ \ e^{-s}=\dfrac{1}{Cosh\,t}=Sech\,t=y$	Generally, candidates scored quite well on this question. Some candidates struggled with part (a)(i) by not realising that sech *t* was (cosh t)$^{-1}$, instead expressing sech *t* in exponential form. This latter method rarely led to a correct solution. However, apart from some sign fudging in part (a)(ii), most candidates were able to recover to answer parts (a)(ii) and (b)(i) correctly. Very few candidates were able to score the three available marks in part (b)(ii), by either ignoring the limits of integration completely or by writing *s* = ln cosh *t* + *c* with no effort to show that the value of *c* was zero. In spite of some inelegant methods, part (b)(iii) was usually answered correctly.

Question 4: continues	Exam report
$c)\, S_x = 2\pi\int_0^t y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt = 2\pi\int_0^t Sech\,t\times Tanh\,t\,dt$ $S_x = 2\pi\int_0^t \frac{Sinh\,t}{Cosh^2\,t}dt = 2\pi\int_0^t Sinh\,t\times Cosh^{-2}t\,dt = 2\pi\left[-Cosh^{-t}t\right]_0^t$ $S_x = 2\pi\left(-Sech\,t+1\right) = 2\pi(1+e^{-s})\quad because\ Sech\,t = e^{-s}\ (Qb)iii))$	Part(c) caused problems, often by candidates attempting to integrate e^{-s} tanh *t* with respect to *t* by regarding e^{-s} either as a constant or else as e^{-t} Of those who correctly integrated to arrive at .secht , again very few candidates recognised the need for limits and so were unable to arrive at the printed result.

Question 5:	Exam report
$a)$ Proposition,P_n : $for\ all\ n$ positive integer, $(Cos\theta+iSin\theta)^n = Cos(n\theta)+iSin(n\theta)$ is to be proven by induction Base case: $n=1,\ (Cos\theta+iSin\theta)^1 = Cos(\theta)+iSin(\theta)$ $Cos(1\theta)+i\sin(1\theta) = Cos(\theta)+i\sin(\theta)$ $P_1\ is\ true$ We suppose that for n=k, the proposition is true $(Cos\theta+iSin\theta)^k = Cos(k\theta)+iSin(k\theta)$ $Let's$ show that P_{k+1} is true, let's show that $(Cos\theta+iSin\theta)^{k+1} = Cos((k+1)\theta)+iSin((k+1)\theta)$ - $(Cos\theta+iSin\theta)^{k+1} = (Cos\theta+iSin\theta)^k\times(Cos\theta+iSin\theta)$ $= (Cos(k\theta)+iSin(k\theta))\left(Cos\theta+iSin\theta\right)$ $= \left[Cos(k\theta)Cos\theta - Sin(k\theta)Sin\theta\right]+i\left[Cos(k\theta)Sin\theta+Sin(k\theta)Cos\theta\right]$ using the trig identities: $Cos(a+b) = Cos(a)Cos(b)-Sin(a)Sin(b)$ and $Sin(a+b) = Sin(a)Cos(b)+Sin(b)Cos(a)$ $we\ have:$ $(Cos\theta+iSin\theta)^{k+1} = Cos(k\theta+\theta)+iSin(k\theta+\theta) = Cos((k+1)\theta)+iSin((k+1)\theta)$ - *Conclusion* : If P_k is true then P_{k+1} is true, because P_1 is true we can conclude, according to the induction principle, that P_n is true for all n positive integer	It was clear that a good number of candidates had not met the proof of de Moivre's Theorem by induction before and there were not many solutions gaining full marks. It was also not uncommon to see expressions such as cos *k*θ +i sin *k*θ + cosθ + i sinθ = cos (*k* +1)θ + i sin (*k* +1)θ

Question 5:continues	Exam report
$b)\left(Cos\frac{\pi}{6}+iSin\frac{\pi}{6}\right)^6 = Cos\frac{6\pi}{6}+iSin\frac{6\pi}{6} = Cos\pi + iSin\pi = -1$ $c)\,(Cos\theta + i\sin\theta)(1+Cos\theta - i\sin\theta) =$ $(Cos\theta+iSin\theta)+(Cos\theta+iSin\theta)(Cos\theta-i\sin\theta)=$ $(Cos\theta+iSin\theta)+Cos^2\theta+Sin^2\theta = Cos\theta+iSin\theta+1$ $d)\,(1+Cos\frac{\pi}{6}+iSin\frac{\pi}{6})^6+(1+Cos\frac{\pi}{6}-iSin\frac{\pi}{6})^6 =$ $\left((Cos\frac{\pi}{6}+i\sin\frac{\pi}{6})(1+Cos\frac{\pi}{6}-i\sin\frac{\pi}{6})\right)^6+(1+Cos\frac{\pi}{6}-iSin\frac{\pi}{6})^6 =$ $(Cos\frac{\pi}{6}+iSin\frac{\pi}{6})^6\left(1+Cos\frac{\pi}{6}-iSin\frac{\pi}{6}\right)^6+(1+Cos\frac{\pi}{6}-iSin\frac{\pi}{6})^6 =$ $-1\times\left(1+Cos\frac{\pi}{6}-iSin\frac{\pi}{6}\right)^6+(1+Cos\frac{\pi}{6}-iSin\frac{\pi}{6})^6 = 0$	Parts (b) and (c) were generally well done, although in part (c) a number of candidates, when multiplying i sinθ by .i sinθ , wrote .i $\sin^2\theta$ and thus were unable to complete this part satisfactorily. Those candidates who spotted the connection between parts (c) and (d) usually went on to write out a correct solution to part (d), but it was disappointing to see $\left(1+Cos\frac{\pi}{6}+iSin\frac{\pi}{6}\right)^6$ *written as* $1^6+Cos\frac{6\pi}{6}+iSin\frac{6\pi}{6}$ with alarming regularity.

Question 6:	Exam report
$a)\,z^3=1$ we write $z=re^{i\theta}$ and $1=1e^{i0}$ $z^3=1$ *becomes* $r^3e^{i3\theta}=1e^{i0}$ $r=1$ and $3\theta=0+k2\pi$ $r=1$ and $\theta=k\frac{2\pi}{3}\quad k=-1,0,1$ $z=e^{-i\frac{2\pi}{3}}$ or $z=1=e^{i0}$ or $z=e^{i\frac{2\pi}{3}}$ $b)$ *we can note* $e^{i\frac{2\pi}{3}}=\omega$ $1+\omega+\omega^2$ is the sum of a geometric series with common ratio $\omega\,(\omega\neq1)$ $1+\omega+\omega^2=\frac{1-\omega^3}{1-\omega}=\frac{1-1}{1-\omega}=0$ $c)\,i)\,1+\omega+\omega^2=0\qquad \omega^2=-(1+\omega)$ $\frac{\omega^2}{1+\omega}=-1\qquad \frac{\omega}{1+\omega}=-\frac{1}{\omega}$ $ii)\,\frac{\omega}{1+\omega}=-\frac{1}{\omega}\qquad \frac{-1-\omega^2}{-\omega^2}=-\frac{1}{\omega}$ $\frac{1+\omega^2}{\omega^2}=-\frac{1}{\omega}\qquad \frac{\omega^2}{1+\omega^2}=-\omega$	It was disappointing to find many candidates unsure of the cube roots of unity and even more unsure of how to obtain them. It was also disappointing to note that few candidates were able to establish the result 1+ω +ω^2 = 0 in part (b), in spite of the variety of ways in which this result could be established. On the whole, parts (c)(i) and (c)(ii) were correctly done in spite of using roundabout methods to obtain the printed results.

Question 6:continues	Exam report
$iii)\left(\frac{\omega}{1+\omega}\right)^k+\left(\frac{\omega^2}{\omega^2+1}\right)^k=\left(-\frac{1}{\omega}\right)^k+\left(-\omega\right)^k$ $=\left(-e^{-i\frac{2\pi}{3}}\right)^k+\left(-e^{i\frac{2\pi}{3}}\right)^k$ $=(-1)^k e^{-i\frac{2k\pi}{3}}+(-1)^k e^{i\frac{2k\pi}{3}}$ $=(-1)^k\left(e^{i\frac{2k\pi}{3}}+e^{-i\frac{2k\pi}{3}}\right)$ $=(-1)^k Cos(\frac{2k\pi}{3})$	In part (c)(iii), however, most solutions ended at $\left(-\frac{1}{\omega}\right)^k+(-\omega)^k$, but of those candidates who attempted this part further, sign errors hindered completely correct solutions.

Question 7:

$$a) Tan(A-B)=Tan((r+1)x-rx)=Tan(x)$$

$$and\ Tan((r+1)x-rx)=\frac{Tan((r+1)x)-Tan(rx)}{1+Tan((r+1)x)Tan(rx)}$$

$$So\ Tan\,x=\frac{Tan((r+1)x)-Tan(rx)}{1-Tan((r+1)x)Tan(rx)}$$

$$1+Tan((r+1)x)Tan(rx)=\frac{Tan((r+1)x)-Tan(rx)}{Tan\,x}$$

$$Tan((r+1)x)Tan(rx)=\frac{Tan((r+1)x)}{Tan\,x}-\frac{Tan(rx)}{Tan\,x}-1$$

$$b) Tan\frac{\pi}{50}Tan\frac{2\pi}{50}+Tan\frac{2\pi}{50}Tan\frac{3\pi}{50}+...+Tan\frac{19\pi}{50}Tan\frac{20\pi}{50}$$

$$=\sum_{r=1}^{19}Tan(r\frac{\pi}{50})Tan((r+1)\frac{\pi}{50})=\sum_{r=1}^{19}\frac{Tan((r+1)\frac{\pi}{50})}{Tan\frac{\pi}{50}}-\frac{Tan(r\frac{\pi}{50})}{Tan\frac{\pi}{50}}-1$$

$$=\cancel{\frac{Tan(\frac{2\pi}{50})}{Tan\frac{\pi}{50}}}-\frac{Tan(\frac{\pi}{50})}{Tan\frac{\pi}{50}}-1+\xcancel{\frac{Tan(\frac{3\pi}{50})}{Tan\frac{\pi}{50}}}-\cancel{\frac{Tan(\frac{2\pi}{50})}{Tan\frac{\pi}{50}}}-1+\xcancel{\frac{Tan(\frac{4\pi}{50})}{Tan\frac{\pi}{50}}}-\cancel{\frac{Tan(\frac{3\pi}{50})}{Tan\frac{\pi}{50}}}-1$$

$$+...+\cancel{\frac{Tan(\frac{19\pi}{50})}{Tan\frac{\pi}{50}}}-\xcancel{\frac{Tan(\frac{18\pi}{50})}{Tan\frac{\pi}{50}}}-1+\frac{Tan(\frac{20\pi}{50})}{Tan\frac{\pi}{50}}-\cancel{\frac{Tan(\frac{19\pi}{50})}{Tan\frac{\pi}{50}}}-1$$

$$\text{all the terms cancel except }-\frac{Tan(\frac{\pi}{50})}{Tan\frac{\pi}{50}}+\frac{Tan(\frac{20\pi}{50})}{Tan\frac{\pi}{50}}-1-1-1-1..-1$$

$$=-1+\frac{Tan(\frac{20\pi}{50})}{Tan\frac{\pi}{50}}-19=\frac{Tan(\frac{2\pi}{5})}{Tan\frac{\pi}{50}}-20$$

Exam report

This question was surprisingly well done and attracted many completely correct solutions. When errors occurred, it was usually in the summation of terms in part (b). Candidates summed 20 terms instead of 19 which in turn led to some faking in arriving at the printed answer, especially the .20. For instance it was not uncommon to see the summation written as

$$\frac{Tan(\frac{21\pi}{50})-Tan(\frac{\pi}{50})}{Tan(\frac{\pi}{50})}-20$$

followed by the correct answer.

Component Code	Component Title	Maximum Scaled Mark	A	B	C	D	E
			Scaled Mark Grade Boundaries				
MFP2	MATHEMATICS UNIT MFP2	75	61	53	45	37	29

General Certificate of Education
June 2007
Advanced Level Examination

MATHEMATICS **MFP2**
Unit Further Pure 2

Tuesday 26 June 2007 1.30 pm to 3.00 pm

For this paper you must have:
- an 8-page answer book
- the **blue** AQA booklet of formulae and statistical tables.

You may use a graphics calculator.

Time allowed: 1 hour 30 minutes

Instructions
- Use blue or black ink or ball-point pen. Pencil should only be used for drawing.
- Write the information required on the front of your answer book. The *Examining Body* for this paper is AQA. The *Paper Reference* is MFP2.
- Answer **all** questions.
- Show all necessary working; otherwise marks for method may be lost.

Information
- The maximum mark for this paper is 75.
- The marks for questions are shown in brackets.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.

Answer **all** questions.

---

1 (a) Given that $f(r) = (r-1)r^2$, show that

$$f(r+1) - f(r) = r(3r+1)$$ *(3 marks)*

(b) Use the method of differences to find the value of

$$\sum_{r=50}^{99} r(3r+1)$$ *(4 marks)*

2 The cubic equation

$$z^3 + pz^2 + 6z + q = 0$$

has roots α, β and γ.

(a) Write down the value of $\alpha\beta + \beta\gamma + \gamma\alpha$. *(1 mark)*

(b) Given that p and q are real and that $\alpha^2 + \beta^2 + \gamma^2 = -12$:

(i) explain why the cubic equation has two non-real roots and one real root; *(2 marks)*

(ii) find the value of p. *(4 marks)*

(c) One root of the cubic equation is $-1 + 3\mathrm{i}$.

Find:

(i) the other two roots; *(3 marks)*

(ii) the value of q. *(2 marks)*

3 Use De Moivre's Theorem to find the smallest positive angle θ for which

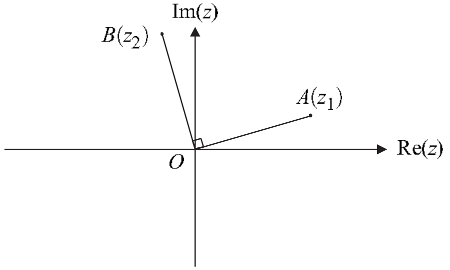
$$(\cos\theta + \mathrm{i}\sin\theta)^{15} = -\mathrm{i}$$ *(5 marks)*

4 (a) Differentiate $x\tan^{-1}x$ with respect to x. *(2 marks)*

(b) Show that

$$\int_0^1 \tan^{-1}x \, dx = \frac{\pi}{4} - \ln\sqrt{2}$$ *(5 marks)*

5 The sketch shows an Argand diagram. The points A and B represent the complex numbers z_1 and z_2 respectively. The angle $AOB = 90°$ and $OA = OB$.



(a) Explain why $z_2 = iz_1$. *(2 marks)*

(b) On a **single** copy of the diagram, draw:

(i) the locus L_1 of points satisfying $|z - z_2| = |z - z_1|$; *(2 marks)*

(ii) the locus L_2 of points satisfying $\arg(z - z_2) = \arg z_1$. *(3 marks)*

(c) Find, in terms of z_1, the complex number representing the point of intersection of L_1 and L_2. *(2 marks)*

6 (a) Show that

$$\left(1 - \frac{1}{(k+1)^2}\right) \times \frac{k+1}{2k} = \frac{k+2}{2(k+1)}$$ *(3 marks)*

(b) Prove by induction that for all integers $n \geqslant 2$

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\ldots\left(1-\frac{1}{n^2}\right) = \frac{n+1}{2n}$$ *(4 marks)*

7 A curve has equation $y = 4\sqrt{x}$.

(a) Show that the length of arc s of the curve between the points where $x = 0$ and $x = 1$ is given by

$$s = \int_0^1 \sqrt{\frac{x+4}{x}} \, dx$$ *(4 marks)*

(b) (i) Use the substitution $x = 4\sinh^2\theta$ to show that

$$\int \sqrt{\frac{x+4}{x}} \, dx = \int 8\cosh^2\theta \, d\theta$$ *(5 marks)*

(ii) Hence show that

$$s = 4\sinh^{-1}0.5 + \sqrt{5}$$ *(6 marks)*

8 (a) (i) Given that $z^6 - 4z^3 + 8 = 0$, show that $z^3 = 2 \pm 2i$. *(2 marks)*

(ii) Hence solve the equation

$$z^6 - 4z^3 + 8 = 0$$

giving your answers in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. *(6 marks)*

(b) Show that, for any real values of k and θ,

$$(z - ke^{i\theta})(z - ke^{-i\theta}) = z^2 - 2kz\cos\theta + k^2$$ *(2 marks)*

(c) Express $z^6 - 4z^3 + 8$ as the product of three quadratic factors with real coefficients. *(3 marks)*

END OF QUESTIONS

MFP2

Q	Solution	Marks	Total	Comments
1(a)	$f(r+1)-f(r)=r(r+1)^2-(r-1)r^2$	M1		
	$=r\left(r^2+2r+1-r^2+r\right)$	A1		any expanded form
	$=r(3r+1)$	A1	3	AG
(b)	$\left.\begin{array}{ll} r=50 & f(51)-f(50) \\ r=51 & f(52)-f(51) \\ r=99 & f(100)-f(99) \end{array}\right\}$ PI	M1A1		OE clearly shown. Accept $\sum_1^{99}-\sum_1^{49}$
	$\sum_{r=50}^{99} r(3r+1)=f(100)-f(50)$	m1		clear cancellation
	$=867500$	A1F	4	cao
	Total		7	
2(a)	$\sum\alpha\beta=6$	B1	1	
(b)(i)	Sum of squares < 0 ∴ not all real	E1		
	Coefficients real ∴ conjugate pair	E1	2	
(ii)	$\left(\sum\alpha\right)^2=\sum\alpha^2+2\sum\alpha\beta$	M1A1		A1 for numerical values inserted
	$\left(\sum\alpha\right)^2=0$	A1F		
	$p=0$	A1F	4	cao
(c)(i)	$-1-3i$ is a root	B1		
	Use of appropriate relationship eg $\sum\alpha=0$	M1		M0 if $\sum\alpha^2$ used unless the root 2 is checked
	Third root 2	A1F	3	incorrect $p\surd$
(ii)	$q=-(-1-3i)(-1+3i)2$	M1		allow even if sign error
	$=-20$	A1F	2	ft incorrect 3rd root
	Total		12	
3	$(\cos\theta+i\sin\theta)^{15}=\cos 15\theta+i\sin 15\theta$	M1		or $=e^{15i\theta}$
	$\cos 15\theta=0$ $\sin 15\theta=-1$	m1A1		or $-i=e^{\frac{3\pi i}{2}}$
	$15\theta=\frac{3\pi}{2}$ or $270°$	A1F		m1 for **both** R&I parts written down
	$\theta=\frac{\pi}{10}$ or $18°$	A1F	5	ft provided the value of 15θ is a correct value
	SC $\cos 15\theta+i\sin 15\theta=i$	(M1)		
	$\sin 15\theta=-1$	(B1)		or for $\cos 15\theta=0$
	$\theta=\frac{\pi}{10}$	(B1)	(3)	
	Total		5	

MFP2 (cont)

Q	Solution	Marks	Total	Comments
4(a)	$\frac{x}{1+x^2}+\tan^{-1}x$	B1B1	2	
(b)	$\int_0^1\tan^{-1}x\,dx=\left[x\tan^{-1}x\right]_0^1-\int_0^1\frac{x\,dx}{1+x^2}$	M1		either use of part (a) or integration by parts. Allow if sign error
	$\int\frac{x\,dx}{1+x^2}=\frac{1}{2}\ln\left(1+x^2\right)$	M1A1F		ft on $\int\frac{x}{1-x^2}dx$
	$I=1\tan^{-1}1-\frac{1}{2}\ln 2$	M1		
	$=\frac{\pi}{4}-\ln\sqrt{2}$	A1	5	AG
	Total		7	
5(a)	Explanation	E2,1,0	2	E1 for $i=e^{\frac{\pi i}{2}}$ or $iz_1=-y_1+ix_1$
(b)(i)	Perpendicular bisector of AB	B1		
	through O	B1	2	
(ii)	half-line	B1		If L_2 is taken to be the line AB give B0
	from B	B1		
	parallel to OA	B1	3	
(c)	$(1+i)z_1$	M1A1	2	ft if L_2 taken as line AB
	Total		9	
6(a)	$\left(1-\frac{1}{(k+1)^2}\right)\times\frac{k+1}{2k}=\frac{(k+1)^2-1}{(k+1)^2}\times\frac{k+1}{2k}$	M1		
	$=\frac{k^2+2k}{(k+1)^2}\times\frac{k+1}{2k}$	A1		
	$=\frac{k+2}{2(k+1)}$	A1	3	AG
(b)	Assume true for $n=k$, then $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\ldots\left(1-\frac{1}{(k+1)^2}\right)$	M1		
	$=\frac{k+2}{2(k+1)}$	A1		
	True for $n=2$ shown $1-\frac{1}{2^2}=\frac{3}{4}$	B1		
	$P_n\Rightarrow P_{n+1}$ and P_2 true	E1	4	only if the other 3 marks earned
	Total		7	

MFP2 (cont)

Q	Solution	Marks	Total	Comments
7(a)	$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2}{\sqrt{x}}$	B1		accept $2x^{-\frac{1}{2}}$ etc
	$\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}=\sqrt{1+\frac{4}{x}}$	M1A1F		ft sign error in $\frac{\mathrm{d}y}{\mathrm{d}x}$
	$=\sqrt{\frac{x+4}{x}}$	A1	4	AG
(b)(i)	$x=4\sinh^2\theta,\ \mathrm{d}x=8\sinh\theta\cosh\theta\,\mathrm{d}\theta$	M1A1		M1 for any attempt at $\frac{\mathrm{d}x}{\mathrm{d}\theta}$
	$\mathrm{I}=\int\sqrt{\frac{4\sinh^2\theta+4}{4\sinh^2\theta}}\,8\sinh\theta\cosh\theta\,\mathrm{d}\theta$	M1		
	$=\int\frac{2\cosh\theta}{2\sinh\theta}\,8\sinh\theta\cosh\theta\,\mathrm{d}\theta$	m1		ie use of $\cosh^2\theta-\sinh^2\theta=1$
	$=\int 8\cosh^2\theta\,\mathrm{d}\theta$	A1	5	AG
(ii)	Use of $2\cosh^2\theta=1+\cosh 2\theta$	M1		allow if sign error
	$\mathrm{I}=\int 4(1+\cosh 2\theta)\,\mathrm{d}\theta$	A1		oe
	$=4\theta+2\sinh 2\theta$	A1F		oe
	Use of $\sinh 2\theta=2\sinh\theta\cosh\theta$	m1		
	$=4\sinh^{-1}\frac{1}{2}+4\times\frac{1}{2}\sqrt{1+\frac{1}{4}}$	A1F		
	$=4\sinh^{-1}\frac{1}{2}+\sqrt{5}$	A1	6	AG
	Total		**15**	

MFP2 (cont)

Q	Solution	Marks	Total	Comments
8(a)(i)	$z^3=\frac{4\pm\sqrt{16-32}}{2}$	M1		
	$=2\pm 2\mathrm{i}$	A1	2	AG
(ii)	$2+2\mathrm{i}=2\sqrt{2}\mathrm{e}^{\frac{\pi\mathrm{i}}{4}},\ 2-2\mathrm{i}=2\sqrt{2}\mathrm{e}^{\frac{-\pi\mathrm{i}}{4}}$	M1 A1A1		M1 for either result or for one of $r=2\sqrt{2},\quad\theta=\pm\frac{\pi}{4}$ $\left(r=2\sqrt{2}\ \text{A1},\ \theta=\pm\frac{\pi}{4}\ \text{A1}\right)$
	$z=\sqrt{2}\mathrm{e}^{\frac{\pi\mathrm{i}}{12}+\frac{2k\pi\mathrm{i}}{3}}$ or $\sqrt{2}\mathrm{e}^{\frac{-\pi\mathrm{i}}{12}+\frac{2k\pi\mathrm{i}}{3}}$	M1		M1 for either
	$z=\sqrt{2}\,\mathrm{e}^{\frac{\pm\pi\mathrm{i}}{12}},\ \sqrt{2}\mathrm{e}^{\frac{\pm3\pi\mathrm{i}}{4}},\ \sqrt{2}\mathrm{e}^{\frac{\pm7\pi\mathrm{i}}{12}}$	A2,1,0 F	6	allow A1 for any 3 correct ft errors in $\pm\frac{\pi}{4}$
(b)	Multiplication of brackets	M1		
	Use of $\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{-\mathrm{i}\theta}=2\cos\theta$	A1	2	AG
(c)	$\left(z-\sqrt{2}\mathrm{e}^{\frac{\pi\mathrm{i}}{12}}\right)\left(z-\sqrt{2}\mathrm{e}^{-\frac{\pi\mathrm{i}}{12}}\right)$			
	$=z^2-2\sqrt{2}\cos\frac{\pi}{12}z+2$	M1A1F		PI
	Product is $\left(z^2-2\sqrt{2}\cos\frac{\pi}{12}z+2\right)$ $\left(z^2-2\sqrt{2}\cos\frac{7\pi}{12}z+2\right)$ $\left(z^2-2\sqrt{2}\cos\frac{3\pi}{4}z+2\right)$	A1F	3	$\left(\text{or } z^2+2z+2\right)$
	Total		**13**	
	TOTAL		**75**	

AQA – Further pure 2 – Jun 2007 – Answers

Question 1:	**Exam report**
$a)\ f(r)=(r-1)r^2$ $f(r+1)-f(r)=r(r+1)^2-(r-1)r^2$ $=r\left[(r+1)^2-r(r-1)\right]$ $=r\left[r^2+2r+1-r^2+r\right]$ $=r(3r+1)$ $b)\ \sum_{r=50}^{99} r(3r+1)=\sum_{r=50}^{99} f(r+1)-f(r)=\cancel{f(51)}-f(50)$ $\cancel{+f(52)}\ \cancel{-f(51)}$ $\cancel{+f(53)}\ \cancel{-f(52)}$ $+\dots\ \cancel{+f(99)}\ \cancel{-f(98)}$ $+f(100)\cancel{-f(99)}$ All the terms cancel except $f(100)-f(50)=99\times100^2-49\times50^2$ $\sum_{r=50}^{99} r(3r+1)=867500$	Almost all candidates were successful with part (a) However, in part (b) a number of candidates used $\sum r^2$ and $\sum r$ to evaluate $\sum_{r=1}^{99} r(3r+1)$ contrary to the requirement of the question and so, even with a correct answer, scored no marks. The most successful candidates for this part of the question were those who carefully wrote out a number of rows including the first and last row, to illustrate the cancellations. Some candidates went awry when writing down the first or last terms of the series.

Question 2:	**Exam report**
$a)\ \alpha\beta+\beta\gamma+\alpha\gamma=6$ $b)\,i)\ \alpha^2+\beta^2+\gamma^2=-12<0$ This can only happens if one of the root is not a real number so if α is a complex number, then $\beta=\alpha^*$ *because p and q are real numbers and* γ *is real* (because otherwise γ^* would be a root too, making 4 roots instead of the expected 3) $ii)\ \alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\alpha\gamma)$ $-12\quad=(\alpha+\beta+\gamma)^2-2\times6$ $-12=(\alpha+\beta+\gamma)^2-12 \qquad \alpha+\beta+\gamma=0$ $So\ \ p=-(\alpha+\beta+\gamma)=0 \qquad p=0$ $c)\ \alpha=-1+3i \qquad \beta=\alpha^*=-1-3i$ $\alpha+\beta+\gamma=0$ $-1+3i-1-3i+\gamma=0 \qquad \gamma=2$ $ii)\ q=-\alpha\beta\gamma=-(-1+3i)(-1-3i)(2)=-2(1+9)=-20$	Whilst part (a) was usually correctly done, part (b)(i) was poorly answered. Some candidates were able to comment on the condition that as the sum of the squares of the roots was less than zero there would have to be complex roots, but few stated the conditions that the coefficients of the cubic equation were all real. The value of *p* in part (b)(ii) was very often correct but in part (c)(i) a very common error as to use $\sum\alpha^2$ = −12 in order to find the third root. This method led to α^2 = 4 from which almost all candidates using this method wrote α =2 without even considering the possibility that α could equal -2 . Part (c)(ii) was usually worked correctly although αβγ = + *q* appeared from time to time.

Question 3:	**Exam report**
$\left(Cos\theta+iSin\theta\right)^{15}=Cos\left(15\theta\right)+iSin\left(15\theta\right)=0-i$ $Cos(15\theta)=0\ \ and\ \ Sin(15\theta)=-1$ $so\ \ 15\theta=\frac{3\pi}{2} \qquad \theta=\frac{3\pi}{30}=\frac{\pi}{10}$	There were many incomplete solutions to this question. Whilst most candidates used the de Moivre's Theorem correctly, many candidates either equated real parts only to arrive at an incorrect answer, or equated imaginary parts. In this latter case, the solution $\theta=-\frac{\pi}{30}$ appeared frequently in spite of the request in the question that θ should be positive, or the correct answer appeared but from an incomplete solution. Some candidates solved cosθ = 0 and sinθ =-1 but gave two different values of θ as their answer, one from each equation.

Question 4 :	**Exam report**
a) $y = xTan^{-1}x \qquad \frac{dy}{dx} = 1\times Tan^{-1}x + x\times\frac{1}{1+x^2}$ $\frac{dy}{dx} = Tan^{-1}x + \frac{x}{1+x^2}$ b) $\int_0^1 Tan^{-1}x\,dx = \int_0^1 (Tan^{-1}x + \frac{x}{1+x^2}) - \frac{x}{1+x^2}dx$ $= \left[xTan^{-1}x\right]_0^1 - \int_0^1 \frac{x}{1+x^2}dx$ $= Tan^{-1}1 - \left[\frac{1}{2}\ln(1+x^2)\right]_0^1$ $= \frac{\pi}{4} - \frac{1}{2}\ln 2 = \frac{\pi}{4} - \ln\sqrt{2}$	This is the first time that a question has been set on inverse trigonometrical functions since this topic was included in the MFP2 specification. It was clear that many candidates did not know what tan.1*x* was. They were able to complete part (a) with the help of the formulae booklet although even then there was confusion between the derivatives of tan^{-1}and tanh^{-1} *x* as the derivative of tan^{-1} was given as $\frac{1}{1-x^2}$. However it was part (b) that revealed the true lack of understanding of inverse trigonometrical functions. Part (b) was either abandoned altogether or when attempted tan^{-1} *x* was frequently written as $\frac{1}{\tan x}$.

Question 5:	**Exam report**
a) $Angle\ AOB = 90^o\ and\ OA = OB$ $In\ complex\ terms\ this\ means\ that\ \lvert z_1\rvert = \lvert z_2\rvert$ $and\ Arg\left(\frac{z_2}{z_1}\right) = \frac{\pi}{2}$ $This\ gives\ \frac{z_2}{z_1} = 1e^{i\frac{\pi}{2}} = i \qquad z_2 = iz_1$ b) i) $A(z_1), B(z_2)\ and\ M(z)$ $\lvert z - z_1\rvert = \lvert z - z_2\rvert\ is\ equivalent\ to$ $AM = BM$ L_1 is the perpendicular bisector of AB. ii) $\arg(z - z_2) = \arg(z_1)$ L_2 is the half line from B, parallel to OA. c) Let's call I the point of intersection and I(z_I) *OBIA is a square :* *Because OA being perpendicular to OB* *we know that IB is also perpendicular to OB* *and by symmetry about the line* L_1, *IA is perpendicular to AO.* *OBIA* is a quadrilateral with 4 right angles and OA=OB *so OBIA is a square* In complex term, $z_I = z_1 + z_2 = z_1 + iz_1 = (1+i)z_1$	Explanations in part (a) were very unclear and generally far from convincing. Candidates generally referred to what had happened to the coordinates of the points represented by z_1 and z_2 , but few made allusion to the significance of *i* in the *iz* . The neatest solutions came from candidates who considered multiplication of a complex number by *i* as a rotation anticlockwise of π/2 Inaccurate copying of the diagram in part (b) caused loss of marks. For in*stance,* although candidates knew that the locus L_1 was the perpendicular bisector of *AB*, poor diagrams meant that their line did not pass through the origin. Again, for the locus L_2 , although the majority of candidates drew a half line through *B*, their line was not always parallel to *OA*. Part (c) proved to be beyond most candidates probably because few realised that the point of intersection of L_1 and L_2 was, in fact, the fourth vertex of the square whose three other vertices were *A, O* and *B*

Question 6:	**Exam report**
$\left(1 - \frac{1}{(k+1)^2}\right)\times\frac{k+1}{2k} = \left(\frac{(k+1)^2 - 1}{(k+1)^2}\right)\times\frac{k+1}{2k}$ $= \frac{k^2+2k}{(k+1)^2}\times\frac{k+1}{2k} = \frac{k(k+2)}{2k(k+1)} = \frac{(k+2)}{2(k+1)}$	Part (a) was usually answered correctly although there were many very long-winded algebraic methods employed including the multiplication out of just about every bracket followed immediately by their re-factorisation.

Question 6:continues

$$\Pr oposition\ P_n : For\ n \geq 2, \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\ldots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}$$

is to be proven by induction.

Base case: n=2 $\quad \left(1-\frac{1}{2^2}\right)=\frac{2^2-1}{2^2}=\frac{3}{4}\ and\ \frac{n+1}{2n}=\frac{2+1}{2\times 2}=\frac{3}{4}$

The proposition is true for n=2

Let's suppose that the propostion is true for n=k,

meaning $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\ldots\left(1-\frac{1}{k^2}\right)=\frac{k+1}{2k}$

Let's show that the proposition is true for n=k+1,

Let's show that $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\ldots\left(1-\frac{1}{(k+1)^2}\right)=\frac{k+2}{2(k+1)}$

– –

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\ldots\left(1-\frac{1}{(k+1)^2}\right)=\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\ldots\left(1-\frac{1}{k^2}\right)\left(1-\frac{1}{(k+1)^2}\right)$$

$$=\frac{k+1}{2k}\times\left(1-\frac{1}{(k+1)^2}\right)=\frac{k+2}{2(k+1)}\ from\ part\ a)$$

– –

Conclusion :If the proposition is true for n=k, then it is true for n=k+1 because the proposition is true for n=2, according to te induction principle I can conclude that the proposition is true for all n ≥ 2:

for all n ≥ 2, $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\ldots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}$

Exam report

There was however much muddled thinking in part (b). Whilst most candidates had some outline of the method of induction many candidates attempted this part with no reference whatever to the series product in question, whilst others tried to **add** the $(k+1)^{th}$ term to the sum of k products. Candidates who did consider the series usually used ∑ rather than Π but this was not penalised.

Question 7:

$y=4\sqrt{x}$

$a)\ s=\int_0^1\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx\quad from\ the\ formulae\ booklet$

$\frac{dy}{dx}=4\times\frac{1}{2\sqrt{x}}=\frac{2}{\sqrt{x}}\qquad\left(\frac{dy}{dx}\right)^2=\left(\frac{2}{\sqrt{x}}\right)^2=\frac{4}{x}$

$s=\int_0^1\sqrt{1+\frac{4}{x}}dx=\int_0^1\sqrt{\frac{x+4}{x}}dx$

Exam report

This question was generally answered well and many candidates were able to score 12 out of the available 15 marks. Part (a) was well answered apart from a few candidates who wrote $\frac{dy}{dx}=2x^{-\frac{1}{2}}$ followed by $\frac{1}{2x^{\frac{1}{2}}}$

Question 7:continues

$$b)i)\ x=4Sinh^2\theta \qquad \frac{dx}{d\theta}=4\times 2\times Cosh\theta\times Sinh\theta$$

$$dx=8Cosh\theta\,Sinh\theta\,d\theta$$

$$when\ \ x=1, Sinh\theta=\frac{1}{2}\, so\, \theta=Sinh^{-1}0.5 \quad and \quad when\ \ x=0, Sinh\theta=0$$

$$s=\int_0^1\sqrt{\frac{x+4}{x}}dx=\int_o^{Sinh^{-1}0.5}\sqrt{\frac{4Sinh^2\theta+4}{4Sinh^2\theta}}\times 8Cosh\theta\,Sinh\theta\,d\theta$$

$$s=\int_0^{Sinh^{-1}0.5}\sqrt{\frac{4Cosh^2\theta}{4Sinh^2\theta}}\times 8Cosh\theta\,Sinh\theta\,d\theta=\int_0^{Sinh^{-1}0.5}\frac{Cos\theta}{Sinh\theta}\times 8Cosh\theta\,Sinh\theta\,d\theta$$

$$s=\int_0^{Sinh^{-1}0.5}8\,Cosh^2\theta\,d\theta$$

$$b)ii)\ Cosh2\theta=2Cosh^2\theta-1\ so\ Cosh^2\theta=\frac{1}{2}+\frac{1}{2}Cosh2\theta$$

$$so\ \ s=\int_0^{Sinh^{-1}0.5}8(\frac{1}{2}+\frac{1}{2}Cosh2\theta)d\theta=\int_0^{Sinh^{-1}0.5}4+4Cosh2\theta d\theta$$

$$s=\left[4\theta+2Sinh2\theta\right]_0^{Sinh^{-1}0.5}=4Sinh^{-1}0.5+2Sinh(2Sinh^{-1}0.5)$$

$$Sinh2\theta=2Sinh\theta Cos\theta=2\times Sinh\theta\times\sqrt{1+Sinh^2\theta}$$

$$with\ \ \theta=Sinh^{-1}0.5\ \ we\ have\ Sinh(2Sinh^{-1}0.5)=2\times\frac{1}{2}\sqrt{1+\frac{1}{4}}=\frac{\sqrt{5}}{2}$$

$$s=4Sinh^{-1}0.5+2Sinh(2Sinh^{-1}0.5)=4Sinh^{-1}0.5+2\frac{\sqrt{5}}{2}=4Sinh^{-1}0.5+\sqrt{5}$$

Exam report

In part (b) there were two main sources of error. The first was to interchange d*x* with dθ without any consideration of $\frac{dx}{d\theta}$; and the second was to write $\sqrt{\frac{4Sinh^2\theta+4}{4Sinh^2\theta}}$ *as* $\frac{2Sinh\theta+2}{2Sinh\theta}$.

There were also a few candidates who were unable to differentiate $4Sinh^2\theta$.

In part (b)(i), most candidates were able to integrate 8cosh^{2}θ correctly but few were able to arrive at the printed result in part (b)(ii). Two factors contributed to this. Candidates either failed to change the limits for *x* to the corresponding limits for θ or else wrote the answer with no evident method. This was unacceptable as the answer for the arc lengths was given.

Question 8:

$$a)i)\ z^6-4z^3+8=0$$

$$Let\ z^3\ be\,t, the\, equation\, becomes\ \ t^2-4t+8=0$$

$$\text{discriminant:(-4)-4}\times 1\times 8\text{=-16=(4i)}^2$$

$$So\ t=z^3=\frac{4\pm 4i}{2}=z^3=2\pm 2i$$

$$ii)\ \text{Let's write } z^3=(re^{i\theta})^3=r^3e^{i3\theta}$$

$$2\pm 2i=2\sqrt{2}\left(\frac{1}{\sqrt{2}}\pm i\frac{1}{\sqrt{2}}\right)=2\sqrt{2}e^{\pm i\frac{\pi}{4}}$$

These complex numbers are equal when

$$r^3=2\sqrt{2}\ \ and\ \ 3\theta=\pm\frac{\pi}{4}+2k\pi$$

$$r=\sqrt{2}\ \ and\ \theta=\pm\frac{\pi}{12}+k\frac{2\pi}{3}\quad k=-1,0,1$$

This gives 6 solutions:

$$\sqrt{2}e^{\pm i\frac{\pi}{12}}\ \ or\ \ \sqrt{2}e^{\pm i\frac{3\pi}{12}}\ \ or\ \sqrt{2}e^{\pm i\frac{7\pi}{12}}$$

$$b)\ (z-ke^{i\theta})(z-ke^{-i\theta})=z^2-zk(e^{i\theta}+e^{-i\theta})+k^2e^0$$

$$=z^2-2zkCos\theta+k^2$$

$$c)\ z^6-4z^3+8=$$

$$(z-\sqrt{2}e^{i\frac{\pi}{12}})(z-\sqrt{2}e^{-i\frac{\pi}{12}})(z-\sqrt{2}e^{i\frac{7\pi}{12}})(z-\sqrt{2}e^{-i\frac{7\pi}{12}})(z-\sqrt{2}e^{i\frac{3\pi}{12}})(z-\sqrt{2}e^{-i\frac{3\pi}{12}})$$

$$=\left(z^2-2\sqrt{2}Cos\frac{\pi}{12}-2\right)\left(z^2-2\sqrt{2}Cos\frac{7\pi}{12}-2\right)\left(z^2-2\sqrt{2}Cos\frac{3\pi}{12}-2\right)$$

Exam report

Candidates were usually able to establish the result in part (a) although the methods used were sometimes somewhat inelegant. Part (a)(ii) was reasonably well done although some carelessness was in evidence in this part. For instance, some candidates although showing that the argument of z^3 was $\pm\frac{\pi}{4}$ continued their solution with only $+\frac{\pi}{4}$ and so arrived at a total of three roots. Others having reached $\left|z^3\right|=\sqrt{8}$ then thought that $\left|z\right|=\sqrt{8}$ also.

A few candidates used a method which, although possible, was not really suitable. They replaced the z^3 in $z^6+4z^3+8=0$ with 2 ±2*i* and so arrived at z^6 = ± 8*i* .

This latter equation gave the twelve roots of z^{12} = -.64 and the method was incomplete unless 6 of the roots were rejected. Part (b) was generally well done, but part (c) was really only completed by candidates who had correctly answered part (a)(ii).

Component Code	Component Title	Maximum Scaled Mark	Scaled Mark Grade Boundaries A	B	C	D	E
MFP2	GCE MATHEMATICS UNIT FP2	75	56	49	42	35	29

General Certificate of Education
January 2008
Advanced Level Examination

MATHEMATICS **MFP2**
Unit Further Pure 2

Thursday 31 January 2008 9.00 am to 10.30 am

For this paper you must have:
- an 8-page answer book
- the blue AQA booklet of formulae and statistical tables.

You may use a graphics calculator.

Time allowed: 1 hour 30 minutes

Instructions
- Use blue or black ink or ball-point pen. Pencil should only be used for drawing.
- Write the information required on the front of your answer book. The *Examining Body* for this paper is AQA. The *Paper Reference* is MFP2.
- Answer **all** questions.
- Show all necessary working; otherwise marks for method may be lost.

Information
- The maximum mark for this paper is 75.
- The marks for questions are shown in brackets.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.

Answer **all** questions.

---

1 (a) Express $4+4\mathrm{i}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\leqslant\pi$. *(3 marks)*

(b) Solve the equation

$$z^5 = 4+4\mathrm{i}$$

giving your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\leqslant\pi$. *(5 marks)*

2 (a) Show that

$$(2r+1)^3-(2r-1)^3 = 24r^2+2$$ *(3 marks)*

(b) Hence, using the method of differences, show that

$$\sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1)$$ *(6 marks)*

3 A circle C and a half-line L have equations

$$|z-2\sqrt{3}-\mathrm{i}| = 4$$

and

$$\arg(z+\mathrm{i}) = \frac{\pi}{6}$$

respectively.

(a) Show that:

(i) the circle C passes through the point where $z=-\mathrm{i}$; *(2 marks)*

(ii) the half-line L passes through the centre of C. *(3 marks)*

(b) On one Argand diagram, sketch C and L. *(4 marks)*

(c) Shade on your sketch the set of points satisfying both

$$|z-2\sqrt{3}-\mathrm{i}| \leqslant 4$$

and

$$0 \leqslant \arg(z+\mathrm{i}) \leqslant \frac{\pi}{6}$$ *(2 marks)*

Formulae | Roots | Complex numbers | De Moivre theorem | Proof by induction | Finite series | Inverse trig functions | Hyperbolic functions | Arc length | Past Papers | Jan 2006 | Jun 2006 | Jan 2007 | Jun 2007 | Jan 2008 | Jun 2008 | Jan 2009 | Jun 2009 | Jan 2010 | Jun 2010

4 The cubic equation

$$z^3 + \mathrm{i}z^2 + 3z - (1 + \mathrm{i}) = 0$$

has roots α, β and γ.

(a) Write down the value of:

(i) $\alpha + \beta + \gamma$; *(1 mark)*

(ii) $\alpha\beta + \beta\gamma + \gamma\alpha$; *(1 mark)*

(iii) $\alpha\beta\gamma$. *(1 mark)*

(b) Find the value of:

(i) $\alpha^2 + \beta^2 + \gamma^2$; *(3 marks)*

(ii) $\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2$; *(4 marks)*

(iii) $\alpha^2\beta^2\gamma^2$. *(2 marks)*

(c) Hence write down a cubic equation whose roots are α^2, β^2 and γ^2. *(2 marks)*

5 Prove by induction that for all integers $n \geqslant 1$

$$\sum_{r=1}^{n} (r^2 + 1)(r!) = n(n+1)!$$ *(7 marks)*

6 (a) (i) By applying De Moivre's theorem to $(\cos\theta + \mathrm{i}\sin\theta)^3$, show that

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$$ *(3 marks)*

(ii) Find a similar expression for $\sin 3\theta$. *(1 mark)*

(iii) Deduce that

$$\tan 3\theta = \frac{\tan^3\theta - 3\tan\theta}{3\tan^2\theta - 1}$$ *(3 marks)*

(b) (i) Hence show that $\tan\frac{\pi}{12}$ is a root of the cubic equation

$$x^3 - 3x^2 - 3x + 1 = 0$$ *(3 marks)*

(ii) Find two other values of θ, where $0 < \theta < \pi$, for which $\tan\theta$ is a root of this cubic equation. *(2 marks)*

(c) Hence show that

$$\tan\frac{\pi}{12} + \tan\frac{5\pi}{12} = 4$$ *(2 marks)*

7 (a) Given that $y = \ln\tanh\frac{x}{2}$, where $x > 0$, show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \operatorname{cosech} x$$ *(6 marks)*

(b) A curve has equation $y = \ln\tanh\frac{x}{2}$, where $x > 0$. The length of the arc of the curve between the points where $x = 1$ and $x = 2$ is denoted by s.

(i) Show that

$$s = \int_1^2 \coth x \,\mathrm{d}x$$ *(2 marks)*

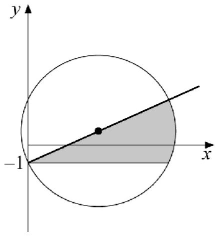
(ii) Hence show that $s = \ln(2\cosh 1)$. *(4 marks)*

END OF QUESTIONS

MFP2

Q	Solution	Marks	Total	Comments
1(a)	Any method for finding r or θ	M1		
	$r = 4\sqrt{2},\ \theta = \frac{\pi}{4}$	A1A1	3	
(b)	$z^5 = 4\sqrt{2}\,\mathrm{e}^{\frac{\pi i}{4}}$			M1 needs some reference to $a + 2k\pi i$
	$z = \sqrt{2}\mathrm{e}^{\frac{\pi i}{20} + \frac{2k\pi i}{5}}$	M1 A1F A1F		$\left.\begin{array}{l}\text{A1 for } r\\ \text{A1 for } \theta\end{array}\right]$ incorrect $r,\ \theta$ part (a)
	$z = \sqrt{2}\,\mathrm{e}^{\frac{\pi i}{20}},\ \sqrt{2}\,\mathrm{e}^{\frac{9\pi i}{20}},\ \sqrt{2}\,\mathrm{e}^{\frac{17\pi i}{20}},$ $\sqrt{2}\,\mathrm{e}^{\frac{-7\pi i}{20}},\ \sqrt{2}\,\mathrm{e}^{\frac{-15\pi i}{20}}$	A2,1,0 F	5	Accept r in any form eg $32^{\frac{1}{10}}$ Correct but some answers outside range allow A1 ft incorrect $r,\ \theta$ in part (a)
	Total		**8**	
2(a)	Attempt to expand $(2r+1)^3 - (2r-1)^3$	M1		
	$(2r+1)^3$ or $(2r-1)^3$ expanded	A1		
	$24r^2 + 2$	A1	3	AG
(b)	$r = 1 \quad 3^3 - 1^3 = 24\times1^2 + 2$ $r = 2 \quad 5^3 - 3^3 = 24\times2^2 + 2$	M1A1		3 rows seen Do not allow M1 for $(2n+1)^3 - 1$not equal to anything
	$r = n \quad (2n+1)^3 - (2n-1)^3 = 24\times n^2 + 2$			
	$(2n+1)^3 - 1 = 24\sum_{r=1}^{n} r^2 + 2n$	A1		
	$8n^3 + 12n^2 + 6n + 1 - 1 - 2n = 24\sum_{r=1}^{n} r^2$	M1		M1 for multiplication of bracket or taking $(2n+1)$out as a factor
	$8n^3 + 12n^2 + 4n = 24\sum_{r=1}^{n} r^2$	A1		CAO
	$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$	A1	6	AG
	Total		**9**	

MFP2 (cont)

Q	Solution	Marks	Total	Comments
3(a)(i)	$z = -i \quad \left\|-2\sqrt{3} - 2i\right\| = \sqrt{12+4} = 4$	M1 A1	2	$\left\|-2\sqrt{3} - 2i\right\|$ 4
(ii)	Centre of circle is $2\sqrt{3} + i$	B1		Do not accept$\left(2\sqrt{3},\ 1\right)$unless attempt to solve using trig
	Substitute into line	M1		
	arg $\left(2\sqrt{3} + 2i\right) = \frac{\pi}{6}$ shown	A1	3	
(b)				
	Circle: centre correct through $(0,-1)$ Half line: through $(0,-1)$ through centre of circle	B1 B1 B1 B1	4	
(c)	Shading inside circle and below line Bounded by $y = -1$	B1F B1	2	
	Total		**11**	
4(a)(i)	$\sum \alpha = -i$	B1	1	
(ii)	$\sum \alpha\beta = 3$	B1	1	
(iii)	$\alpha\beta\gamma = 1 + i$	B1	1	
(b)(i)	$\sum \alpha^2 = \left(\sum \alpha\right)^2 - 2\sum \alpha\beta$ used	M1		Allow if sign error or 2 missing
	$= (-i)^2 - 2\times3$	A1F		
	$= -7$	A1F	3	ft errors in (a)
(ii)	$\sum \alpha^2\beta^2 = \left(\sum \alpha\beta\right)^2 - 2\sum \alpha\beta\cdot\beta\gamma$	M1		Allow if sign error in 2 missing
	$= \left(\sum \alpha\beta\right)^2 - 2\alpha\beta\gamma\sum \alpha$	A1		
	$= 9 - 2(1+i)(-i)$	A1F		ft errors in (a)
	$= 7 + 2i$	A1F	4	ft errors in (a)
(iii)	$\alpha^2\beta^2\gamma^2 = (1+i)^2 = 2i$	M1 A1F	2	ft sign error in $\alpha\beta\gamma$
(c)	$z^3 + 7z^2 + (7+2i)z - 2i = 0$	B1F B1F	2	Correct numbers in correct places Correct signs
	Total		**14**	

MFP2 (cont)

Q	Solution	Marks	Total	Comments
5	Assume result true for $n=k$			
	Then $\sum_{r=1}^{k+1}(r^2+1)r!$			
	$=\left((k+1)^2+1\right)(k+1)!+k(k+1)!$	M1A1		
	Taking out $(k+1)!$ as factor	m1		
	$=(k+1)!\left(k^2+2k+1+1+k\right)$	A1		
	$=(k+1)(k+2)!$	A1		
	$\left.\begin{array}{l}k=1 \text{ shown } (1^2+1)1!=2\\ \qquad 1\times 2!=2\end{array}\right\rbrace$	B1		
	$P_k \Rightarrow P_{k+1}$ and P_1 true	E1	7	If all 6 marks earned
	Total		7	
6(a)(i)	$\cos 3\theta+\mathrm{i}\sin 3\theta=(\cos\theta+\mathrm{i}\sin\theta)^3$	M1		
	$=\cos^3\theta+3\mathrm{i}\cos^2\theta\sin\theta+3\mathrm{i}^2\cos\theta\sin^2\theta$ $+\mathrm{i}^3\sin^3\theta$	A1		
	Real parts: $\cos 3\theta=\cos^3\theta-3\cos\theta\sin^2\theta$	A1	3	AG
(ii)	Imaginary parts: $\sin 3\theta=3\cos^2\theta\sin\theta-\sin^3\theta$	A1F	1	
(iii)	$\tan 3\theta=\dfrac{\sin 3\theta}{\cos 3\theta}$	M1		Used
	$=\dfrac{3\cos^2\theta\sin\theta-\sin^3\theta}{\cos^3\theta-3\sin^2\theta\cos\theta}$	A1F		Error in $\sin 3\theta$
	$=\dfrac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}$			
	$=\dfrac{\tan^3\theta-3\tan\theta}{3\tan^2\theta-1}$	A1	3	AG
(b)(i)	$\tan\dfrac{3\pi}{12}=1$	B1		Used (possibly implied)
	$\tan\dfrac{\pi}{12}$ is a root of $1=\dfrac{x^3-3x}{3x^2-1}$	M1		Must be hence
	$x^3-3x^2-3x+1=0$	A1	3	
(ii)	Other roots are $\tan\dfrac{5\pi}{12}$, $\tan\dfrac{9\pi}{12}$	B1B1	2	
(c)	$\tan\dfrac{\pi}{12}+\tan\dfrac{5\pi}{12}+\tan\dfrac{9\pi}{12}=3$	M1		Must be hence
	$\tan\dfrac{\pi}{12}+\tan\dfrac{5\pi}{12}=4$	A1	2	
	Total		14	

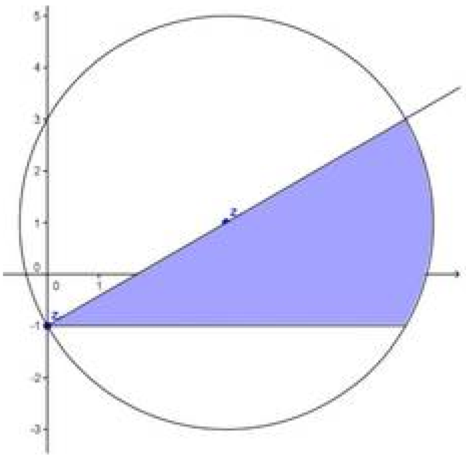
MFP2 (cont)

Q	Solution	Marks	Total	Comments
7(a)	$\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{1}{\tanh\frac{x}{2}}\cdots$	B1		
	$\operatorname{sech}^2\dfrac{x}{2}\cdots$	B1		
	$\dfrac{1}{2}$	B1		
	$=\dfrac{1}{2\dfrac{\sinh\frac{x}{2}}{\cosh\frac{x}{2}}\cosh^2\dfrac{x}{2}}$	M1		OE ie expressing in $\sinh\frac{x}{2}$ and $\cosh\frac{x}{2}$
	$=\dfrac{1}{2\sinh\frac{x}{2}\cosh\frac{x}{2}}$			
	$=\dfrac{1}{\sinh x}$	M1		ie use of $\sinh 2A=2\sinh A\cosh A$
	$=\operatorname{cosech} x$	A1	6	AG
	Alternative			
	$\ln\sinh\frac{x}{2}-\ln\cosh\frac{x}{2}$	(B1)		
	$\dfrac{1}{2}\dfrac{\cosh\frac{x}{2}}{\sinh\frac{x}{2}}-\dfrac{1}{2}\dfrac{\sinh\frac{x}{2}}{\cosh\frac{x}{2}}$	(B1B1)		
	$\dfrac{\cosh^2\frac{x}{2}-\sinh^2\frac{x}{2}}{2\sinh\frac{x}{2}\cosh\frac{x}{2}}$	(M1)		
	Use of $\sinh 2A$ = $2\sinh A\cosh A$	(M1)		
	result	(A1)		
(b)(i)	$s=\int_1^2\sqrt{1+\operatorname{cosech}^2 x}\,\mathrm{d}x$	M1		
	$=\int_1^2\coth x\,\mathrm{d}x$	A1	2	AG
(ii)	$s=[\ln\sinh x]_1^2$	M1		needs to be correct
	$=\ln\sinh 2-\ln\sinh 1$	A1		
	$=\ln\dfrac{2\sinh 1\cosh 1}{\sinh 1}$	A1F		must be seen
	$=\ln(2\cosh 1)$	A1	4	AG
	Total		12	
	TOTAL		75	

AQA – Further pure 2 – Jan 2008 – Answers

Question 1:	**Exam report**
$a)\ 4+4i=4\sqrt{2}\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)=4\sqrt{2}e^{i\frac{\pi}{4}}$ $b)\ \text{Let's write } z^5=(re^{i\theta})^5=r^5e^{i5\theta}$ $z^5=4+4i\ \ becomes$ $r^5e^{i5\theta}=4\sqrt{2}e^{i\frac{\pi}{4}}$ $so\ \ r^5=4\sqrt{2}\ \ and\ \ 5\theta=\frac{\pi}{4}+k\times 2\pi$ $r=\sqrt{2}\ \ and\ \ \theta=\frac{\pi}{20}+k\times\frac{2\pi}{5}\quad k=-2,-1,0,1,2$ $r=\sqrt{2}\ \ and\ \ \theta=-\frac{15\pi}{20},-\frac{7\pi}{20},\frac{\pi}{20},\frac{9\pi}{20},\frac{17\pi}{20}$ $The\ 5^{th}\ roots\ of\ \ 4+4i\ are:$ $\sqrt{2}e^{-i\frac{3\pi}{4}},\ \sqrt{2}e^{-i\frac{7\pi}{20}},\ \sqrt{2}e^{i\frac{\pi}{20}},\ \sqrt{2}e^{i\frac{9\pi}{20}},\ \sqrt{2}e^{i\frac{17\pi}{20}}$	Part (a) was done well by the majority of candidates. However, responses to part (b) were less successful. A number of candidates gave the roots of z^5 =1 as their answer to part (b), whilst others left the modulus of the roots as $4\sqrt{2}$ instead of $\sqrt{2}$, and others again gave solutions outside the range of θ as specified in the question. Some candidates yet again were either unable to handle $\frac{1}{5}\left(\frac{\pi}{4}+k\times 2\pi\right)$ or, when taking the fifth root of $e^{\left(\frac{\pi}{4}+k\times 2\pi\right)i}$, wrote $e^{\left(\frac{\pi}{4}+k\times\frac{2\pi}{5}\right)i}$

Question 2:	**Exam report**
$(2r+1)^3-(2r-1)^3=(8r^3+12r^2+6r+1)-(8r^3-12r^2+6r-1)$ $24r^2+2$ $b)\ 24\sum_{r=1}^{n}r^2=\sum_{r=1}^{n}\left((2r+1)^3-(2r-1)^3-2\right)=\sum_{r=1}^{n}\left((2r+1)^3-(2r-1)^3\right)-\sum_{r=1}^{n}2$ $=3^3-1+5^3-3^3+7^3-5^3+\ldots+$ $(2n-1)^3-(2n-3)^3+(2n+1)^3-(2n-1)^3-2n$ $=-1+(2n+1)^3-2n=-1+8n^3+12n^2+6n+1-2n$ $=8n^3+12n^2+4n$ $24\sum_{r=1}^{n}r^2=4n\left(2n^2+3n+1\right)=4n(2n+1)(n+1)$ $\sum_{r=1}^{n}r^2=\frac{1}{6}n(n+1)(2n+1)$	Again part (a) was answered well, but solutions to part (b) were mixed. Generally speaking, the best solutions came from candidates who rewrote part (a) as $r^2=\frac{1}{24}\left((2r+1)^3-(2r-1)^3-2\right)$ before making their summation. Those candidates who preferred to use part (a) in the form in which it was printed either forgot to sum the 2's to make 2*n* or only partially divided by 24. A small number of candidates used the method of induction either through confusing the two methods of summation or by deliberately choosing an alternative method. Either way, no credit could be given.

Question 3:	**Exam report**
$a)\,i)\lvert -i-2\sqrt{3}-i\rvert=\lvert -2\sqrt{3}-2i\rvert=\sqrt{(-2\sqrt{3})^2+(-2)^2}=\sqrt{12+4}=\sqrt{16}=4$ The circle C passes through the point where $z=-i$ $ii)$ The centre of C is the point where $z=2\sqrt{3}+i$ $\arg(z+i)=\arg(2\sqrt{3}+i+i)=\arg(2\sqrt{3}+2i)$ $Tan^{-1}(\frac{2}{2\sqrt{3}})=\frac{\pi}{6}.$ The half-line L passes through the centre of C. $b)$ $c)$ 	Lack of clear evidence that candidates understood what they were doing in part (a) caused a loss of marks for this part of the question. Methods varied. Those candidates who turned this part of the question into a coordinate geometry exercise probably provided the clearest solutions. Those candidates who evaluated $\lvert -2i-2\sqrt{3}\rvert$ *and* $\arg(2\sqrt{3}+2i)$ provided less convincing solutions and in some cases evident error. For instance it was not uncommon to see $\lvert -2i-2\sqrt{3}\rvert$ *written as* $\sqrt{(2\sqrt{3})^2-(2i)^2}$ Part (b) on the whole was done well except that in some instances not all the results of part (a) were incorporated in the candidate's Argand diagram. Part (c) was done well but, if a mistake did occur, it was almost always that the shaded area would be bounded by the real axis rather than by a line parallel to the real axis through the point represented by the complex number *z* = – i.

Question 4:	**Exam report**
$z^3+iz^2+3z-(1+i)=0$ *has roots* $\alpha,\beta,\gamma.$ $a)\,i)\alpha+\beta+\gamma=-i$ $ii)\alpha\beta+\alpha\gamma+\beta\gamma=3$ $iii)\alpha\beta\gamma=1+i$ $b)\,i)\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)$ $=(-i)^2-2\times 3$ $\alpha^2+\beta^2+\gamma^2=-7$ $ii)\alpha^2\beta^2+\alpha^2\gamma^2+\beta^2\gamma^2=(\alpha\beta+\alpha\gamma+\beta\gamma)^2-2(\alpha^2\beta\gamma+\beta^2\alpha\gamma+\gamma^2\alpha\beta)$ $=(\alpha\beta+\alpha\gamma+\beta\gamma)^2-2\alpha\beta\gamma(\alpha+\beta+\gamma)$ $=(3)^2-2\times(1+i)(-i)=9+2i+2i^2$ $\alpha^2\beta^2+\alpha^2\gamma^2+\beta^2\gamma^2=7+2i$ $iii)\alpha^2\beta^2\gamma^2=(\alpha\beta\gamma)^2=(1+i)^2=$ $\alpha^2\beta^2\gamma^2=2i$ $c)\,z^3-(-7)z^2+(7+2i)z-2i=0$ $z^3+7z^3+(7+2i)z-2i=0$	This question was probably the most popular question on this paper and certainly showed candidates well prepared to answer questions on this part of the specification. There were many fully correct solutions or correct apart from the odd sign error, the most common of which was to write down $(-i)^2$ as +1 instead of –1. If there was a major loss of marks, it was usually in the inability of a candidate to evaluate $\Sigma\alpha^2\beta^2$ and in this case the candidate started by considering $\left(\sum\alpha^2\right)^2$ only to find that the evaluation of $\Sigma\alpha^4$ posed a serious problem.

Question 5:	**Exam report**
The proposition P_n, $for\ all\ n \ge 1, \sum_{r=1}^{n}(r^2+1)(r!) = n(n+1)!$ is to be proven by induction Base case: $n=1$, $\sum_{r=1}^{1}(r^2+1)(r!) = (1^2+1)(1!) = 2$ $and\ 1(1+1)! = 2! = 2$ the proposition P_1 is true Let's supppose that P_k is true, ie $\sum_{r=1}^{k}(r^2+1)(r!) = k(k+1)!$ Let's show that P_{k+1} is true, Let's show that $\sum_{r=1}^{k+1}(r^2+1)(r!) = (k+1)(k+2)!$ – $\sum_{r=1}^{k+1}(r^2+1)(r!) = \sum_{r=1}^{k}(r^2+1)(r!) + \left((k+1)^2+1\right)(k+1)!$ $= k(k+1)! + \left(k^2+2k+2\right)(k+1)!$ $= (k+1)!\left(k+k^2+2k+2\right)$ $= (k+1)!\left(k^2+3k+2\right)$ $= \underbrace{(k+1)!(k+2)}(k+1)$ $= \quad (k+2)!\ \ (k+1)$ – *Conclusion* : If the proposition is true for $n=k$, then it is true for $n=k+1$ and because it true for $n=1$, we can conclude according to the induction principal that the proposition is true for all n. For all $n \ge 1, \sum_{r=1}^{n}(r^2+1)(r!) = n(n+1)!$	Responses to this question varied considerably. It was not, in general, that candidates did not understand the method of induction but rather that the algebraic manipulation especially in the handling of factorials proved to be a stumbling block. For instance k (k +1)! would be written as (k^2 + k)! and in a significant number of solutions candidates, having managed to reach (k +1)(k + 2)(k +1)!, abandoned their solutions not realising that the result was, in fact, (k +1)(k + 2)! .

Question 6:	**Exam report**
$a)\,i)\left(Cos\theta + iSin\theta\right)^3 = Cos3\theta + iSin3\theta$ $and\ \ also$ $\left(Cos\theta + iSin\theta\right)^3 = Cos^3\theta + 3iCos^2\theta\,Sin\theta - 3Cos\theta\,Sin^2\theta - iSin^3\theta$ $= \left(Cos^3\theta - 3Cos\theta\,Sin^2\theta\right) + i\left(3Cos^2\theta\,Sin\theta - Sin^3\theta\right)$ By identifying the real and imaginary parts, we have $Cos3\theta = Cos^3\theta - 3Cos\theta\,Sin^2\theta$ $ii)$ $Sin3\theta = 3Cos^2\theta\,Sin\theta - Sin^3\theta$ $iii)\,Tan3\theta = \frac{Sin3\theta}{Cos3\theta} = \frac{3Cos^2\theta\,Sin\theta - Sin^3\theta}{Cos^3\theta - 3Cos\theta\,Sin^2\theta}$ Now divide the numerator and the denominator by $Cos^3\theta$ $Tan3\theta = \frac{3\frac{Sin\theta}{Cos\theta} - \frac{Sin^3\theta}{Cos^3\theta}}{1-3\frac{Sin^2\theta}{Cos^2\theta}} = \frac{3Tan\theta - Tan^3\theta}{1-3Tan^2\theta} = \frac{Tan^3\theta - 3Tan\theta}{3Tan^2\theta - 1}$	Responses to this question were rather disappointing. In part (a)(i), although most candidates correctly quoted cos3θ + i sin 3θ = (cosθ + i sinθ)^3 , some immediately went on to use the multiple angle formulae instead of expanding (cosθ + i sinθ)^3. Some of those candidates who expanded (cosθ + i sinθ)^3 did not seem to realise that the answers to parts (a)(i) and (a)(ii) were obtained by simply equating real and imaginary parts. Other candidates wrote i^3 as +i and so were unable to reach the correct result of part (a)(ii) and the printed result in part (a)(iii). Even those candidates who worked parts (a)(i) and (a)(ii) correctly in terms of sinθ and cosθ , having written $\frac{Sin3\theta}{Cos3\theta}$, did not realise that the division of numerator and denominator by – cos3θ would give the printed result, but rather chose to use sin^{2}θ + cos^{2}θ =1 to express numerator and denominator in a different form, with no hope of reaching the printed result.

Question 6:continues	**Exam report**
$b)i)\text{For } \theta=\frac{\pi}{12}, we\ have$ $Tan\frac{3\pi}{12}=\frac{Tan^3\frac{\pi}{12}-3Tan\frac{\pi}{12}}{3Tan^2\frac{\pi}{12}-1}\quad .Let\ x\ be\ \tan\frac{\pi}{12}$ $then\quad 1=\frac{x^3-3x}{3x^2-1}\qquad x^3-3x=3x^2-1$ $x^3-3x^2-3x+1=0$ $ii)$ the other values of θ can be obtained by solving $Tan3\theta=1\ :$ $3\theta=\frac{\pi}{4}+k\pi$ $\theta=\frac{\pi}{12}+k\times\frac{\pi}{3}\qquad for\ k=0,\ \theta=\frac{\pi}{12}$ $k=1,\ \theta=\frac{5\pi}{12}$ $k=2,\ \theta=\frac{9\pi}{12}=\frac{3\pi}{4}$ c) The three roots of the equation $x^3-3x^2-3x+1=0$ $are\ \alpha=Tan\frac{\pi}{12}\ ,\ \beta=Tan\frac{5\pi}{12}\ ,\ \gamma=Tan\frac{3\pi}{4}=-1$ $\alpha+\beta+\gamma=3\qquad (x^3-3x^2-3x+1=0)$ $Tan\frac{\pi}{12}+Tan\frac{5\pi}{12}-1=3$ $Tan\frac{\pi}{12}+Tan\frac{5\pi}{12}=4$	Although the question in part (b)(i) started with the word ‘hence’ few candidates took up the hint and replaced θ by $\frac{\pi}{12}$ in part (a)(iii). If this part was attempted it was often done by solving the cubic equation in *x* to find its three roots and then by quoting that $Tan\frac{\pi}{12}=2-\sqrt{3}$ and a corresponding result for $Tan\frac{5\pi}{12}$ and consequently using these results in part (c), a method not indicated by the question.

Question 7:	**Exam report**
$a)\ y=\ln\left(Tanh\frac{x}{2}\right)\quad x>0$ $\frac{dy}{dx}=\frac{\frac{1}{2}\left(\sec h^2\frac{x}{2}\right)}{Tanh\frac{x}{2}}=\frac{1}{2}\times\frac{1}{Cosh^2\frac{x}{2}}\times\frac{Cosh\frac{x}{2}}{Sinh\frac{x}{2}}$ $\frac{dy}{dx}=\frac{1}{2Cosh\frac{x}{2}Sinh\frac{x}{2}}=\frac{1}{Sinh(2\times\frac{x}{2})}=\frac{1}{Sinhx}$ $\frac{dy}{dx}=\text{Cosech}\,x$	Although many candidates were able to write down $\frac{1}{\tanh\frac{x}{2}}$ multiplied by $\frac{1}{2}\text{sech}^2\frac{x}{2}$, fewer were able to combine these results to obtain cosech x . Even those candidates who expressed $\frac{dy}{dx}$ entirely in terms of $\cosh\frac{x}{2}\ and\ \sinh\frac{x}{2}$ seemed to baulk at the algebra which led to $\frac{1}{2\sinh\frac{x}{2}\cosh\frac{x}{2}}$.

Question 7:continues	**Exam report**
$b)\,i)\,s=\int_1^2 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\int_1^2\sqrt{1+\text{Cosech}^2x}\,dx$ $1+\text{Cosech}^2x=1+\frac{1}{Sinh^2x}=\frac{Sinh^2x+1}{Sinh^2x}=\frac{Cosh^2x}{Sinh^2x}=Coth^2x$ $s=\int_1^2 Coth\,x\,dx$ $ii)\,s=\int_1^2 Coth\,x\,dx=\int_1^2\frac{Cosh\,x}{Sinh\,x}dx=\left[\ln\left(Sinh\,x\right)\right]_1^2$ $s=\ln(\sinh 2)-\ln(\sinh 1)$ $s=\ln\left(\frac{Sinh2}{Sinh1}\right)$ using $Sinh2x=2\sinh x\cosh x,$ *we have* $\sinh 2=2\sinh 1\cosh 1$ $s=\ln\left(\frac{2\sinh 1\cosh 1}{\sinh 1}\right)$ $s=\ln\left(2\cosh 1\right)$	Part (b)(i) was done well and many candidates were able to arrive at $s = \ln\sinh 2 - \ln\sinh 1$ in part (b)(ii) but were unable to reach the printed answer. If the integral of coth x was performed incorrectly, it was often by coth x being replaced by $\frac{1}{\tanh x}$ followed by ln tanh x or ln cosh x as the integral.

Grade boundaries

Component Code	Component Title	Maximum Scaled Mark	Scaled Mark Grade Boundaries				
			A	B	C	D	E
MFP2	GCE MATHEMATICS UNIT FP2	75	60	52	45	38	31

June 2008

Answer **all** questions.

1 (a) Express

$$5\sinh x + \cosh x$$

in the form $Ae^{x} + Be^{-x}$, where A and B are integers. *(2 marks)*

(b) Solve the equation

$$5\sinh x + \cosh x + 5 = 0$$

giving your answer in the form $\ln a$, where a is a rational number. *(4 marks)*

2 (a) Given that

$$\frac{1}{r(r+1)(r+2)} = \frac{A}{r(r+1)} + \frac{B}{(r+1)(r+2)}$$

show that $A = \frac{1}{2}$ and find the value of B. *(3 marks)*

(b) Use the method of differences to find

$$\sum_{r=10}^{98} \frac{1}{r(r+1)(r+2)}$$

giving your answer as a rational number. *(4 marks)*

3 The cubic equation

$$z^3 + qz + (18 - 12\mathrm{i}) = 0$$

where q is a complex number, has roots α, β and γ.

(a) Write down the value of:

(i) $\alpha\beta\gamma$; *(1 mark)*

(ii) $\alpha + \beta + \gamma$. *(1 mark)*

(b) Given that $\beta + \gamma = 2$, find the value of:

(i) α; *(1 mark)*

(ii) $\beta\gamma$; *(2 marks)*

(iii) q. *(3 marks)*

(c) Given that β is of the form $k\mathrm{i}$, where k is real, find β and γ. *(4 marks)*

4 (a) A circle C in the Argand diagram has equation

$$|z + 5 - \mathrm{i}| = \sqrt{2}$$

Write down its radius and the complex number representing its centre. *(2 marks)*

(b) A half-line L in the Argand diagram has equation

$$\arg(z + 2\mathrm{i}) = \frac{3\pi}{4}$$

Show that $z_1 = -4 + 2\mathrm{i}$ lies on L. *(2 marks)*

(c) (i) Show that $z_1 = -4 + 2\mathrm{i}$ also lies on C. *(1 mark)*

(ii) Hence show that L touches C. *(3 marks)*

(iii) Sketch L and C on one Argand diagram. *(2 marks)*

(d) The complex number z_2 lies on C and is such that $\arg(z_2 + 2\mathrm{i})$ has as great a value as possible.

Indicate the position of z_2 on your sketch. *(2 marks)*

5 (a) Use the definition $\cosh x = \frac{1}{2}(e^x + e^{-x})$ to show that $\cosh 2x = 2\cosh^2 x - 1$. *(2 marks)*

(b) (i) The arc of the curve $y = \cosh x$ between $x = 0$ and $x = \ln a$ is rotated through 2π radians about the x-axis. Show that S, the surface area generated, is given by

$$S = 2\pi \int_0^{\ln a} \cosh^2 x \, dx$$ *(3 marks)*

(ii) Hence show that

$$S = \pi\left(\ln a + \frac{a^4 - 1}{4a^2}\right)$$ *(5 marks)*

6 By using the substitution $u = x - 2$, or otherwise, find the exact value of

$$\int_{-1}^{5} \frac{dx}{\sqrt{32 + 4x - x^2}}$$ *(5 marks)*

7 (a) Explain why $n(n+1)$ is a multiple of 2 when n is an integer. *(1 mark)*

(b) (i) Given that

$$f(n) = n(n^2 + 5)$$

show that $f(k+1) - f(k)$, where k is a positive integer, is a multiple of 6. *(4 marks)*

(ii) Prove by induction that $f(n)$ is a multiple of 6 for all integers $n \geqslant 1$. *(4 marks)*

8 (a) (i) Expand

$$\left(z + \frac{1}{z}\right)\left(z - \frac{1}{z}\right)$$ *(1 mark)*

(ii) Hence, or otherwise, expand

$$\left(z + \frac{1}{z}\right)^4\left(z - \frac{1}{z}\right)^2$$ *(3 marks)*

(b) (i) Use De Moivre's theorem to show that if $z = \cos\theta + i\sin\theta$ then

$$z^n + \frac{1}{z^n} = 2\cos n\theta$$ *(3 marks)*

(ii) Write down a corresponding result for $z^n - \frac{1}{z^n}$. *(1 mark)*

(c) Hence express $\cos^4\theta \sin^2\theta$ in the form

$$A\cos 6\theta + B\cos 4\theta + C\cos 2\theta + D$$

where A, B, C and D are rational numbers. *(4 marks)*

(d) Find $\int \cos^4\theta \sin^2\theta \, d\theta$. *(2 marks)*

END OF QUESTIONS

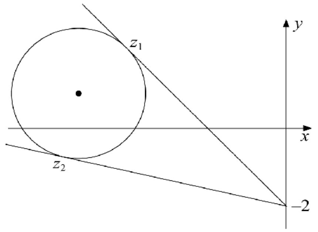
MFP2

Q	Solution	Marks	Total	Comments
1(a)	$5\left(\frac{e^x - e^{-x}}{2}\right)+\left(\frac{e^x + e^{-x}}{2}\right)$	M1		M0 if no 2s in denominator
	$= 3e^x - 2e^{-x}$	A1	2	
(b)	$3e^x - 2e^{-x} + 5 = 0$			
	$3e^{2x} + 5e^x - 2 = 0$	M1		ft if 2s missing in (a)
	$(3e^x - 1)(e^x + 2) = 0$	A1F		
	$e^x \neq -2$	E1		any **indication** of rejection
	$e^x = \frac{1}{3} \qquad x = \ln\frac{1}{3}$	A1F	4	provided quadratic factorises into real factors
	Total		**6**	
2(a)	$1 = A(r+2) + Br$	M1		
	$2A = 1, \quad A = \frac{1}{2}$	A1		
	$A + B = 0, \quad B = -\frac{1}{2}$	A1	3	
(b)	$r = 10 \qquad \frac{1}{2}\left(\frac{1}{10.11} - \frac{1}{11.12}\right)$			if (a) is incorrect but $A = \frac{1}{2}$ and $B = -\frac{1}{2}$ used, allow full marks for (b)
	$r = 11 \qquad \frac{1}{2}\left(\frac{1}{11.12} - \frac{1}{12.13}\right)$			
	$\cdots\cdots\cdot$			
	$r = 98 \qquad \frac{1}{2}\left(\frac{1}{98.99} - \frac{1}{99.100}\right)$	M1A1		3 relevant rows seen
	$S = \frac{1}{2}\left(\frac{1}{10.11} - \frac{1}{99.100}\right)$	m1		if split into $\frac{1}{2r} - \frac{1}{r+1} + \frac{1}{2(r+2)}$, follow mark scheme, in which case $\frac{1}{2.10} - \frac{1}{2.11} + \frac{1}{2.100} - \frac{1}{2.99}$ scores m1
	$= \frac{89}{19800}$	A1	4	
	Total		**7**	

MFP2 (cont)

Q	Solution	Marks	Total	Comments
3(a)(i)	$\alpha\beta\gamma = -18 + 12i$	B1	1	accept $-(18 - 12i)$
(ii)	$\alpha + \beta + \gamma = 0$	B1	1	
(b)(i)	$\alpha = -2$	B1F	1	
(ii)	$\beta\gamma = \frac{\alpha\beta\gamma}{\alpha} = 9 - 6i$	M1 A1F	2	ft sign errors in (a) or (b)(i) or slips such as miscopy
(iii)	$q = \sum\alpha\beta = \alpha(\beta+\gamma) + \beta\gamma$	M1		
	$= -2\times 2 + 9 - 6i$	A1F		ft incorrect $\beta\gamma$ or α
	$= 5 - 6i$	A1F	3	
(c)	$\beta = ki, \quad \gamma = 2 - ki$	B1		
	$ki(2 - ki) = 9 - 6i$	M1		
	$2k = -6 \quad (k^2 = 9) \quad k = -3$	m1		imaginary parts
	$\beta = -3i, \quad \gamma = 2 + 3i$	A1	4	
	Total		**12**	

MFP2 (cont)

Q	Solution	Marks	Total	Comments
4(a)	radius $\sqrt{2}$ centre $-5+\mathrm{i}$	B1,B1	2	condone $(-5, 1)$ for centre do not accept $(-5, \mathrm{i})$
(b)	$\arg(z_1+2\mathrm{i}) = \arg(-4+4\mathrm{i})$	M1		
	$= \frac{3\pi}{4}$	A1	2	clearly shown eg $\tan^{-1}\left(-\frac{1}{1}\right)$
(c)(i)	$\lvert z_1+5-\mathrm{i}\rvert = \lvert 1+\mathrm{i}\rvert = \sqrt{2}$	B1	1	
(ii)	Gradient of line from $(-5, 1)$ to $(-4, 2)$ is 1 $\left(\frac{\pi}{4}\right)$	M1A1		M1 for a complete method
	radius $\perp$ line $\therefore$ tangent	E1	3	
(iii)				
	Circle correct	B1F		ft incorrect centre or radius
	Half line correct	B1	2	line must touch C generally above the circle
(d)	z_2 in correct place	B1		B0 if z_2 is directly below the centre of C
	with tangent shown	B1	2	
	Total		**12**	

MFP2 (cont)

Q	Solution	Marks	Total	Comments
5(a)	$\left(\mathrm{e}^x+\mathrm{e}^{-x}\right)^2$ expanded correctly	B1		$\mathrm{e}^{2x}+2\mathrm{e}^0+\mathrm{e}^{-2x}$ is acceptable
	Result	B1	2	AG
(b)(i)	$\frac{\mathrm{d}y}{\mathrm{d}x} = \sinh x$	B1		
	$\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} = \sqrt{1+\sinh^2 x}$			
	$= \cosh x$	M1		use of $\cosh^2 x - \sinh^2 x = 1$
	$S = 2\pi\int_0^{\ln a} \cosh^2 x \,\mathrm{d}x$	A1	3	AG (clearly derived)
(ii)	Use of $\cosh^2 x = \frac{1}{2}(1+\cosh 2x)$	M1		allow one slip in formula M0 if $\int \cosh^2 x \,\mathrm{d}x$ is given as $\sinh^2 x$
	$S = \pi\left[x + \frac{1}{2}\sinh 2x\right]_0^{\ln a}$	A1		
	$= \pi\left[\ln a + \frac{1}{2}\left(\frac{\mathrm{e}^{2\ln a} - \mathrm{e}^{-2\ln a}}{2}\right)\right]$	M1		
	$= \pi\left[\ln a + \frac{1}{4}\left(a^2 - a^{-2}\right)\right]$	A1F		
	$= \pi\left[\ln a + \frac{1}{4a^2}\left(a^4 - 1\right)\right]$	A1	5	AG
	Total		**10**	
6	$u = x - 2$			
	$\mathrm{d}u = \mathrm{d}x$ or $\frac{\mathrm{d}u}{\mathrm{d}x} = 1$	B1		clearly seen
	$32 + 4x - x^2 = 36 - u^2$	B1		if $32+4x-x^2$ is written as $36-(x-2)^2$, give B2
	$\int \frac{\mathrm{d}u}{\sqrt{36-u^2}} = \sin^{-1}\frac{u}{6}$	M1		allow if $\mathrm{d}x$ is used instead of $\mathrm{d}u$
	limits −3 and 3 or substitute back to give $\sin^{-1}\frac{x-2}{6}$	A1		
	$\mathrm{I} = \frac{\pi}{6} + \frac{\pi}{6} = \frac{\pi}{3}$	A1	5	
	Total		**5**	

MFP2 (cont)

Q	Solution	Marks	Total	Comments
7(a)	Clear reason given	E1	1	Minimum O × E = E
(b)(i)	$(k+1)\left((k+1)^2+5\right)-k\left(k^2+5\right)$	M1		
	$=3k^2+3k+6$	A1		
	$k^2+k=k(k+1)=M(2)$	E1		Must be shown
	$\mathrm{f}(k+1)-\mathrm{f}(k)=M(6)$	E1	4	
(ii)	Assume true for $n=k$			
	$\mathrm{f}(k+1)-\mathrm{f}(k)=M(6)$	M1		Clear method
	$\therefore \mathrm{f}(k+1)=M(6)+\mathrm{f}(k)$			
	$=M(6)+M(6)$	A1		
	$=M(6)$			
	True for $n=1$	B1		
	$P(n)\to P(n+1)$ and P(1) true	E1	4	Provided all other marks earned in (b)(ii)
	Total		9	

MFP2 (cont)

Q	Solution	Marks	Total	Comments
8(a)(i)	$\left(z+\frac{1}{z}\right)\left(z-\frac{1}{z}\right)=z^2-\frac{1}{z^2}$	B1	1	
(ii)	$\left(z^2-\frac{1}{z^2}\right)^2\left(z+\frac{1}{z}\right)^2$			
	$=\left(z^4-2+\frac{1}{z^4}\right)\left(z^2+2+\frac{1}{z^2}\right)$	M1A1		Alternatives for M1A1: $\left(z^4+4z^2+6+\frac{4}{z^2}+\frac{1}{z^4}\right)\left(z^2-2+\frac{1}{z^2}\right)$ or $\left(z^3-\frac{1}{z^3}\right)^2-2\left(z^3-\frac{1}{z^3}\right)\left(z-\frac{1}{z}\right)+\left(z-\frac{1}{z}\right)^2$
	$=z^6+\frac{1}{z^6}+2\left(z^4+\frac{1}{z^4}\right)-\left(z^2+\frac{1}{z^2}\right)-4$	A1	3	CAO (not necessarily in this form)
(b)(i)	$z^n+\frac{1}{z^n}=\cos n\theta+\mathrm{i}\sin n\theta$	M1A1		
	$+\cos(-n\theta)+\mathrm{i}\sin(-n\theta)$			
	$=2\cos n\theta$	A1	3	AG SC: if solution is incomplete and $(\cos\theta+\mathrm{i}\sin\theta)^{-n}$ is written as $\cos n\theta-\mathrm{i}\sin n\theta$, award M1A0A1
(ii)	$z^n-z^{-n}=2\mathrm{i}\sin n\theta$	B1	1	
(c)	RHS = $2\cos 6\theta+4\cos 4\theta-2\cos 2\theta-4$	M1 A1F		ft incorrect values in (a)(ii) provided they are cosines
	LHS = $-64\cos^4\theta\sin^2\theta$	M1		
	$\cos^4\theta\sin^2\theta$			
	$=-\frac{1}{32}\cos 6\theta-\frac{1}{16}\cos 4\theta+\frac{1}{32}\cos 2\theta+\frac{1}{16}$	A1	4	
(d)	$-\frac{\sin 6\theta}{192}-\frac{\sin 4\theta}{64}+\frac{\sin 2\theta}{64}+\frac{\theta}{16}(+k)$	M1 A1F	2	ft incorrect coefficients but not letters A, B, C, D
	Total		**14**	
	TOTAL		**75**	

AQA – Further pure 2 – Jun 2008 – Answers

Question 1:

Exam report

$$a)\, 5\sinh x + \cosh x = \frac{5}{2}\left(e^{x} - e^{-x}\right) + \frac{1}{2}\left(e^{x} + e^{-x}\right)$$

$$= \frac{5}{2}e^{x} + \frac{1}{2}e^{x} - \frac{5}{2}e^{-x} + \frac{1}{2}e^{-x}$$

$$5\sinh x + \cosh x = 3e^{x} - 2e^{-x}$$

$$b)\ 5\sinh x + \cosh x + 5 = 0 \quad becomes$$

$$3e^{x} - 2e^{-x} + 5 = 0 \qquad (\times e^{x})$$

$$3e^{2x} - 2 + 5e^{x} = 0$$

$$3e^{2x} + 5e^{x} - 2 = 0$$

$$(3e^{x} - 1)(e^{x} + 2) = 0$$

$$e^{x} = \frac{1}{3} \ or \ e^{x} = -2\ (no\ \ solution)$$

$$x = \ln\left(\frac{1}{3}\right)$$

Question 2:

Exam report

$$\frac{A}{r(r+1)} + \frac{B}{(r+1)(r+2)} = \frac{A(r+2)+Br}{r(r+1)(r+2)} = \frac{(A+B)r+2A}{r(r+1)(r+2)}$$

$$This\ expression\ is\ equal\ to\ \frac{1}{r(r+1)(r+2)}\ for$$

$$A + B = 0 \quad and \quad 2A = 1$$

$$A = \frac{1}{2}\ and\ \ B = -\frac{1}{2}$$

$$b)\sum_{r=10}^{98}\frac{1}{r(r+1)(r+2)} = \sum_{r=10}^{98}\frac{1}{2r(r+1)} - \frac{1}{2(r+1)(r+2)}$$

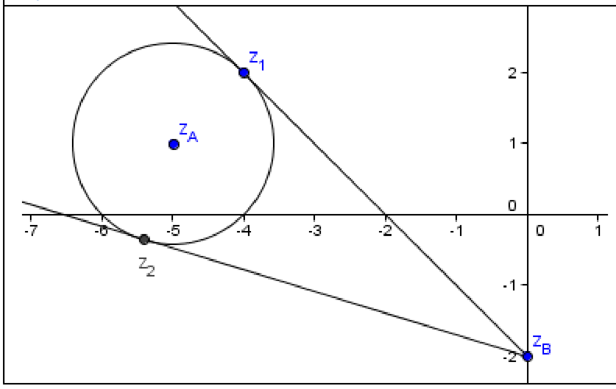
$$= \frac{1}{220} - \frac{1}{264} + \frac{1}{264} - \frac{1}{312} + \frac{1}{312} - \frac{1}{364} +$$

$$\ldots + \frac{1}{19012} - \frac{1}{19404} + \frac{1}{19404} - \frac{1}{19800}$$

All the terms cancel except $\frac{1}{220} - \frac{1}{19800} = \frac{89}{19800}$

Question 3:	**Exam report**
$z^3+qz+18-12i=0$ has roots α,β,γ $a)\ i)\ \alpha\beta\gamma=-18+12i$ $ii)\ \alpha+\beta+\gamma=0$ $b)\ \beta+\gamma=2$ $i)\ \alpha+\beta+\gamma=0$ $\alpha+2=0$ $\alpha=-2$ $ii)\ \alpha\beta\gamma=-18+12i$ $-2\beta\gamma=-18+12i$ $\beta\gamma=9-6i$ $iii)\ q=\alpha\beta+\alpha\gamma+\beta\gamma=\alpha(\beta+\gamma)+\beta\gamma$ $q=-2\times(2)+9-6i=5-6i$ $c)\ \beta=ki$ and it is a root of $z^3+qz+18-12i=0$ so $(ki)^3+(5-6i)\times(ki)+18-12i=0$ $-ik^3+5ki+6k+18-12i=0$ $(6k+18)+i(-k^3+5k-12)=0$ $6k+18=0$ and $-k^3+5k-12=0$ $k=-3$ and $-(-3)^3+5\times-3-12=27-15-12=27-27=0$ so $\alpha=-2,\beta=-3i\ ,\ \gamma=2-\beta=2+3i$	

Question 4:	**Exam report**
$a)\ \lvert z+5-i\rvert=\sqrt{2}$ Let $z_A=-5+i$ and $A(z_A)$ C is the circle centre A, radius $r=\sqrt{2}$ $b)\ \arg(z+2i)=\arg(-4+2i+2i)=\arg(-4+4i)$ $Tan^{-1}\left(\frac{4}{-4}\right)=Tan^{-1}(-1)=\frac{3\pi}{4}$ $z_1=-4+2i$ lies on L $c)\ i)\ \lvert -4+2i+5-i\rvert=\lvert 1+i\rvert=\sqrt{(1)^2+(1)^2}=\sqrt{2}$ z_1 lies on C $ii)\ L$ touches (is tangent to) C if L is perpendicular to the radius $\arg(z_1-z_A)=\arg(1+i)=\frac{\pi}{4}$ $\arg(z_1+2i)=\frac{3\pi}{4}$ $\arg(z_1+2i)-\arg(z_1-z_A)=\frac{3\pi}{4}-\frac{\pi}{4}=\frac{\pi}{2}$ L is perpendicular to the radius, L is tangent to the circle C.	

Question 4:continues	Exam report
 	

Question 5:	Exam report
$a)\ \cosh x=\frac{1}{2}\left(e^{x}+e^{-x}\right)$ $\cosh^2 x=\frac{1}{4}\left(e^{x}+e^{-x}\right)^2=\frac{1}{4}\left(e^{2x}+e^{-2x}+2\right)$ $\cosh^2 x=\frac{1}{2}\times\frac{1}{2}(e^{2x}+e^{-2x})+\frac{1}{2}$ $\cosh^2 x=\frac{1}{2}\cosh 2x+\frac{1}{2}$ $\cosh 2x=2\cosh^2 x-1$ $b)i)\ S=2\pi\int_0^{\ln a} y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=2\pi\int_0^{\ln a}\cosh x\times\sqrt{1+\mathrm{sech}^2 x}dx$ $S=2\pi\int_0^{\ln a}\cosh x\times\cosh x dx=2\pi\int_0^{\ln a}\cosh^2 x\ dx$ $ii)\ S=2\pi\int_0^{\ln a}\frac{1}{2}\cosh 2x+\frac{1}{2}dx=2\pi\left[\frac{1}{4}\sinh 2x+\frac{1}{2}x\right]_0^{\ln a}$ $=2\pi\left(\frac{1}{4}\sinh(2\ln a)+\frac{1}{2}\ln a-0\right)$ $S=\pi\left(\frac{1}{2}\sinh(\ln a^2)+\ln a\right)=\pi\left(\frac{1}{2}\times\frac{1}{2}\left(e^{\ln a^2}-e^{-\ln a^2}\right)+\ln a\right)$ $S=\pi\left(\frac{1}{4}\left(a^2-\frac{1}{a^2}\right)+\ln a\right)=\pi\left(\frac{a^4-1}{4a^2}+\ln a\right)$	

Question 6:	Exam report
$I=\int_{-1}^{5}\frac{dx}{\sqrt{32+4x-x^2}}$ $32+4x-x^2=-(x-2)^2+4+32$ $=36-(x-2)^2$ $I=\int_{-1}^{5}\frac{dx}{\sqrt{32+4x-x^2}}=\int_{-1}^{5}\frac{dx}{\sqrt{36-(x-2)^2}}$ $\quad u=x-2\ \ and\ \ dx=du$ $when\ x=-1, u=-3$ $when\ x=5, u=3$ $I=\int_{-3}^{3}\frac{du}{\sqrt{36-u^2}}=\left[Sin^{-1}\left(\frac{u}{6}\right)\right]_{-3}^{3}=Sin^{-1}\left(\frac{1}{2}\right)-Sin^{-1}\left(-\frac{1}{2}\right)=\frac{\pi}{6}+\frac{\pi}{6}=\frac{\pi}{3}$	

Question 7:

Exam report

$a)\, n\, and\, n+1\; are\; CONSECUTIVE\; numbers\; so\; one\; of\; them\; is\; EVEN$

$therefore\;\; n(n+1)\; is\; a\; multiple\; of\; 2$

$b)\, i)\, f(n) = n(n^2+5)$

$$f(k+1) - f(k) = (k+1)((k+1)^2+5) - k(k^2+5)$$
$$= (k+1)\left(k^2+2k+1+5\right) - k(k^2+5)$$
$$= (k+1)(k^2+2k+6) - k(k^2+5)$$
$$= k^3+2k^2+6k+k^2+2k+6-k^3-5k$$
$$= 3k^2+3k+6 = 3(k+1)(k+2)$$

$(k+1)(k+2)$ is a multiple of 2 as the product of two consecutive numbers

so $f(k+1) - f(k)$ is a multiple of 6

ii) Proposition P_n, $for\; all\; n \geq 1, f(n)$ is a multiple of 6 is to be proven by induction

base case: $n = 1$

$$f(1) = 1\times(1^2+5) = 6\; which\; is\; a\; multiple\; of\; 6.$$

$P_1\; is\; true.$

Let's suppose that for $n = k, f(k)$ is a multiple of 6

Let's sahow that $f(k+1)$ is then a multiple of 6.

$-------------------------$

$f(k+1) = 3(k+1)(k+2) - f(k)$

$3(k+1)(k+2)$ is a multiple of 6

$f(k)$ is a multiple of 6 (by hypothesis)

so $f(k+1)$ is a multiple of 6 (as the difference of two multiples of 6)

$-------------------------$

Conclusion:

If P_k is true then P_{k+1} is true. because P_1 is true,

we can conclude according to the induction principle

that the proposition is true for all $n \geq 1$.

Question 8:

Exam report

$$a)\, i)\left(z+\frac{1}{z}\right)\left(z-\frac{1}{z}\right) = z^2-1+1-\frac{1}{z^2} = z^2-\frac{1}{z^2}$$

$$ii)\left(z+\frac{1}{z}\right)^4\left(z-\frac{1}{z}\right)^2 = \left(z+\frac{1}{z}\right)^2 \times \left[\left(z+\frac{1}{z}\right)\left(z-\frac{1}{z}\right)\right]^2$$
$$= \left(z^2+\frac{1}{z^2}+2\right)\left(z^2-\frac{1}{z^2}\right)^2$$
$$= \left(z^2+\frac{1}{z^2}+2\right)\left(z^4+\frac{1}{z^4}-2\right)$$
$$= z^6+\frac{1}{z^2}-2z^2+z^2+\frac{1}{z^6}-\frac{2}{z^2}+2z^4+\frac{2}{z^4}-4$$
$$= \left(z^6-\frac{1}{z^6}\right)-\left(z^2+\frac{1}{z^2}\right)+2\left(z^4+\frac{1}{z^4}\right)-4$$

Question 8:continues	**Exam report**
$b)i)\ z^n+\dfrac{1}{z^n}=\left(\cos n\theta+i\sin n\theta\right)+\left(\cos n\theta-i\sin n\theta\right)=2\cos n\theta$ $ii)\ z^n-\dfrac{1}{z^n}=\left(\cos n\theta+i\sin n\theta\right)-\left(\cos n\theta-i\sin n\theta\right)=2i\sin n\theta$ $c)\cos^4\theta\sin^2\theta=\left(\dfrac{1}{2^4}\left(z+\dfrac{1}{z}\right)^4\right)\left(\dfrac{1}{(2i)^2}\left(z-\dfrac{1}{z}\right)^2\right)$ $=-\dfrac{1}{64}\left(z+\dfrac{1}{z}\right)^4\left(z-\dfrac{1}{z}\right)^2$ $=-\dfrac{1}{64}\left[\left(z^6-\dfrac{1}{z^6}\right)-\left(z^2+\dfrac{1}{z^2}\right)+2\left(z^4+\dfrac{1}{z^4}\right)-4\right]$ $=-\dfrac{1}{64}\left(2\cos 6\theta-2\cos 2\theta+4\cos 4\theta-4\right)$ $=-\dfrac{1}{32}\cos 6\theta-\dfrac{1}{16}\cos 4\theta+\dfrac{1}{32}\cos 2\theta+\dfrac{1}{16}$ $d)\int\cos^4\theta\sin^2\theta d\theta=\int-\dfrac{1}{32}\cos 6\theta-\dfrac{1}{16}\cos 4\theta+\dfrac{1}{32}\cos 2\theta+\dfrac{1}{16}d\theta$ $=-\dfrac{1}{192}\sin 6\theta-\dfrac{1}{64}\sin 4\theta+\dfrac{1}{64}\sin 2\theta+\dfrac{1}{16}\theta+c$	

Component Code	Component Title	Maximum Scaled Mark	Scaled Mark Grade Boundaries				
			A	B	C	D	E
MFP2	GCE MATHEMATICS UNIT FP2	75	58	51	44	37	30

General Certificate of Education
January 2009
Advanced Level Examination

MATHEMATICS **MFP2**
Unit Further Pure 2

Monday 19 January 2009 1.30 pm to 3.00 pm

For this paper you must have:
- an 8-page answer book
- the blue AQA booklet of formulae and statistical tables.

You may use a graphics calculator.

Time allowed: 1 hour 30 minutes

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Write the information required on the front of your answer book. The *Examining Body* for this paper is AQA. The *Paper Reference* is MFP2.
- Answer **all** questions.
- Show all necessary working; otherwise marks for method may be lost.

Information
- The maximum mark for this paper is 75.
- The marks for questions are shown in brackets.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.

Answer **all** questions.

---

1 (a) Use the definitions $\sinh\theta = \frac{1}{2}(e^{\theta} - e^{-\theta})$ and $\cosh\theta = \frac{1}{2}(e^{\theta} + e^{-\theta})$ to show that

$$1 + 2\sinh^2\theta = \cosh 2\theta$$ *(3 marks)*

(b) Solve the equation

$$3\cosh 2\theta = 2\sinh\theta + 11$$

giving each of your answers in the form $\ln p$. *(6 marks)*

2 (a) Indicate on an Argand diagram the region for which $|z - 4i| \leqslant 2$. *(4 marks)*

(b) The complex number z satisfies $|z - 4i| \leqslant 2$. Find the range of possible values of $\arg z$. *(4 marks)*

3 (a) Given that $f(r) = \frac{1}{4}r^2(r+1)^2$, show that

$$f(r) - f(r-1) = r^3$$ *(3 marks)*

(b) Use the method of differences to show that

$$\sum_{r=n}^{2n} r^3 = \frac{3}{4}n^2(n+1)(5n+1)$$ *(5 marks)*

4 It is given that α, β and γ satisfy the equations

$$\begin{aligned}\alpha + \beta + \gamma &= 1\\ \alpha^2 + \beta^2 + \gamma^2 &= -5\\ \alpha^3 + \beta^3 + \gamma^3 &= -23\end{aligned}$$

(a) Show that $\alpha\beta + \beta\gamma + \gamma\alpha = 3$. *(3 marks)*

(b) Use the identity

$$(\alpha + \beta + \gamma)(\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta - \beta\gamma - \gamma\alpha) = \alpha^3 + \beta^3 + \gamma^3 - 3\alpha\beta\gamma$$

to find the value of $\alpha\beta\gamma$. *(2 marks)*

(c) Write down a cubic equation, with integer coefficients, whose roots are α, β and γ. *(2 marks)*

(d) Explain why this cubic equation has two non-real roots. *(2 marks)*

(e) Given that α is real, find the values of α, β and γ. *(4 marks)*

5 (a) Given that $u = \cosh^2 x$, show that $\frac{\mathrm{d}u}{\mathrm{d}x} = \sinh 2x$. *(2 marks)*

(b) Hence show that

$$\int_0^1 \frac{\sinh 2x}{1 + \cosh^4 x}\,\mathrm{d}x = \tan^{-1}(\cosh^2 1) - \frac{\pi}{4}$$ *(5 marks)*

6 Prove by induction that

$$\frac{2 \times 1}{2 \times 3} + \frac{2^2 \times 2}{3 \times 4} + \frac{2^3 \times 3}{4 \times 5} + \ldots + \frac{2^n \times n}{(n+1)(n+2)} = \frac{2^{n+1}}{n+2} - 1$$

for all integers $n \geqslant 1$. *(7 marks)*

7 (a) Show that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\cosh^{-1}\frac{1}{x}\right) = \frac{-1}{x\sqrt{1 - x^2}}$$ *(3 marks)*

(b) A curve has equation

$$y = \sqrt{1 - x^2} - \cosh^{-1}\frac{1}{x} \qquad (0 < x < 1)$$

Show that:

(i) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sqrt{1 - x^2}}{x}$; *(4 marks)*

(ii) the length of the arc of the curve from the point where $x = \frac{1}{4}$ to the point where $x = \frac{3}{4}$ is $\ln 3$. *(5 marks)*

8 (a) Show that

$$(z^4 - \mathrm{e}^{\mathrm{i}\theta})(z^4 - \mathrm{e}^{-\mathrm{i}\theta}) = z^8 - 2z^4 \cos\theta + 1$$ *(2 marks)*

(b) Hence solve the equation

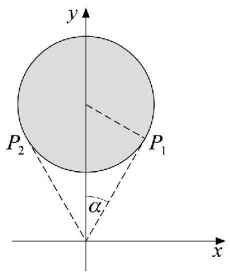
$$z^8 - z^4 + 1 = 0$$

giving your answers in the form $\mathrm{e}^{\mathrm{i}\phi}$, where $-\pi < \phi \leqslant \pi$. *(6 marks)*

(c) Indicate the roots on an Argand diagram. *(3 marks)*

END OF QUESTIONS

MFP2

Q	Solution	Marks	Total	Comments
1(a)	LHS = $1+\frac{1}{2}\left(e^{2\theta}-2+e^{-2\theta}\right)$	M1		Expansion of $\frac{1}{2}(e^{\theta}-e^{-\theta})^2$ correctly
		A1		Any form
	$=\frac{1}{2}\left(e^{2\theta}+e^{-2\theta}\right)=\cosh 2\theta$	A1	3	AG
(b)	$3+6\sinh^2\theta=2\sinh\theta+11$	M1		
	$3\sinh^2\theta-\sinh\theta-4=0$	A1		OE
	$(3\sinh\theta-4)(\sinh\theta+1)=0$	M1		Attempt to factorise or formula
	$\sinh\theta=\frac{4}{3}$ or -1	A1F		ft if factorises or real roots found
	$\theta=\ln 3$	A1F		
	$\theta=\ln\left(\sqrt{2}-1\right)$	A1F	6	
	Total		9	
2(a)	 	B1 B1 B1 B1F	 4	Circle Correct centre Correct radius Inside shading
(b)	Correct points P_1 and P_2 indicated	B1F		Possibly by tangents drawn ft mirror image of circle in x-axis
	$\sin\alpha=\frac{2}{4}$	M1		
	$\alpha=\frac{\pi}{6}$	A1		
	Range is $\frac{\pi}{3}\leqslant \arg z\leqslant\frac{2\pi}{3}$	A1	4	Deduct 1 for angles in degrees
	Total		8	

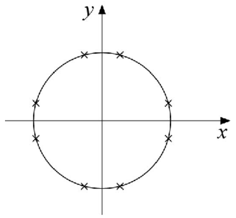
MFP2 (cont)

Q	Solution	Marks	Total	Comments
3(a)	$f(r)-f(r-1)$			
	$=\frac{1}{4}r^2(r+1)^2-\frac{1}{4}(r-1)^2r^2$	M1		
	$=\frac{1}{4}r^2\left(r^2+2r+1-r^2+2r-1\right)$	A1		Correct expansions of $(r+1)^2$ and $(r-1)^2$
	$=r^3$	A1	3	AG
(b)	$r=n:\ \ n^3=\frac{1}{4}n^2(n+1)^2-\frac{1}{4}(n-1)^2n^2$	M1 A1		For either $r=n$ or $r=2n$. PI
	$r=2n$:			
	$(2n)^3=\frac{1}{4}(2n)^2(2n+1)^2-\frac{1}{4}(2n-1)^2(2n)^2$	A1		
	$\sum_{r=n}^{2n}r^3=\frac{1}{4}.4n^2(2n+1)^2-\frac{1}{4}(n-1)^2n^2$	M1		
	$=\frac{3}{4}n^2(5n+1)(n+1)$	A1	5	AG **Alternatively** $\sum_{r=1}^{2n}r^3$ and $\sum_{r=1}^{n-1}r^3$ stated M1A1A1 (M1 for either) Difference M1 Answer A1
	Total		8	
4(a)	Use of $\left(\sum\alpha\right)^2=\sum\alpha^2+2\sum\alpha\beta$	M1		
	$1=-5+2\sum\alpha\beta$	A1		
	$\sum\alpha\beta=3$	A1	3	AG
(b)	$1(-5-3)=-23-3\alpha\beta\gamma$	M1		For use of identity
	$\alpha\beta\gamma=-5$	A1	2	
(c)	$z^3-z^2+3z+5=0$	M1 A1F	2	For correct signs and "= 0"
(d)	$\alpha^2+\beta^2+\gamma^2<0\Rightarrow$ non real roots	B1		
	Coefficients real $\therefore$ conjugate pair	B1	2	
(e)	$f(-1)=0\Rightarrow z+1$ is a factor	M1A1		
	$(z+1)\left(z^2-2z+5\right)=0$	A1		
	$z=-1,\ 1\pm 2i$	A1	4	
	Total		13	

MFP2 (cont)

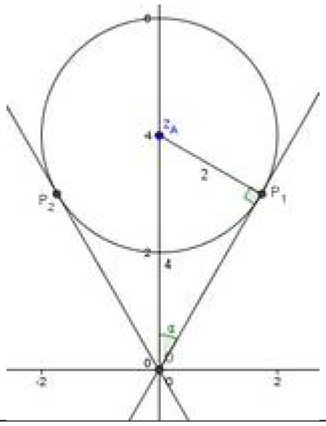
Q	Solution	Marks	Total	Comments
5(a)	$\frac{du}{dx} = 2\cosh x \sinh x$	M1		Any correct method
	$= \sinh 2x$	A1	2	AG
(b)	$I = \int_{x=0}^{x=1} \frac{du}{1+u^2}$	M1A1		Ignore limits here
	$= \left[\tan^{-1}u\right]_{x=0}^{x=1}$	A1		
	$= \left[\tan^{-1}\left(\cosh^2 x\right)\right]_0^1$	A1		Or A1 for change of limits
	$= \tan^{-1}\left(\cosh^2 1\right) - \tan^{-1}\left(\cosh^2 0\right)$			
	$= \tan^{-1}\left(\cosh^2 1\right) - \frac{\pi}{4}$	A1	5	AG
	Total		7	
6	Assume result true for $n - k$			
	Then $\sum_{r=1}^{k+1} \frac{2^r \times r}{(r+1)(r+2)}$			**SC** If no series at all indicated on LHS, deduct 1 and give E0 at end
	$= \frac{2^{k+1}}{k+2} + \frac{2^{k+1}(k+1)}{(k+2)(k+3)} - 1$	M1A1		
	$= \frac{2^{k+1}(k+3+k+1)}{(k+2)(k+3)} - 1$	M1		Putting over common denominator (not including the –1, unless separated later)
	$= \frac{2^{k+1}2(k+2)}{(k+2)(k+3)} - 1$	A1		
	$= \frac{2^{k+2}}{k+3} - 1$	A1		
	$k = 1$: LHS $= \frac{1}{3}$, RHS $= \frac{2^2}{3} - 1$	B1		
	$P_k \Rightarrow P_{k+1}$ and P_1 true	E1	7	Must be completely correct
	Total		7	

MFP2 (cont)

Q	Solution	Marks	Total	Comments
7(a)	$\frac{d}{dx}\left(\cosh^{-1}\frac{1}{x}\right) = \frac{1}{\sqrt{\frac{1}{x^2}-1}}\left(-\frac{1}{x^2}\right)$	M1A1		M0 if $\frac{dy}{dx} = f(y)$ and no attempt to substitute back to x
	$= \frac{-1}{x\sqrt{1-x^2}}$	A1	3	AG
(b)(i)	$\frac{d}{dx}\left(\sqrt{1-x^2}\right) = \frac{-2x}{2\sqrt{1-x^2}}$	B1 B1		For numerator For denominator (not $(1-x^2)^{\frac{1}{2}}$)
	$\frac{dy}{dx} = \frac{-x}{\sqrt{1-x^2}} + \frac{1}{x\sqrt{1-x^2}}$			
	$= \frac{1-x^2}{x\sqrt{1-x^2}} = \frac{\sqrt{1-x^2}}{x}$	M1		For attempt to put over a common denominator
		A1	4	AG
(ii)	$s = \int_{\frac{1}{4}}^{\frac{3}{4}} \sqrt{1+\frac{1-x^2}{x^2}}\,dx = \int_{\frac{1}{4}}^{\frac{3}{4}} \frac{1}{x}\,dx$	M1 A1A1		For use of $\sqrt{1+\left(\frac{dy}{dx}\right)^2}$
	$= [\ln x]_{\frac{1}{4}}^{\frac{3}{4}}$	M1		
	$= \ln\frac{3}{4} - \ln\frac{1}{4} = \ln 3$	A1	5	AG
	Total		**12**	
8(a)	Correct multiplication of brackets	M1		
	$e^{i\theta} + e^{-i\theta} = 2\cos\theta$	A1	2	Clearly shown
(b)	$2\cos\theta = 1$	M1		**SC** If 'hence' not used and, say, $z^8 - z^4 + 1 = 0$ is solved by formula, lose M1A1, but then continue M1m1 etc if $\frac{\pi}{3}$ is obtained
	$\theta = \frac{\pi}{3}$	A1		
	$z^4 = e^{\frac{\pi i}{3}}$ or $e^{\frac{-\pi i}{3}}$	M1		
	$z = e^{\frac{\pi i}{12} + \frac{2k\pi i}{4}}$ or $e^{\frac{-\pi i}{12} + \frac{2k\pi i}{4}}$	m1		
	$e^{\pm\frac{\pi i}{12}}, e^{\pm\frac{7\pi i}{12}}, e^{\pm\frac{5\pi i}{12}}, e^{\pm\frac{11\pi i}{12}}$	A2, 1, 0F	6	A1 if 3 roots correct
(c)		B2,1,0		B1 for 4 roots indicated correctly on a circle. CAO
	Indication that $r = 1$	B1	3	
	Total		**11**	
	TOTAL		**75**	

AQA – Further pure 2 – Jan 2009 – Answers

Question 1:	**Exam report**
$a)1+2\sinh^2\theta=1+2\times\frac{1}{4}\left(e^{\theta}-e^{-\theta}\right)^2=1+\frac{1}{2}\left(e^{2\theta}+e^{-2\theta}-2\right)$ $1+2\sinh^2\theta=\frac{1}{2}\left(e^{2\theta}+e^{-2\theta}\right)=\cosh 2\theta$ $b)3\cosh 2\theta=2\sinh\theta+11$ $3(1+2\sinh^2\theta)=2\sinh\theta+11$ $3+6\sinh^2\theta-2\sinh\theta-11=0$ $6\sinh^2\theta-2\sinh\theta-8=0$ $3\sinh^2\theta-\sinh\theta-4=0$ $(3\sinh\theta-4)(\sinh\theta+1)=0$ $\sinh\theta=\frac{4}{3}\ or\ \sinh\theta=-1$ $\theta=\sinh^{-1}\left(\frac{4}{3}\right)=\ln\left(\frac{4}{3}+\sqrt{1+\left(\frac{4}{3}\right)^2}\right)=\ln 3$ $or\,\theta=\sinh^{-1}(-1)=\ln\left(-1+\sqrt{1+(-1)^2}\right)=\ln\left(-1+\sqrt{2}\right)$	Part (a) was reasonably well done, although in a number of cases candidates quoted other relationships between coshθ and sinhθ and so were unable to gain credit. In part (b), two points are worthy of note: firstly some candidates thought that sinhθ = −1 had no solutions as a negative sign occurred, and secondly a substantial number of candidates solved sinhθ = $\frac{4}{3}$ and sinhθ = −1 by using the exponential form for sinhθ followed by the solving of a quadratic equation in e^{θ} instead of merely quoting the formula for $\sinh^{-1}$ (*x*) given on page 5 of the formulae booklet.

Question 2:	**Exam report**
$a)\lvert z-4i\rvert\le 2$ is the region inside the circle centre A(0,4) and radius $r=2$. $b)$ *Draw* the two tangents to the circle from the *originO.* We call the points of contact $\mathrm{P}_1(z_1)$ *and* $P_2(z_2)$. *Use* trig.properties to work out the argument of z_1 and z_2 : In the right-angle triangle $\mathrm{OAP}_1,\sin\alpha=\frac{opp}{hyp}=\frac{2}{4}=\frac{1}{2}$ $so\,\alpha=\sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{6}$ $\arg(z_1)=\frac{\pi}{2}-\frac{\pi}{6}=\frac{\pi}{3}\quad and\quad \arg(z_2)=\arg(z_1)+2\alpha=\frac{2\pi}{3}$ $\frac{\pi}{3}\le\arg(z)\le\frac{2\pi}{3}$  	Part (a) was well done apart from a few candidates who drew their circle at the mirror image of its correct position in the *x*-axis. In part (b), although candidates realised that the tangents needed to be drawn from the origin to the circle in order to find the possible value of arg *z*, few were able to manage the trigonometry involved to reach the correct range, and it was not uncommon to see $\tan^{-1}\left(\frac{4}{2}\right)$ appearing, suggesting that candidates thought that these points were in fact the points of intersection of the circle with the line through its centre parallel to the *x*-axis.

Question 3:	**Exam report**
$a)\ f(r)-f(r-1)=\frac{1}{4}r^2(r+1)^2-\frac{1}{4}(r-1)^2r^2$ $=\frac{1}{4}r^2\left[(r+1)^2-(r-1)^2\right]$ $=\frac{1}{4}r^2\left(r^2+2r+1-r^2+2r-1\right)$ $=\frac{1}{4}r^2\left(4r\right)$ $f(r)-f(r-1)=r^3$ $b)\sum_{r=n}^{2n}r^3=\sum_{r=n}^{2n}f(r)-f(r-1)=f(n)-f(n-1)+$ $f(n+1)-f(n)+$ $f(n+2)-f(n+1)+$ $....+$ $f(2n-1)-f(2n-2)+$ $f(2n)-f(2n-1)$ all the terms cancel except $f(2n)-f(n-1)$ $\sum_{r=n}^{2n}r^3=f(2n)-f(n-1)$ $=\frac{1}{4}(2n)^2\left(2n+1\right)^2-\frac{1}{4}\left(n-1\right)^2n^2$ $=\frac{1}{4}n^2\left[4(2n+1)^2-(n-1)^2\right]$ $=\frac{1}{4}n^2\left(16n^2+4+16n-n^2+2n-1\right)$ $=\frac{1}{4}n^2\left(15n^2+18n+3\right)=\frac{3}{4}n^2\left(5n^2+6n+1\right)$ $\sum_{r=n}^{2n}r^3=\frac{3}{4}n^2\left(5n+1\right)\left(n+1\right)$	There were many good and completely correct solutions to this question. Virtually all candidates completed part (a) correctly, and in part (b), if an error occurred, it was usually in the selection of an incorrect value for *r* at one end. For instance, $\sum_{r=n}^{2n}r^3$ was taken to be f (2*n*) – f (*n*) rather than f (2*n*) – f (*n* – 1) .

Question 4:	**Exam report**
$a)\left(\alpha+\beta+\gamma\right)^2=\alpha^2+\beta^2+\gamma^2+2\left(\alpha\beta+\alpha\gamma+\gamma\beta\right)$ $1^2=-5+2\left(\alpha\beta+\alpha\gamma+\gamma\beta\right)$ $so\ \ \alpha\beta+\alpha\gamma+\gamma\beta=3$ $b)\left(\alpha+\beta+\gamma\right)\left(\alpha^2+\beta^2+\gamma^2-\alpha\beta-\alpha\gamma-\gamma\beta\right)=\alpha^3+\beta^3+\gamma^3-3\alpha\gamma\beta$ $1\times\left(-5-3\right)=-23-3\alpha\beta\gamma$ $so\ \ \alpha\beta\gamma=-5$ $c)z^3-(\alpha+\beta+\gamma)z^2+\left(\alpha\beta+\alpha\gamma+\gamma\beta\right)z-\alpha\beta\gamma=0$ $z^3-z^2+3z+5=0$ $d)\alpha^2+\beta^2+\gamma^2=-5<0$ so at least one of the root is complex; And because the coefficients of the equation are REAL, its conjugate is also a root. $e)z^3-z^2+3z+5=0\ \ has\ \ an\ "obvious"root\ :\alpha=-1$ $indeed:(-1)^3-(-1)^2+3\times(-1)+5=-1-1-3+5=0$ Factorise the polynomial $(z+1)(z^2-2z+5)=0$ Discriminant of z^2-2z+5: $(-2)^2-4\times1\times5=-16=(4i)^2$ $\beta=\frac{2+4i}{2}\ \ and\ \gamma=\frac{2-4i}{2}$ $\alpha=-1,\ \beta=1+2i\ ,\ \gamma=1-2i$	Parts (a), (b) and (c) of this question were well done apart from odd sign errors here and there. However, there was much woolly thinking in part (d). For instance, it was not uncommon to see statements such as 'the cubic equation has real coefficients so it must have one real root and a conjugate pair of non-real roots' or 'since $\alpha^2+\beta^2+\gamma^2=-5$, two of α, β and γ must be nonreal'. Part (e) was poorly answered: it just did not seem to occur to candidates to use the factor theorem to find the real root. Instead they tried to use the symmetric relations between the roots in order to find them. This in turn led to heavy algebra with final abandonment.

Question 5:	**Exam report**
$a)u=\cosh^2 x,\ \ \frac{du}{dx}=2\sinh x\cosh x=\sinh 2x$ $du=\sinh 2x\ dx$ $when\ \ x=0,u=1$ $x=1,u=\cosh^2 1$ $I=\int_0^1\frac{\sinh 2x}{1+\cosh^4 x}dx=\int_1^{\cosh^2 1}\frac{du}{1+u^2}=\left[\tan^{-1}u\right]_1^{\cosh^2 1}$ $I=\tan^{-1}\left(\cosh^2 1\right)-\tan^{-1}(1)$ $I=\tan^{-1}\left(\cosh^2 1\right)-\frac{\pi}{4}$	Part (a) was usually correctly done. However, responses to part (b) were mixed. It was clear that not all candidates were familiar with $\int\frac{du}{1+u^2}$, even though it is given on page 8 of the formulae booklet. A significant number of candidates gave this integral as ln(1+ cosh^4 x).

Question 6:

The proposition P_n :

$$for\ n \geq 1,\ \frac{2\times 1}{2\times 3}+\frac{2^2\times 2}{3\times 4}+\frac{2^3\times 3}{4\times 5}+\ldots+\frac{2^n\times n}{(n+1)\times(n+2)}=\frac{2^{n+1}}{n+2}-1$$

is to be proven by induction

Base case: $n=1$

$$LHS:\frac{2\times 1}{2\times 3}=\frac{2}{6}=\frac{1}{3}\quad and\ \ RHS:\frac{2^{1+1}}{1+2}-1=\frac{4}{3}-1=\frac{1}{3}$$

$P_1\ is\ true$

Suppose that for $n=k$ the propsotion P_k is true.

Let's show that the proposition P_{k+1} is true

$$\text{ie let's show that }\frac{2\times 1}{2\times 3}+\frac{2^2\times 2}{3\times 4}+\frac{2^3\times 3}{4\times 5}+\ldots+\frac{2^{k+1}\times(k+1)}{(k+2)\times(k+3)}=\frac{2^{k+2}}{k+3}-1$$

– –

$$\frac{2\times 1}{2\times 3}+\frac{2^2\times 2}{3\times 4}+\frac{2^3\times 3}{4\times 5}+\ldots+\frac{2^{k+1}\times(k+1)}{(k+2)\times(k+3)}$$

$$=\underbrace{\frac{2\times 1}{2\times 3}+\frac{2^2\times 2}{3\times 4}+\frac{2^3\times 3}{4\times 5}+\ldots+\frac{2^k\times k}{(k+1)\times(k+2)}}+\frac{2^{k+1}\times(k+1)}{(k+2)\times(k+3)}$$

$$=\qquad\frac{2^{k+1}}{k+2}-1\qquad+\qquad\frac{2^{k+1}\times(k+1)}{(k+2)\times(k+3)}$$

$$=\frac{2^{k+1}(k+3)+2^{k+1}(k+1)}{(k+2)(k+3)}-1=\frac{2^{k+1}\left(k+3+k+1\right)}{(k+2)(k+3)}-1$$

$$=\frac{2^{k+1}\left(2k+4\right)}{(k+2)(k+3)}-1=\frac{2^{k+2}\left(k+2\right)}{(k+2)(k+3)}-1=\frac{2^{k+2}}{(k+3)}-1\qquad Q.E.D$$

– –

Conclusion :

If the proposition is true for $n=k$, then it is true for $n=k+1$.

Because it is true for $n=1$, according to the induction principal

we can conclude that is true true for all $\mathrm{n}\geq 1$.

Exam report

One thing which became evident in the marking of this question was that although candidates were able to perform the mechanics of proof by induction they did not really understand the theory behind it. In a significant number of solutions not one reference to a series or the use of the Σ symbol occurred. Solutions started 'assume result true for $n = k$' followed by $\frac{2^{k+1}}{n+2}+1+\frac{2^{k+1}(k+1)}{(k+2)(k+3)}$

which was duly shown to be $\frac{2^{k+2}}{k+3}-1$

Question 7:

$$a)\frac{d}{dx}\left(\cosh^{-1}\frac{1}{x}\right)=-\frac{1}{x^2}\times\frac{1}{\sqrt{\left(\frac{1}{x}\right)^2-1}}=-\frac{1}{x^2\sqrt{\frac{1-x^2}{x^2}}}\times\frac{\sqrt{x^2}}{\sqrt{x^2}}$$

$$=-\frac{x}{x^2\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\cosh^{-1}\frac{1}{x}\right)=-\frac{1}{x\sqrt{1-x^2}}$$

$$b)\ y=\sqrt{1-x^2}-\cosh^{-1}\frac{1}{x}\qquad(0<x<1)$$

$$i)\ \frac{dy}{dx}=\frac{1}{2}\times-2x\times\frac{1}{\sqrt{1-x^2}}+\frac{1}{x\sqrt{1-x^2}}=\frac{-x}{\sqrt{1-x^2}}+\frac{1}{x\sqrt{1-x^2}}$$

$$\frac{dy}{dx}=\frac{1-x^2}{x\sqrt{1-x^2}}=\frac{\sqrt{1-x^2}}{x}$$

$$ii)\ s=\int_{\frac{1}{4}}^{\frac{3}{4}}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\int_{\frac{1}{4}}^{\frac{3}{4}}\sqrt{1+\frac{1-x^2}{x^2}}\,dx=\int_{\frac{1}{4}}^{\frac{3}{4}}\sqrt{\frac{1}{x^2}}$$

$$s=\int_{\frac{1}{4}}^{\frac{3}{4}}\frac{1}{x}\,dx=\left[\ln x\right]_{\frac{1}{4}}^{\frac{3}{4}}=\ln\left(\frac{3}{4}\right)-\ln\left(\frac{1}{4}\right)=\ln(3)$$

Exam report

Few candidates were able to supply a correct proof of part (a). The derivative of $\cosh^{-1}\left(\frac{1}{x}\right)$ was almost invariably given as $\frac{1}{\sqrt{\left(\frac{1}{x}\right)^2-1}}$ followed by the correct answer after a small amount of spurious algebra.

Candidates fared better in part (b)(i), although it was a little surprising how many candidates were unable to differentiate $\sqrt{1-x^2}$ correctly. Those candidates who continued and attempted part (b)(ii) frequently produced a correct solution. Solutions to this part which went awry were those in which candidates wrote $\sqrt{1+\left(\frac{\sqrt{1-x^2}}{x}\right)^2}$ followed by $\sqrt{1+\frac{1-x}{x}}$ (which incidentally gave the correct answer!), or in some cases where candidates arrived at $\sqrt{\frac{1}{x^2}}$ or $\sqrt{x^{-2}}$ but were unable to take the square root correctly.

Question 8:

$$a)\left(z^4-e^{i\theta}\right)\left(z^4-e^{-i\theta}\right)=z^8-z^4e^{-i\theta}-z^4e^{i\theta}+1=z^8-z^4\left(e^{i\theta}+e^{-i\theta}\right)+1$$

$$=z^8-z^4\times2\cos\theta+1$$

$$\left(z^4-e^{i\theta}\right)\left(z^4-e^{-i\theta}\right)=z^8-2z^4\cos\theta+1$$

$$b)\ \text{for }\cos\theta=\frac{1}{2}\ \left(\theta=\frac{\pi}{3}\right),\ z^8-2z^4\cos\theta+1\ \textit{becomes}\ z^8-z^4+1=0$$

We can factorise as $\left(z^4-e^{i\frac{\pi}{3}}\right)\left(z^4-e^{-i\frac{\pi}{3}}\right)=0$

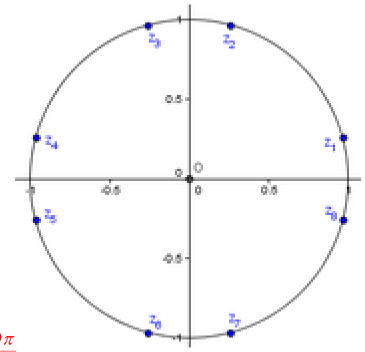
We need to solve $z^4=e^{\pm i\frac{\pi}{3}}$

$$(re^{i\phi})^4=e^{\pm i\frac{\pi}{3}}$$

$$r^4e^{4i\phi}=e^{\pm i\frac{\pi}{3}}$$

$$r=1\ \ or\ \ 4\phi=\pm\frac{\pi}{3}+k\times2\pi$$

$$\phi=\pm\frac{\pi}{12}+\frac{k\pi}{2}$$



This gives $z=e^{\pm i\frac{11\pi}{12}},\ e^{\pm i\frac{5\pi}{12}},\ e^{\pm i\frac{\pi}{12}},\ e^{\pm i\frac{9\pi}{12}}$

c)

Exam report

Part (a) was quite well done, although for some reason the multiplication of the brackets sometimes resulted in $z^8-2(e^{i\theta}+e^{-i\theta})z^4+1$ followed by the printed result. In part (b), a number of candidates lost some marks through attempting to use an ‘otherwise’ method instead of the ‘hence’ method as directed.

Those candidates arriving at $z^4=e^{i\frac{\pi}{3}}$ usually went on to solve the given equation correctly.

The Argand diagram, however, was poorly drawn. Frequently no circle was indicated and roots appeared at different distances from the origin, and in many cases candidates seemed to think that the eight roots were equally spaced round the origin

Component Code	Component Title	Maximum Scaled Mark	A	B	C	D	E
			Scaled Mark Grade Boundaries				
MFP2	GCE MATHEMATICS UNIT FP2	75	60	52	44	36	29

General Certificate of Education
June 2009
Advanced Level Examination

MATHEMATICS **MFP2**
Unit Further Pure 2

Friday 5 June 2009 1.30 pm to 3.00 pm

For this paper you must have:
- an 8-page answer book
- the blue AQA booklet of formulae and statistical tables.

You may use a graphics calculator.

Time allowed: 1 hour 30 minutes

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Write the information required on the front of your answer book. The *Examining Body* for this paper is AQA. The *Paper Reference* is MFP2.
- Answer **all** questions.
- Show all necessary working; otherwise marks for method may be lost.

Information
- The maximum mark for this paper is 75.
- The marks for questions are shown in brackets.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.

Answer **all** questions.

1 Given that $z = 2e^{\frac{\pi i}{12}}$ satisfies the equation

$$z^4 = a(1 + \sqrt{3}\,i)$$

where a is real:

(a) find the value of a; *(3 marks)*

(b) find the other three roots of this equation, giving your answers in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. *(5 marks)*

2 (a) Given that

$$\frac{1}{4r^2 - 1} = \frac{A}{2r - 1} + \frac{B}{2r + 1}$$

find the values of A and B. *(2 marks)*

(b) Use the method of differences to show that

$$\sum_{r=1}^{n} \frac{1}{4r^2 - 1} = \frac{n}{2n + 1}$$ *(3 marks)*

(c) Find the least value of n for which $\sum_{r=1}^{n} \frac{1}{4r^2 - 1}$ differs from 0.5 by less than 0.001. *(3 marks)*

3 The cubic equation

$$z^3 + pz^2 + 25z + q = 0$$

where p and q are real, has a root $\alpha = 2 - 3i$.

(a) Write down another non-real root, β, of this equation. *(1 mark)*

(b) Find:

(i) the value of $\alpha\beta$; *(1 mark)*

(ii) the third root, γ, of the equation; *(3 marks)*

(iii) the values of p and q. *(3 marks)*

4 (a) Sketch the graph of $y = \tanh x$. *(2 marks)*

(b) Given that $u = \tanh x$, use the definitions of $\sinh x$ and $\cosh x$ in terms of e^{x} and e^{-x} to show that

$$x = \tfrac{1}{2}\ln\left(\frac{1+u}{1-u}\right)$$ *(6 marks)*

(c) (i) Show that the equation

$$3\,\text{sech}^2 x + 7\tanh x = 5$$

can be written as

$$3\tanh^2 x - 7\tanh x + 2 = 0$$ *(2 marks)*

(ii) Show that the equation

$$3\tanh^2 x - 7\tanh x + 2 = 0$$

has only one solution for x.

Find this solution in the form $\tfrac{1}{2}\ln a$, where a is an integer. *(5 marks)*

5 (a) Prove by induction that, if n is a positive integer,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$ *(5 marks)*

(b) Hence, given that

$$z = \cos\theta + i\sin\theta$$

show that

$$z^n + \frac{1}{z^n} = 2\cos n\theta$$ *(3 marks)*

(c) Given further that $z + \frac{1}{z} = \sqrt{2}$, find the value of

$$z^{10} + \frac{1}{z^{10}}$$ *(4 marks)*

6 (a) Two points, A and B, on an Argand diagram are represented by the complex numbers $2 + 3i$ and $-4 - 5i$ respectively. Given that the points A and B are at the ends of a diameter of a circle C_1, express the equation of C_1 in the form $|z - z_0| = k$. *(4 marks)*

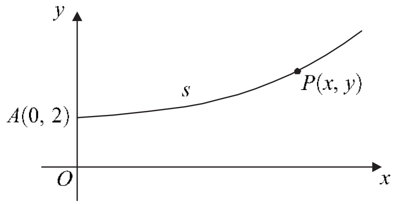
(b) A second circle, C_2, is represented on the Argand diagram by the equation $|z - 5 + 4i| = 4$. Sketch on one Argand diagram both C_1 and C_2. *(3 marks)*

(c) The points representing the complex numbers z_1 and z_2 lie on C_1 and C_2 respectively and are such that $|z_1 - z_2|$ has its maximum value. Find this maximum value, giving your answer in the form $a + b\sqrt{5}$. *(5 marks)*

7 The diagram shows a curve which starts from the point A with coordinates (0, 2). The curve is such that, at every point P on the curve,

$$\frac{dy}{dx} = \tfrac{1}{2}s$$

where s is the length of the arc AP.



(a) (i) Show that

$$\frac{ds}{dx} = \tfrac{1}{2}\sqrt{4 + s^2}$$ *(3 marks)*

(ii) Hence show that

$$s = 2\sinh\frac{x}{2}$$ *(4 marks)*

(iii) Hence find the cartesian equation of the curve. *(3 marks)*

(b) Show that

$$y^2 = 4 + s^2$$ *(2 marks)*

END OF QUESTIONS

Key to mark scheme and abbreviations used in marking

M	mark is for method
m or dM	mark is dependent on one or more M marks and is for method
A	mark is dependent on M or m marks and is for accuracy
B	mark is independent of M or m marks and is for method and accuracy
E	mark is for explanation

√or ft or F	follow through from previous incorrect result	MC	mis-copy
CAO	correct answer only	MR	mis-read
CSO	correct solution only	RA	required accuracy
AWFW	anything which falls within	FW	further work
AWRT	anything which rounds to	ISW	ignore subsequent work
ACF	any correct form	FIW	from incorrect work
AG	answer given	BOD	given benefit of doubt
SC	special case	WR	work replaced by candidate
OE	or equivalent	FB	formulae book
A2,1	2 or 1 (or 0) accuracy marks	NOS	not on scheme
$-x$ EE	deduct x marks for each error	G	graph
NMS	no method shown	c	candidate
PI	possibly implied	sf	significant figure(s)
SCA	substantially correct approach	dp	decimal place(s)

No Method Shown

Where the question specifically requires a particular method to be used, we must usually see evidence of use of this method for any marks to be awarded. However, there are situations in some units where part marks would be appropriate, particularly when similar techniques are involved. Your Principal Examiner will alert you to these and details will be provided on the mark scheme.

Where the answer can be reasonably obtained without showing working and it is very unlikely that the correct answer can be obtained by using an incorrect method, we must award **full marks**. However, the obvious penalty to candidates showing no working is that incorrect answers, however close, earn **no marks.**

Where a question asks the candidate to state or write down a result, no method need be shown for full marks.

Where the permitted calculator has functions which reasonably allow the solution of the question directly, the correct answer without working earns **full marks**, unless it is given to less than the degree of accuracy accepted in the mark scheme, when it gains **no marks.**

Otherwise we require evidence of a correct method for any marks to be awarded.

MFP2

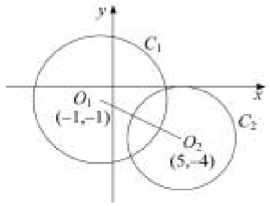
Q	Solution	Marks	Total	Comments
1(a)	$z^4 = 16e^{\frac{4\pi i}{12}}$	M1		Allow M1 if $z^4 = 2e^{\frac{4\pi i}{12}}$
	$= 16\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$	A1		OE could be $2ae^{\frac{\pi i}{3}}$ or $2a\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$
	$= 8+8\sqrt{3}i$; $a = 8$	A1F	3	ft errors in 2^4
(b)	For other roots, $r = 2$	B1		for realising roots are of form $2\times e^{i\theta}$
	$\theta = \frac{\pi}{12} + \frac{2k\pi}{4}$	M1A1		M1 for strictly correct θ i.e must be $\left(\text{their } \frac{\pi}{3} + 2k\pi\right)\times\frac{1}{4}$
	Roots are $2e^{\frac{7\pi i}{12}}$, $2e^{\frac{-5\pi i}{12}}$, $2e^{\frac{-11\pi i}{12}}$	A2,1, 0 F	5	ft error in $\frac{\pi}{12}$ or r $\left[\text{accept } 2e^{\left(\frac{\pi}{12}+\frac{2k\pi}{4}\right)i} \quad k = -1, -2, 1\right]$
	Total		**8**	
2(a)	$A = \frac{1}{2}$, $B = -\frac{1}{2}$	B1, B1F	2	For either A or B For the other
(b)	Method of differences clearly shown	M1		
	$\text{Sum} = \frac{1}{2}\left(1 - \frac{1}{2n+1}\right)$	A1		
	$= \frac{n}{2n+1}$	A1	3	AG
(c)	$\frac{1}{2(2n+1)} < 0.001$ or $\frac{n}{2n+1} > 0.499$	M1		Condone use of equals sign
	$1 < 0.004n + 0.002$ or $n > 0.998n + 0.499$			
	$n > \frac{0.998}{0.004}$ or $0.004n > 0.998$	A1		OE
	$n = 250$	A1F	3	ft if say 0.4999 used If method of trial and improvement used, award full marks for a completely correct solution showing working
	Total		**8**	

MFP2 (cont)

Q	Solution	Marks	Total	Comments
3(a)	$2+3\mathrm{i}$	B1	1	
(b)(i)	$\alpha\beta=13$	B1	1	
(ii)	$\alpha\beta+\beta\gamma+\gamma\alpha=25$	M1		M1A0 for -25 (no ft)
	$\gamma(\alpha+\beta)=12$	A1F		
	$\gamma=3$	A1F	3	ft error in $\alpha\beta$
(iii)	$p=-\sum\alpha=-7$	M1 A1F		M1 for a correct method for either p or q
	$q=-\alpha\beta\gamma=-39$	A1F	3	ft from previous errors p and q must be real for sign errors in p and q allow M1 but A0
	Alternative for (b)(ii) and (iii):			
(ii)	Attempt at $(z-2+3\mathrm{i})(z-2-3\mathrm{i})$	(M1)		
	$z^2-4z+13$	(A1)		
	cubic is $(z^2-4z+13)(z-3)\therefore\gamma=3$	(A1)	(3)	
(iii)	Multiply out or pick out coefficients	(M1)		
	$p=-7,\ q=-39$	(A1, A1)	(3)	
	Total		**8**	
4(a)	Sketch, approximately correct shape	B1		B0 if curve touches asymptotes
	Asymptotes at $y=\pm1$	B1	2	lines of answer booklet could be used for asymptotes be strict with sketch
(b)	Use of $u=\dfrac{\sinh x}{\cosh x}$	M1		
	$=\dfrac{\mathrm{e}^x-\mathrm{e}^{-x}}{\mathrm{e}^x+\mathrm{e}^{-x}}$ or $\dfrac{\mathrm{e}^{2x}-1}{\mathrm{e}^{2x}+1}$	A1		
	$u\left(\mathrm{e}^x+\mathrm{e}^{-x}\right)=\mathrm{e}^x-\mathrm{e}^{-x}$	M1		M1 for multiplying up
	$\mathrm{e}^{-x}(1+u)=\mathrm{e}^x(1-u)$	A1		A1 for factorizing out e's or M1 for attempt at $1+u$ and $1-u$ in terms of e^x
	$\mathrm{e}^{2x}=\dfrac{1+u}{1-u}$	m1		
	$x=\dfrac{1}{2}\ln\left(\dfrac{1+u}{1-u}\right)$	A1	6	AG

Q	Solution	Marks	Total	Comments
4(c)(i)	Use of $\tanh^2x=1-\operatorname{sech}^2x$	M1		
	Printed answer	A1	2	
(ii)	$(3\tanh x-1)(\tanh x-2)=0$	M1		Attempt to factorise
	$\tanh x\neq2$	E1		Accept tanh $x\neq2$ written down but not ignored or just crossed out
	$\tanh x=\dfrac{1}{3}$	A1		
	$x=\dfrac{1}{2}\ln 2$	M1 A1F	5	ft
	Total		**15**	
5(a)	$(\cos\theta+\mathrm{i}\sin\theta)^{k+1}=$			
	$(\cos k\theta+\mathrm{i}\sin k\theta)(\cos\theta+\mathrm{i}\sin\theta)$	M1		
	Multiply out	A1		Any form
	$=\cos(k+1)\theta+\mathrm{i}\sin(k+1)\theta$	A1		Clearly shown
	True for $n=1$ shown	B1		
	$P(k)\Rightarrow P(k+1)$ and $P(1)$ true	E1	5	provided previous 4 marks earned
(b)	$\dfrac{1}{z^n}=\dfrac{1}{\cos n\theta+\mathrm{i}\sin n\theta}=\cos n\theta-\mathrm{i}\sin n\theta$	M1A1		or $z^{-n}=\cos(-n\theta)+\mathrm{i}\sin(-n\theta)$ SC $(\cos\theta+\mathrm{i}\sin\theta)^{-n}$ quoted as $\cos n\theta-\mathrm{i}\sin n\theta$ earns M1A1 only
	$z^n+\dfrac{1}{z^n}=2\cos n\theta$	A1	3	AG
(c)	$z+\dfrac{1}{z}=\sqrt{2}$			
	$2\cos\theta=\sqrt{2}$	M1		
	$\theta=\dfrac{\pi}{4}$	A1		
	$z^{10}+\dfrac{1}{z^{10}}=2\cos\left(\dfrac{10\pi}{4}\right)$	M1		M0 for merely writing $z^{10}+\dfrac{1}{z^{10}}=2\cos10\theta$
	$=0$	A1F	4	
	Total		**12**	

MFP2 (cont)

Q	Solution	Marks	Total	Comments
6(a)	Centre $-1-\mathrm{i}$ or $(-1,-1)$	B1		
	Radius 5	M1 A1F		ft incorrect centre if used
	$\lvert z+1+\mathrm{i}\rvert=5$ or $\lvert z-(-1-i)\rvert=5$	A1F	4	ft $\lvert z+1+\mathrm{i}\rvert=10$ earns M0B1
(b)				
	C_1 correct centre, correct radius	B1F		ft errors in (a) but fit circles need to intersect and C_1 enclose $(0,0)$
	C_2 correct centre, correct radius	B1		
	Touching x-axis	B1F	3	error in plotting centre
(c)	$O_1O_2=3\sqrt{5}$	M1A1		allow if circles misplaced but O_1O_2 is still $3\sqrt{5}$
	Correct length identified	m1		
	Length is $9+3\sqrt{5}$	M1 A1F	5	ft if r is taken as 10
	Total		**12**	

MFP2 (cont)

Q	Solution	Marks	Total	Comments
7(a)(i)	$\frac{\mathrm{d}s}{\mathrm{d}x}=\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}=\sqrt{1+\left(\frac{s}{2}\right)^2}$	M1A1		Allow M1 for $s=\int\sqrt{1+\left(\frac{s}{2}\right)^2}\,\mathrm{d}x$ then A1 for $\frac{\mathrm{d}y}{\mathrm{d}x}$
	$=\frac{1}{2}\sqrt{4+s^2}$	A1	3	AG
(ii)	$\int\frac{\mathrm{d}s}{\sqrt{4+s^2}}=\int\frac{1}{2}\,\mathrm{d}x$	M1		For separation of variables; allow without integral sign
	$\sinh^{-1}\frac{s}{2}=\frac{1}{2}x+C$	A1		Allow if C is missing
	$C=0$	A1		
	$s=2\sinh\frac{1}{2}x$	A1	4	AG if C not mentioned allow $\frac{3}{4}$ **SC** incomplete proof of (a)(ii), differentiating $s=2\sinh\frac{x}{2}$ to arrive at $\frac{\mathrm{d}s}{\mathrm{d}x}=\frac{1}{2}\sqrt{4+s^2}$ allow M1A1 only $\left(\frac{2}{4}\right)$
(iii)	$\frac{\mathrm{d}y}{\mathrm{d}x}=\sinh\frac{1}{2}x$	M1		
	$y=2\cosh\frac{1}{2}x+C$	A1		Allow if C is missing
	$C=0$	A1	3	Must be shown to be zero and CAO
(b)	$y^2=4\left(1+\sinh^2\frac{x}{2}\right)$	M1		Use of $\cosh^2=1+\sinh^2$
	$=4+s^2$	A1	2	AG
	Total		**12**	
	TOTAL		**75**	

AQA – Further pure 2 – Jun 2009 – Answers

Question 1:	Exam report
a) $z=2e^{i\frac{\pi}{12}}$ so $z^4=\left(2e^{i\frac{\pi}{12}}\right)^4=16e^{i\frac{\pi}{3}}$ $z^4=16\left(\cos\frac{\pi}{3}+iSin\frac{\pi}{3}\right)=16\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)$ $z^4=8(1+i\sqrt{3})$ $\quad a=8$ b) let's write $z=re^{i\theta}$, $z^4=r^4e^{i4\theta}$ z is a solution of this equation when $r^4=16$ and $4\theta=\frac{\pi}{3}+k2\pi$ $r=2$ and $\theta=\frac{\pi}{12}+k\frac{\pi}{2}$ $\quad k=-2,-1,0,1$ Solutions are: $2e^{i\frac{\pi}{12}}$, $2e^{-i\frac{5\pi}{12}}$, $2e^{-i\frac{11\pi}{12}}$,$2e^{i\frac{7\pi}{12}}$	This question proved to be slightly more demanding for candidates than had been anticipated. The main difficulty in part (a) was that, having written down $2^4e^{i\frac{\pi}{3}}$ (or frequently incorrectly $2e^{i\frac{\pi}{3}}$), candidates were unsure about how to proceed, and they either abandoned this part of the question at that point or then tried to manipulate $a(1+i\sqrt{3})$ with little success.

Question 2:	Exam report
a) $\frac{A}{2r-1}+\frac{B}{2r+1}=\frac{A(2r-1)+B(2r+1)}{(2r+1)(2r-1)}$ $=\frac{(2A+2B)r+A-B}{4r^2-1}$ this is equal to $\frac{1}{4r^2-1}$ when $A-B=1$ and $2A+2B=0$ $A=\frac{1}{2}$ and $B=-\frac{1}{2}$ b) $\sum_{r=1}^{n}\frac{1}{4r^2-1}=\sum_{r=1}^{n}\frac{1}{2(2r-1)}-\frac{1}{2(2r+1)}=\frac{1}{2}-\frac{1}{6}+\frac{1}{6}-\frac{1}{10}+$ $\frac{1}{10}-\frac{1}{14}+\ldots+$ $\frac{1}{4n-6}-\frac{1}{4n-2}+$ $\frac{1}{4n-2}-\frac{1}{4n+2}$ All the terms cancel except $\frac{1}{2}-\frac{1}{4n+2}$ $\sum_{r=1}^{n}\frac{1}{4r^2-1}=\frac{1}{2}-\frac{1}{4n+2}=\frac{4n+2-2}{2(4n+2)}=\frac{4n}{4(2n+1)}=\frac{n}{2n+1}$ c) $\frac{1}{2}-\sum_{r=1}^{n}\frac{1}{4r^2-1}<0.001$ $\frac{1}{2}-\left(\frac{1}{2}-\frac{1}{4n+2}\right)<0.001$ $\qquad \frac{1}{4n+2}<0.001$ $4n+2>\frac{1}{0.001}$ $\qquad 4n>998$ $n>249.5$ $\qquad n=250$	Almost all candidates produced correct solutions to parts (a) and (b), apart from the odd arithmetical slip. There was, however, less success with part (c). Few candidates worked with inequalities (although the use of the equals sign was condoned) and the lack of ability to solve an equation in *n* with decimals involved led to the solutions for *n* which common sense should have told candidates was impossible. It was not infrequent to see *n* as a decimal less than unity and, even when candidates, using equalities, arrived at 249.5, they left it as their final answer, not considering that *n* had to be integral.

Question 3:	**Exam report**
$z^3+pz^2+25z+q=0$ *has roots* α,β,γ p *and* q *are real numbers.* $a)\alpha=2-3i.$ Because the coefficients of the equation are REAL numbers, α^* is also a root : $\beta=2+3i$ $b)i)\alpha\beta=(2-3i)(2+3i)=4+9=13$ $ii)\alpha\beta+\alpha\gamma+\beta\gamma=25$ $\alpha\beta+\gamma(\alpha+\beta)=25$ $13+\gamma\times4=25 \qquad \gamma=3$ $iii)\alpha\beta\gamma=-q=13\times3=39 \qquad q=-39$ $\alpha+\beta+\gamma=-p=4+3=7 \qquad p=-7$	Responses to this question were good, and the vast majority of candidates produced a completely correct solution. If errors did occur they were usually arithmetic, although occasionally *p* and *q* were given as Σα and αβγ respectively with no consideration being given to their sign.

Question 4:	**Exam report**
$a)\ y=\tanh x$ $b)u=\tanh x=\dfrac{\sinh x}{\cosh x}=\dfrac{\frac{1}{2}(e^x-e^{-x})}{\frac{1}{2}(e^x+e^{-x})}=\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$ Factorise num. and den. by e^{-x} $u=\tanh x=\dfrac{e^{-x}(e^{2x}-1)}{e^{-x}(e^{2x}+1)}=\dfrac{e^{2x}-1}{e^{2x}+1}=u$ Now make e^{2x} the subject of the expression $e^{2x}-1=u(e^{2x}+1)$ $e^{2x}-ue^{2x}=u+1$ $e^{2x}(1-u)=1+u$ $e^{2x}=\dfrac{1+u}{1-u} \qquad so\ 2x=\ln\left(\dfrac{1+u}{1-u}\right)$ $x=\dfrac{1}{2}\ln\left(\dfrac{1+u}{1-u}\right)$ $c)i)3\operatorname{sech}^2x+7\tanh x=5 \qquad \operatorname{sech}^2x=\dfrac{1}{\cosh^2 x}=1-\tanh^2 x$ $3(1-\tanh^2 x)+7\tanh x=5$ $3-3\tanh^2 x+7\tanh x-5=0$ $3\tanh^2 x-7\tanh x+2=0$ $ii)3\tanh^2 x-7\tanh x+2=0$ $(3\tanh x-1)(\tanh x-2)=0$ $\tanh x=\dfrac{1}{3}\ or\ \tanh x=2\ (no\ solution\ for\ all\ x,-1\le\tanh x\le1)$ $x=\dfrac{1}{2}\ln\left(\dfrac{1+\frac{1}{3}}{1-\frac{1}{3}}\right)=x=\dfrac{1}{2}\ln(2)$	Sketches were poor in part (a). Sometimes asymptotes were not drawn and even when they were sketches crossed or mingled with their asymptotes. It was not uncommon to see $\frac{\pi}{2}$ or π on candidates' diagrams showing some confusion with the graph of *y* = tan *x* . In part (b), provided that candidates knew what to do when they reached $u=\dfrac{e^{2x}-1}{e^{2x}+1}$, they almost always went on to complete this part correctly, but a substantial number of solutions petered out at this point. Part (c) was well done apart from the rejection of tanh *x* = 2 where lack of adequate reasoning for its rejection was often apparent.

Question 5:

$a)$ The proposition P_n : for $n \ge 0, (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ is to be proven by induction

base case: $n = 0$

$LHS: \ (\cos\theta + i\sin\theta)^0 = 1$

$RHS: \cos 0 + i\sin 0 = 1$

P_1 *is true*

Let's suppose that for $n = k$ the proposition is true

Let's show that the proposition is true for $n = k+1$

$i.e$, Let's show that $(\cos\theta + i\sin\theta)^{k+1} = \cos(k+1)\theta + i\sin(k+1)\theta$

– –

$$(\cos\theta + i\sin\theta)^{k+1} = (\cos\theta + i\sin\theta)^k \times (\cos\theta + i\sin\theta)$$
$$= (\cos k\theta + i\sin k\theta) \times (\cos\theta + i\sin\theta)$$
$$= (\cos k\theta\cos\theta - \sin k\theta\sin\theta) + i(\sin k\theta\cos\theta + \cos k\theta\sin\theta)$$

Using the formulae $\cos(A+B) = \cos A\cos B - \sin A\sin B$

and $\sin(A+B) = \sin A\cos B + \cos A\sin B$

$$(\cos\theta + i\sin\theta)^{k+1} = \cos(k\theta + \theta) + i\sin\left(k\theta + \theta\right)$$
$$= Cos(k+1)\theta + i\sin(k+1)\theta \ \ Q.E.D$$

– –

Conclusion:

If the proposition is true for n=k, then it is true for n=k+1.

Because it is true for n=0, we can conclude, according to the induction principal that it is true for all $n \ge 0$.

for all $n \ge 0$, $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

$b)\ z = \cos\theta + i\sin\theta \qquad so\ \ z^n = \cos n\theta + i\sin n\theta$

$$\frac{1}{z} = \cos(-\theta) + i\sin(-\theta) \qquad so\ \frac{1}{z^n} = \frac{1}{\cos(n\theta) + i\sin(n\theta)} = \frac{\cos(n\theta) - i\sin(n\theta)}{\cos^2(n\theta) + \sin^2(n\theta)}$$
$$= \cos(n\theta) - i\sin(n\theta)$$

$$z^n + \frac{1}{z^n} = 2\cos n\theta$$

$$c)\ z + \frac{1}{z} = \sqrt{2} = 2 \times \frac{\sqrt{2}}{2} = 2\cos\theta \quad with\ \ \theta = \frac{\pi}{4}$$

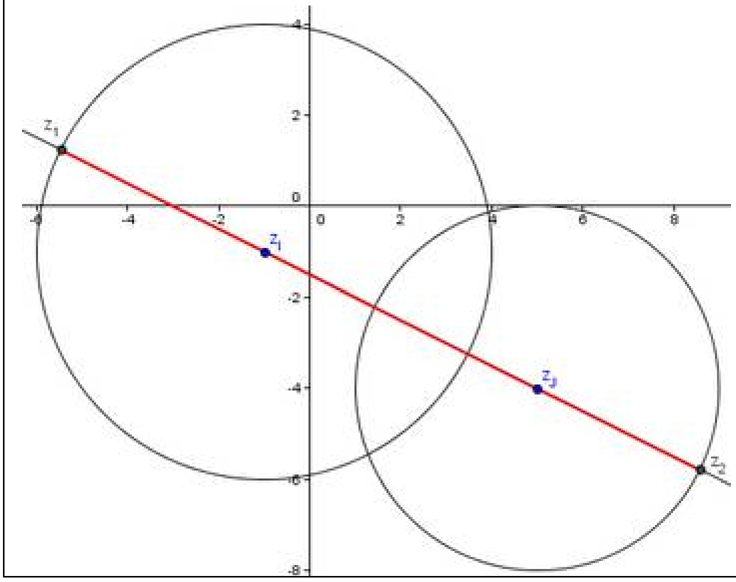
$$so\ \ z^n + \frac{1}{z^n} = 2\cos 10 \times \frac{\pi}{4} = 2\cos\frac{5\pi}{2} = 2\cos\frac{\pi}{2} = 0$$

Exam report

The general rules applicable to proof by induction in part (a) were usually understood, but because candidates realised that the product of cosθ + i sinθ with cos *k*θ +i sin *k*θ had to result in cos (*k* +1)θ + i sin (*k* +1)θ , many lost marks through omitting some of the intermediate steps.

In part (b), many candidates lost a mark by assuming that $(\cos\theta\ i\sin\theta)^{-n}$ was equal to cos *n*θ – i sin *n*θ without any justification.

Part (c) was almost invariably correctly done.

Question 6:	**Exam report**
a) $A(z_A)$ with $z_A = 2+3i$ $B(z_B)$ with $z_B = -4-5i$ • The mid point of AB, I(z_I) with $z_I = \frac{1}{2}(z_A + z_B) = -1-i$ • the radius of the circle is $\|z_I - z_A\|$ $r_1 = \|-1-i-2-3i\| = \|-3-4i\| = \sqrt{9+16} = 5$ $C_1 : \|z - z_I\| = r_1$ $\|z+1+i\| = 5$ b) $C_2 : \|z-5+4i\| = 4$ is the circle centre J(z_J) with $z_J = 5-4i$ and radius $r_2 = 4$ c) Draw the line IJ, joining the centres. This line crosses the circles at M_1 and M_2 $M_1M_2 = IJ + r_1 + r_2$ $= \|z_I - z_J\| + 4 + 5 = \|-1-i-5+4i\| + 9$ $= \|-6+3i\| + 9 = \sqrt{36+9} + 9$ $M_1M_2 = 9 + 3\sqrt{5}$  	The coordinates of the centre of the circle in part (a) were usually obtained, but the notation was often poor and it was not uncommon to see the centre of the circle C_1 written as (−1,−i) and, on the diagram, the scale on the *y*-axis written as i, 2i, 3i and so on. Also, radius and diameter were commonly confused. Sketches in part (b) varied considerably, the best being those who used compasses for their circles. These were generally readable with centre and radius indicated. However, some candidates chose to draw their circles by plotting points and joining up by freehand. These sketches turned out to be very poor. The circle C_2 was sometimes mistakenly drawn in the incorrect quadrant through choice of centre as (–5, 4) whilst others either failed to realise that the circle C_2 touched the *x*-axis or drew a circle touching both axes. In part (c), although many candidates placed z_1 and z_2 on their diagram in the approximately correct positions, not all realised that these points were at the intersections of C_1 and C_2 and the line O_1O_2, and even when they did, the finding of the length O_1O_2 proved to be beyond many.

Question 7:	**Exam report**
$a)\,i)\ \frac{ds}{dx}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\sqrt{1+\left(\frac{s}{2}\right)^2}=\sqrt{\frac{4+s^2}{4}}$ $\frac{ds}{dx}=\frac{1}{2}\sqrt{4+s^2}$ $ii)\ \frac{1}{\sqrt{4+s^2}}\frac{ds}{dx}=\frac{1}{2}$ $\int\frac{1}{\sqrt{4+s^2}}ds=\int\frac{1}{2}dx$ $\sinh^{-1}\left(\frac{s}{2}\right)=\frac{1}{2}x+c$ *When* $x=0, s=0$ *so* $c=0$ $\frac{s}{2}=\sinh\frac{x}{2}$ $\qquad s=2\sinh\frac{x}{2}$ $iii)\ \frac{dy}{dx}=\frac{1}{2}s=\sinh\frac{x}{2}$ $y=2\cosh\frac{x}{2}+k$ $A(0,2)$ belongs to the curve so $2=2\cosh 0+k \qquad k=0$ $y=2\cosh\frac{x}{2}$ $b)\ y^2=4\cosh^2\frac{x}{2}=4(1+\sinh^2\frac{x}{2})$ *from a)ii) we know that* $\sinh\frac{x}{2}=\frac{s}{2}$ $y^2=4(1+\frac{s^2}{4})$ $y^2=4+s^2$	Responses to part (a)(i) of this question were reasonable, although candidates starting with $s=\int\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx$ tended to flounder. Part (a)(ii) was very poorly attempted with the majority of candidates failing to realise that the way forward was to separate the variables in order to integrate. It was very common to see attempts at $\int\sqrt{4+s^2}dx$ treated as if it were $\int\sqrt{4+s^2}ds$. Of the few that did manage to separate the variables, virtually no one considered the boundary conditions but merely assumed that the constant of integration was zero. Candidates were more successful with part (a)(iii) and, although the constant of integration was omitted in many cases, more candidates considered the boundary conditions than in part (a)(ii). Those candidates who managed part (a)(iii) usually went on to work part (b) correctly.

Grade boundaries

Component Code	Component Title	Maximum Scaled Mark	Scaled Mark Grade Boundaries A	B	C	D	E
MFP2	MATHEMATICS UNIT MFP2	75	59	51	43	36	29

January 2010

Answer **all** questions.

1 (a) Use the definitions $\cosh x = \frac{1}{2}(\mathrm{e}^x + \mathrm{e}^{-x})$ and $\sinh x = \frac{1}{2}(\mathrm{e}^x - \mathrm{e}^{-x})$ to show that

$$\cosh^2 x - \sinh^2 x = 1$$ *(3 marks)*

(b) (i) Express

$$5\cosh^2 x + 3\sinh^2 x$$

in terms of $\cosh x$. *(1 mark)*

(ii) Sketch the curve $y = \cosh x$. *(1 mark)*

(iii) Hence solve the equation

$$5\cosh^2 x + 3\sinh^2 x = 9.5$$

giving your answers in logarithmic form. *(4 marks)*

2 (a) On the same Argand diagram, draw:

(i) the locus of points satisfying $|z - 4 + 2\mathrm{i}| = 4$; *(3 marks)*

(ii) the locus of points satisfying $|z| = |z - 2\mathrm{i}|$. *(3 marks)*

(b) Indicate on your sketch the set of points satisfying both

$$|z - 4 + 2\mathrm{i}| \leqslant 4$$

and $$|z| \geqslant |z - 2\mathrm{i}|$$ *(2 marks)*

3 The cubic equation

$$2z^3 + pz^2 + qz + 16 = 0$$

where p and q are real, has roots α, β and γ.

It is given that $\alpha = 2 + 2\sqrt{3}\,\mathrm{i}$.

(a) (i) Write down another root, β, of the equation. *(1 mark)*

(ii) Find the third root, γ. *(3 marks)*

(iii) Find the values of p and q. *(3 marks)*

(b) (i) Express α in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < 0 \leqslant \pi$. *(2 marks)*

(ii) Show that

$$(2 + 2\sqrt{3}\,\mathrm{i})^n = 4^n\left(\cos\frac{n\pi}{3} + \mathrm{i}\sin\frac{n\pi}{3}\right)$$ *(2 marks)*

(iii) Show that

$$\alpha^n + \beta^n + \gamma^n = 2^{2n+1}\cos\frac{n\pi}{3} + \left(-\frac{1}{2}\right)^n$$

where n is an integer. *(3 marks)*

4 A curve C is given parametrically by the equations

$$x = \tfrac{1}{2}\cosh 2t, \qquad y = 2\sinh t$$

(a) Express

$$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2$$

in terms of $\cosh t$. *(6 marks)*

(b) The arc of C from $t = 0$ to $t = 1$ is rotated through 2π radians about the x-axis.

(i) Show that S, the area of the curved surface generated, is given by

$$S = 8\pi\int_0^1 \sinh t\cosh^2 t\,\mathrm{d}t$$ *(2 marks)*

(ii) Find the exact value of S. *(2 marks)*

5 The sum to r terms, S_r, of a series is given by

$$S_r = r^2(r+1)(r+2)$$

Given that u_r is the rth term of the series whose sum is S_r, show that:

(a) (i) $u_1 = 6$; *(1 mark)*

(ii) $u_2 = 42$; *(1 mark)*

(iii) $u_n = n(n+1)(4n-1)$. *(3 marks)*

(b) Show that

$$\sum_{r=n+1}^{2n} u_r = 3n^2(n+1)(5n+2)$$ *(3 marks)*

6 (a) Show that the substitution $t = \tan\theta$ transforms the integral

$$\int \frac{d\theta}{9\cos^2\theta + \sin^2\theta}$$

into $$\int \frac{dt}{9+t^2}$$ *(3 marks)*

(b) Hence show that

$$\int_0^{\frac{\pi}{3}} \frac{d\theta}{9\cos^2\theta + \sin^2\theta} = \frac{\pi}{18}$$ *(3 marks)*

7 The sequence u_1, u_2, u_3,... is defined by

$$u_1 = 2\,, \quad u_{k+1} = 2u_k + 1$$

(a) Prove by induction that, for all $n \geqslant 1$,

$$u_n = 3 \times 2^{n-1} - 1$$ *(5 marks)*

(b) Show that

$$\sum_{r=1}^{n} u_r = u_{n+1} - (n+2)$$ *(3 marks)*

8 (a) (i) Show that $\omega = e^{\frac{2\pi i}{7}}$ is a root of the equation $z^7 = 1$. *(1 mark)*

(ii) Write down the five other non-real roots in terms of ω. *(2 marks)*

(b) Show that

$$1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 = 0$$ *(2 marks)*

(c) Show that:

(i) $\omega^2 + \omega^5 = 2\cos\frac{4\pi}{7}$; *(3 marks)*

(ii) $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = -\frac{1}{2}$. *(4 marks)*

END OF QUESTIONS

Key to mark scheme and abbreviations used in marking

M	mark is for method		
m or dM	mark is dependent on one or more M marks and is for method		
A	mark is dependent on M or m marks and is for accuracy		
B	mark is independent of M or m marks and is for method and accuracy		
E	mark is for explanation		
√or ft or F	follow through from previous incorrect result	MC	mis-copy
CAO	correct answer only	MR	mis-read
CSO	correct solution only	RA	required accuracy
AWFW	anything which falls within	FW	further work
AWRT	anything which rounds to	ISW	ignore subsequent work
ACF	any correct form	FIW	from incorrect work
AG	answer given	BOD	given benefit of doubt
SC	special case	WR	work replaced by candidate
OE	or equivalent	FB	formulae book
A2,1	2 or 1 (or 0) accuracy marks	NOS	not on scheme
$-x$ EE	deduct x marks for each error	G	graph
NMS	no method shown	c	candidate
PI	possibly implied	sf	significant figure(s)
SCA	substantially correct approach	dp	decimal place(s)

No Method Shown

Where the question specifically requires a particular method to be used, we must usually see evidence of use of this method for any marks to be awarded. However, there are situations in some units where part marks would be appropriate, particularly when similar techniques are involved. Your Principal Examiner will alert you to these and details will be provided on the mark scheme.

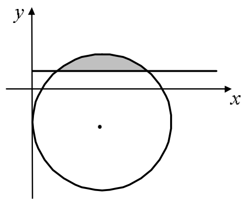
Where the answer can be reasonably obtained without showing working and it is very unlikely that the correct answer can be obtained by using an incorrect method, we must award **full marks**. However, the obvious penalty to candidates showing no working is that incorrect answers, however close, earn **no marks.**

Where a question asks the candidate to state or write down a result, no method need be shown for full marks.

Where the permitted calculator has functions which reasonably allow the solution of the question directly, the correct answer without working earns **full marks**, unless it is given to less than the degree of accuracy accepted in the mark scheme, when it gains **no marks.**

Otherwise we require evidence of a correct method for any marks to be awarded.

MFP2

Q	Solution	Marks	Total	Comments
1(a)	LHS = $\frac{1}{4}\left(e^x+e^{-x}\right)^2-\frac{1}{4}\left(e^x-e^{-x}\right)^2$	M1		
	Correct expansion of either square	A1		
	Shown equal to 1	A1	3	AG
(b)(i)	$8\cosh^2 x-3$	B1	1	
(ii)	Sketch of $y=\cosh x$	B1	1	Must cross y-axis above x-axis
(iii)	$\cosh x=(\pm)1.25$	B1F		OE; ft errors in (b)(i); allow ± missing
	$x=\ln\left(1.25+\sqrt{1.25^2-1}\right)$	M1		
	$=\ln 2$	A1F		
	$\ln\frac{1}{2}$ by symmetry	A1F	4	Accept $-\ln 2$ written straight down Alternatively, if solved by using $e^{2x}-2.5e^x+1=0$, allow M1 for $x=\ln\left(\frac{2.5\pm\sqrt{2.5^2-4}}{2}\right)$
	Total		9	
2				
(a)(i)	Circle	B1		
	Correct centre	B1		x-coordinate ≈ $-2\times$ y-coordinate in correct quadrant; condone $(4,-2\text{i})$
	Touching y-axis	B1	3	
(ii)	Straight line	B1		
	parallel to x-axis	B1		
	through (0, 1)	B1	3	Assume (0, 1) if distance up y-axis is half distance to top of circle; no other shading outside circle
(b)	Shading: inside circle	B1F		
	above line	B1F	2	Whole question reflected in x-axis loses 2 marks
	Total		**8**	

MFP2 (cont)

Q	Solution	Marks	Total	Comments
3(a)(i)	$\beta = 2 - 2\sqrt{3}\,\mathrm{i}$	B1	1	
(ii)	$\alpha\beta\gamma = -8$	M1		Allow for +8 but not ±16
	$\alpha\beta = 16$	B1		
	$\gamma = -\frac{1}{2}$	A1	3	
(iii)	Either $\frac{-p}{2} = \alpha + \beta + \gamma$ or $\frac{q}{2} = \alpha\beta + \beta\gamma + \gamma\alpha$	M1		**SC** if failure to divide by 2 throughout, allow M1A1 for either p or q correct ft
	$p = -7,\ \ q = 28$	A1F, A1F	3	ft incorrect γ
	Alternative to (a)(ii) and (a)(iii):			
	$(z^2 - 4z + 16)(az + b)$	(M1)		
	$\alpha\beta = 16$	(B1)		
	$a = 2,\ b = +1,\ \gamma = -\frac{1}{2}$	(A1)		
	Equating coefficients	(M1)		
	$p = -7$	(A1F)		
	$q = 28$	(A1F)		
(b)(i)	$r = 4,\ \ \theta = \frac{\pi}{3}$	B1,B1	2	
(ii)	$\left(2 + 2\sqrt{3}\,\mathrm{i}\right)^n = \left(4\mathrm{e}^{\frac{\pi\mathrm{i}}{3}}\right)^n$	M1		
	$= 4^n\left(\cos\frac{n\pi}{3} + \mathrm{i}\sin\frac{n\pi}{3}\right)$	A1	2	AG
(iii)	$\left(2 - 2\sqrt{3}\,\mathrm{i}\right)^n = 4^n\left(\cos\frac{n\pi}{3} - \mathrm{i}\sin\frac{n\pi}{3}\right)$	B1		
	$\alpha^n + \beta^n + \gamma^n = 4^n\left(\cos\frac{n\pi}{3} + \mathrm{i}\sin\frac{n\pi}{3}\right)$ $+ 4^n\left(\cos\frac{n\pi}{3} - \mathrm{i}\sin\frac{n\pi}{3}\right) + \left(-\frac{1}{2}\right)^n$	M1		
	$= 2^{2n+1}\cos\frac{n\pi}{3} + \left(-\frac{1}{2}\right)^n$	A1	3	AG
	Total		**14**	

MFP2 (cont)

Q	Solution	Marks	Total	Comments
4(a)	$\frac{\mathrm{d}x}{\mathrm{d}t} = \sinh 2t$	B1		
	$\frac{\mathrm{d}y}{\mathrm{d}t} = 2\cosh t$	B1		
	$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = \sinh^2 2t + 4\cosh^2 t$	M1		
	Use of $\sinh 2t = 2\sinh t\cosh t$	m1		Or other correct formula for double angle
	$= 4\cosh^2 t\left(\sinh^2 t + 1\right)$	A1		For taking out factor
	$= 4\cosh^4 t$	A1F	6	ft errors of sign in $\frac{\mathrm{d}x}{\mathrm{d}t}$ or $\frac{\mathrm{d}y}{\mathrm{d}t}$
(b)(i)	$S = 2\pi\int_0^1 2\sinh t.2\cosh^2 t\,\mathrm{d}t$	M1		Using the value obtained in (a)
	$= 8\pi\int_0^1 \sinh t.\cosh^2 t\,\mathrm{d}t$	A1	2	AG
(ii)	$S = 8\pi\left[\frac{\cosh^3 t}{3}\right]_0^1$	M1		
	$= \frac{8\pi}{3}\left[\cosh^3 1 - 1\right]$	A1	2	OE eg $\frac{\pi}{3}\left(\left(\mathrm{e} + \frac{1}{\mathrm{e}}\right)^3 - 8\right)$
	Total		**10**	
5(a)(i)	$u_1 = S_1 = 1^2.2.3 = 6$	B1	1	AG
(ii)	$u_2 = S_2 - S_1 = 42$	B1	1	AG
(iii)	$u_n = S_n - S_{n-1}$	M1		
	$= n^2(n+1)(n+2) - (n-1)^2 n(n+1)$	A1		
	$= n(n+1)(4n-1)$	A1	3	AG
(b)	$\sum_{r=n+1}^{2n} u_r = S_{2n} - S_n$	M1		
	$= (2n)^2(2n+1)(2n+2) - n^2(n+1)(n+2)$	A1		
	$= 3n^2(n+1)(5n+2)$	A1	3	AG
	Total		**8**	

MFP2 (cont)

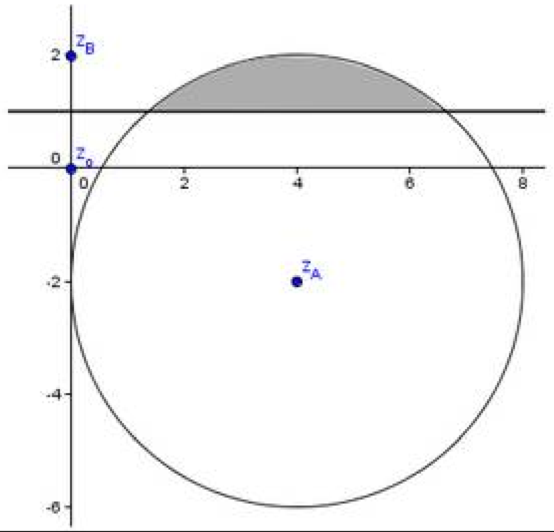
Q	Solution	Marks	Total	Comments
6(a)	$t = \tan\theta \qquad \mathrm{d}t = \sec^2\theta\,\mathrm{d}\theta$	B1		OE
	$I = \int \frac{\mathrm{d}t}{(9\cos^2\theta + \sin^2\theta)\sec^2\theta}$	M1		OE
	$= \int \frac{\mathrm{d}t}{t^2+9}$	A1	3	AG
(b)	$I = \left[\frac{1}{3}\tan^{-1}\frac{t}{3}\right]_0^{\sqrt{3}}$	M1		M1 for $\tan^{-1}$
	$\frac{1}{3}\tan^{-1}\frac{\sqrt{3}}{3}$ or $\frac{1}{3}\tan^{-1}\frac{1}{\sqrt{3}}$	A1		
	$= \frac{\pi}{18}$	A1	3	AG
	Total		**6**	
7(a)	Assume true for $n = k$			
	$u_{k+1} = 2(3\times 2^{k-1} - 1) + 1$	M1A1		
	$= 3\times 2^k - 1$	A1		$2^{(k-1)+1}$ not necessarily seen
	True for $n = 1$ shown	B1		
	Method of induction clearly expressed	E1	5	Provided all 4 previous marks earned
(b)	$\sum_{r=1}^{n} u_r = \sum_{r=1}^{n} 3\times 2^{r-1} - n$			
	$= 3(2^n - 1) - n$	M1A1		M1 for summation, ie recognition of a GP
	$= u_{n+1} - (n+2)$	A1	3	AG
	Total		**8**	

MFP2 (cont)

Q	Solution	Marks	Total	Comments
8(a)(i)	$\left(\mathrm{e}^{\frac{2\pi\mathrm{i}}{7}}\right)^7 = \mathrm{e}^{2\pi\mathrm{i}} = 1$	B1	1	Or $z^7 = \mathrm{e}^{2k\pi\mathrm{i}} \quad z = \mathrm{e}^{\frac{2k\pi\mathrm{i}}{7}} \quad k = 1$
(ii)	Roots are $\omega^2, \omega^3, \omega^4, \omega^5, \omega^6$	M1A1	2	OE; M1A0 for incomplete set **SC** B1 for a set of correct roots in terms of $\mathrm{e}^{\mathrm{i}\theta}$
(b)	Sum of roots considered	M1		$\left\{ \text{or } \sum_{r=0}^{6}\omega^6 = \frac{\omega^7 - 1}{\omega - 1} = 0 \right.$
	$= 0$	A1	2	
(c)(i)	$\omega^2 + \omega^5 = \mathrm{e}^{\frac{4\pi\mathrm{i}}{7}} + \mathrm{e}^{\frac{10\pi\mathrm{i}}{7}}$	M1		
	$= \mathrm{e}^{\frac{4\pi\mathrm{i}}{7}} + \mathrm{e}^{\frac{-4\pi\mathrm{i}}{7}}$	A1		Or $\cos\frac{4\pi}{7} + \mathrm{i}\sin\frac{4\pi}{7} + \cos\frac{4\pi}{7} - \mathrm{i}\sin\frac{4\pi}{7}$
	$= 2\cos\frac{4\pi}{7}$	A1	3	AG
(ii)	$\omega + \omega^6 = 2\cos\frac{2\pi}{7}$; $\omega^3 + \omega^4 = 2\cos\frac{6\pi}{7}$	B1,B1		Allow these marks if seen earlier in the solution
	Using part (b)	M1		
	Result	A1	4	AG
	Total		**12**	
	TOTAL		**75**	

AQA – Further pure 2 – Jan 2010 – Answers

Question 1:	**Exam report**
$a)\cosh^2 x-\sinh^2 x=\frac{1}{4}\left(e^x+e^{-x}\right)^2-\frac{1}{4}\left(e^x-e^{-x}\right)^2$ $=\frac{1}{4}\left(e^{2x}+e^{-2x}+2\right)-\frac{1}{4}\left(e^{2x}+e^{-2x}-2\right)$ $=\frac{2}{4}+\frac{2}{4}=1$ $b)i)5\cosh^2 x+3\sinh^2 x=5\cosh^2 x+3(\cosh^2 x-1)$ $=8\cosh^2 x-3$ $ii)\ y=\cosh x\ graph$ $iii)5\cosh^2 x+3\sinh^2 x=9.5$ $8\cosh^2 x-3=9.5$ $\cosh^2 x=1.5625$ $\cosh x=1.25\ or\ \ \cosh x=-1.25\ \ (no\ solution)$ $x=\cosh^{-1}(1.25)\ or\ x=-\cosh^{-1}(1.25)$ $x=\ln\left(1.25+\sqrt{1.25^2-1}\right)=\ln 2\ or\ -\ln 2$	Part (a) was generally well done apart from a few candidates who were unable to square $\frac{1}{2}\left(e^x+e^{-x}\right)$ successfully. Parts (b)(i) and (b)(ii) likewise were well done although $\pm\frac{1}{2}$ did appear on the *x*-axis of some sketches in part (b)(i). However in part (b)(iii), those candidates using the logarithmic formula for $\cosh^{-1} x$ from the formulae booklet arrived at a single value of *x*, namely ln2, but failed to realise that the sketch in part (b)(ii) was intended to give them a hint that – ln 2 was also a solution of the equation. On the other hand, candidates who used the exponential form of either cosh *x* or cosh2 *x* automatically produced both answers provided their working was correct, but some of these candidates were unable to handle the algebra leading to a quadratic equation in either e^x or e^{2x} .

Question 2:	**Exam report**
$a)i)\lvert z-4+2i\rvert=4$ this is the circle centre $A(z_A)$ *with* $z_A=4-2i$ *and radius* $r=4$ $ii)\lvert z\rvert=\lvert z-2i\rvert$ This is the perpendicular bisector of the line OB with $z_B=2i$ *and* $z_O=0$ b) The region is the intersection of the inside of the circle and the half-plane containing B. 	This question was well done overall and many candidates scored all of the eight available marks. Those candidates using mathematical instruments, i.e. ruler and compasses, produced superior solutions. Errors, when they did occur, were either the misplotting of the centre of the circle, or more commonly, the misplotting of the line. The commonest mistakes were either to draw the line through the point (0, 2) or more frequently through (0, –1). If serious errors were made in the plotting of the line, loss of marks in the shading were almost inevitable.

Question 3:	**Exam report**
$2z^3 + pz^2 + qz + 16 = 0$ has roots α, β, γ. p and q are REAL numbers $\alpha = 2 + 2i\sqrt{3}$ a)i) Since the coefficients of the equation are real numbers, α^* is also a root so $\beta = 2 - 2i\sqrt{3}$ ii) $\alpha\beta\gamma = -\frac{16}{2} = -8$ $\alpha\beta\gamma = (2+2i\sqrt{3})(2-2i\sqrt{3})\gamma = (4+12)\gamma = 16\gamma$ so $\gamma = -\frac{1}{2}$ iii) $-\frac{p}{2} = \alpha + \beta + \gamma = 2 + 2i\sqrt{3} + 2 - 2i\sqrt{3} - \frac{1}{2}$ $p = -2(4 - \frac{1}{2}) = p = -7$ $\frac{q}{2} = \alpha\beta + \alpha\gamma + \beta\gamma = \alpha\beta + \gamma(\alpha + \beta) = 16 - \frac{1}{2} \times 4$ $q = 28$ b)i) $\alpha = 2 + 2i\sqrt{3} = 4\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = 4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ $\alpha = 4e^{i\frac{\pi}{3}}$ ii) $\alpha^n = \left(4e^{i\frac{\pi}{3}}\right)^n = 4^n e^{i\frac{n\pi}{3}} = 4^n\left(\cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3}\right)$ iii) $\alpha^n + \beta^n + \gamma^n =$ $= 4^n\left(\cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3}\right) + 4^n\left(\cos\frac{n\pi}{3} - i\sin\frac{n\pi}{3}\right) + \left(-\frac{1}{2}\right)^n$ $= 4^n\left(2\cos\frac{n\pi}{3}\right) + \left(-\frac{1}{2}\right)^n$ $= 2^{2n} \times 2 \times \cos\frac{n\pi}{3} + \left(-\frac{1}{2}\right)^n = 2^{2n+1}\cos\frac{n\pi}{3} + \left(-\frac{1}{2}\right)^n$	This question provided a good source of marks for many candidates. The commonest error in part(a) was to write αβγ as -16, overlooking the fact that the coefficient of x^3 was not unity and, of course, leading to an incorrect value for γ . This error perpetuated in part (a)(ii) with α + β +γ written as *p* or –*p* and the same for *q*. Parts (b)(i) and (b)(ii) were generally well done, although it should be stated that when answers are printed, candidates are expected to provide sufficient detail to show clearly how their answers are arrived at. Part (b)(iii) was also quite well done and it was pleasing to note that in some cases where candidates had not arrived at $\gamma = -\frac{1}{2}$, they went back to part (a)(ii) to identify their error. Just occasionally in part (b)(iii), some candidates made blatant errors in their attempt to convert $4^n\cos\frac{n\pi}{3} + 4^n\cos\frac{n\pi}{3}$ into $2^{2n+1}\cos\frac{n\pi}{3}$

Question 4:	Exam report
$x=\frac{1}{2}\cosh 2t \quad and \quad y=2\sinh t$ $a)\ \frac{dx}{dt}=\sinh 2t \quad and \quad \frac{dy}{dt}=2\cosh t$ $\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=(\sinh^2 2t+4\cosh^2 t)$ $=\cosh^2 2t-1+4\cosh^2 t$ Using the identity $\cosh t=2\cosh^2 t-1$, *we have* $\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=\left(2\cosh^2 t-1\right)^2-1+4\cosh^2 t$ $=4\cosh^4 t-4\cosh^2 t+1-1+4\cosh^2 t$ $=4\cosh^4 t$ $\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=4\cosh^4 t$ $b)\,i)\ S=2\pi\int_0^1 y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$ $S=2\pi\int_0^1 2\sinh t\times(2\cosh^2 t)dt$ $S=8\pi\int_0^1 \sinh t\times\cosh^2 t\,dt$ $ii)\ S=8\pi\left[\frac{1}{3}\cosh^3 t\right]_0^1=\frac{8\pi}{3}\left(\cosh^3 1-1\right)$	Although a good number of candidates answered part (a) correctly, quite a few stalled at the handling of $\sinh^2 2t$ either by misquoting a formula for the double angle or by using long winded algebraic methods in which they lost direction. Consequently these candidates were unable to score full marks in part (b)(i). Part (b)(ii) proved to be beyond the abilities of the majority of candidates. The usual attempts were either to express replace $\cosh^2 t$ by $1+\sinh^2 t$ or to express $\cosh^2 t$ in terms of $\cosh 2t$, thereby making no progress. Few thought of using a simple substitution.

Question 5:	Exam report
$S_r=r^2(r+1)(r+2)$ $a)\,i)\ S_r=u_1+u_2+u_3+\ldots+u_{r-1}+u_r$ $so \quad for \quad r=1, \quad S_1=u_1$ $u_1=1^2\times(2)\times(3)=u_1=6$ $ii)\ S_2=u_1+u_2=6+u_2$ $and \quad S_2=2^2\times(3)\times(4)=48$ $u_2=48-6=$ $u_2=42$ $iii)\ u_n=S_n-S_{n-1}$ $=n^2(n+1)(n+2)-(n-1)^2(n)(n+1)$ $=n(n+1)\left[n(n+2)-(n-1)^2\right]$ $=n\left(n+1\right)\left(n^2+2n-n^2-1+2n\right)$ $u_n=n(n+1)\left(4n-1\right)$	This question proved to be quite discriminating. Either candidates realised what they were asked to do and scored full marks, or the notation puzzled them and they were unable to proceed beyond part (a)(ii), thinking that the answer to this part should be 48 rather than 42. Some of the weaker candidates, whilst realising what to do, failed to take out common factors in their algebraic manipulation in parts (a)(iii) and (b) with the result that correct answers were written down after incorrect algebra.

Question 5:continues	Exam report
$b)\ \sum_{r=n+1}^{2n} u_r = \sum_{r=n+1}^{2n} S_r - S_{r-1} = S_{n+1} - S_n +$ $S_{n+2} - S_{n+1} +$ $S_{n+3} - S_{n+2} + \ldots$ $+ S_{2n} - S_{2n-1}$ $\sum_{r=n+1}^{2n} u_r = S_{2n} - S_n = (2n)^2(2n+1)(2n+2) - n^2(n+1)(n+2)$ $= 8n^2(2n+1)(n+1) - n^2(n+1)(n+2)$ $= n^2(n+1)\left[8(2n+1) - (n+2)\right]$ $= n^2(n+1)\left(15n+6\right) = 3n^2(n+1)(5n+2)$	

Question 6:	Exam report
$a)\ t = \tan\theta \quad so\ \frac{dt}{d\theta} = 1 + \tan^2\theta = 1 + t^2$ $\frac{dt}{1+t^2} = d\theta$ $\bullet \cos^2\theta = \frac{1}{1+\tan^2\theta} = \frac{1}{1+t^2}$ $\bullet \sin^2\theta = 1 - \cos^2\theta = 1 - \frac{1}{1+t^2} = \frac{t^2}{1+t^2}$ $\int \frac{d\theta}{9\cos^2\theta + \sin^2\theta} = \int \frac{1}{\frac{9}{1+t^2} + \frac{t^2}{1+t^2}} \times \frac{dt}{1+t^2} = \int \frac{dt}{9+t^2}$ $b)\ when\ \theta = 0, t = \tan 0 = 0$ $\theta = \frac{\pi}{3}, t = \tan\frac{\pi}{3} = \sqrt{3}$ $I = \int_0^{\frac{\pi}{3}} \frac{d\theta}{9\cos^2\theta + \sin^2\theta} = \int_0^{\sqrt{3}} \frac{dt}{9+t^2} = \left[\frac{1}{3}\tan^{-1}\left(\frac{t}{3}\right)\right]_0^{\sqrt{3}}$ $I = \frac{1}{3} \times \tan^{-1}\left(\frac{\sqrt{3}}{3}\right) - \frac{1}{3}\tan^{-1}(0) = \frac{1}{3} \times \frac{\pi}{6} = \frac{\pi}{18}$	In part (a), whilst $\frac{dt}{d\theta}$ was expressed correctly, the manipulation required to obtain the integral in terms of t was frequently faulty. Also in part (b), whilst many candidates were able to write down the correct definite integral $\frac{1}{3}\tan^{-1}\frac{t}{3}$, full marks were not awarded unless it was clear how the answer $\frac{\pi}{18}$ was arrived at, as this answer was given in the question.

Question 7	**Exam report**
$u_1 = 2\ ,\ u_{k+1} = 2u_k + 1$ $a)$ the proposition P_n :for all n $\geq 1, u_n = 3\times 2^{n-1} - 1$ is to be proven by induction Base case: $n = 1$ $LHS:\ u_1 = 2$ $RHS: 3\times 2^{1-1} - 1 = 3\times 2^0 - 1 = 3 - 1 = 2$ P_1 *is true.* Let's suppose that for $n = k$, the propostion is true. Let's show that it is true for $n = k+1$. *i.e.* Let's show that $u_{k+1} = 3\times 2^k - 1$ — $u_{k+1} = 2u_k + 1 = 2\left(3\times 2^{k-1} - 1\right) + 1 = 3\times 2^k - 2 + 1$ $u_{k+1} = 3\times 2^k - 1 \qquad Q.E.D$ — Conclusion: If the proposition is true for $n = k$, then it is true for $n = k+1$. Because it is true for $n = 1$, we can conclude, according to the induction principle that it is true for all n ≥ 1. b) $\sum_{r=1}^{n} u_r = \sum_{r=1}^{n} 3\times 2^{r-1} - 1 = 3\sum_{r=1}^{n} 2^{r-1} - n = 3\times\frac{2^n - 1}{2-1} - n$ $remember\ core2?: \sum_{r=1}^{n} 2^{r-1} = 1 + 2 + 4 + 8 + ... + 2^{n-1} = 1\frac{2^n - 1}{2-1}$ $\sum_{r=1}^{n} u_r = 3\times 2^n - 3 - n = (3\times 2^n - 1) - (n+2) = u_{n+1} - (n+2)$	Candidates generally had difficulty in using the recurrence relationship in their proof by induction, so that responses to this question were rather poor. Proper detail is essential in the proof by induction using a sequence and a sequence relationship so that relatively few candidates scored full marks for part (a). Very few candidates indeed were successful in part (b). Only a handful of candidates recognised that a Geometric Progression was involved. If they did, they usually went on to obtain a correct solution. It should perhaps be added that one method of providing an excellent solution was to rewrite $u_{k+1} = 2u_k + 1\ as\ u_{k+1} - u_k = u_k + 1$ and then to use the method of differences to sum the series; but this method of solution was extremely rare.

Question 8:	**Exam report**
$a)\, i)\ \omega = e^{i\frac{2\pi}{7}}$ $so\ \omega^7 = \left(e^{i\frac{2\pi}{7}}\right)^7 = e^{2i\pi} = \cos 2\pi + i\sin 2\pi = 1$ $\omega\ is\ a\ solution\ of\ z^7 = 1$ $ii)\ 7\theta = k\times 2\pi \qquad \theta = k\times\frac{2\pi}{7}$ $the\ other\ non-real\ solutions\ are$ $for\ k = 2,\ e^{i\frac{4\pi}{7}} = \omega^2 \qquad for\ k = 3,\ e^{i\frac{6\pi}{7}} = \omega^3$ $for\ k = 4,\ e^{i\frac{8\pi}{7}} = \omega^4 \qquad for\ k = 5, e^{i\frac{10\pi}{7}} = \omega^5$ $for\ k = 6,\ e^{i\frac{12\pi}{7}} = \omega^6$	Whilst part (a) was generally well done, relatively few candidates expressed the other roots in terms of ω , but rather gave them in the form $re^{i\theta}$

Question 8:continues	Exam report
$b)1+\omega+\omega^2+\omega^3+\omega^4+\omega^5+\omega^6$ is a geometric series with common ration ω $1+\omega+\omega^2+\omega^3+\omega^4+\omega^5+\omega^6=\frac{1-\omega^7}{1-\omega}=\frac{1-1}{1-\omega}=0$ $c)i)\,\omega^2+\omega^5=\omega^2+\omega^{-2}=e^{i\frac{4\pi}{7}}+e^{-\frac{4\pi}{7}}=2\cos\frac{4\pi}{7}$ $ii)1+\omega+\omega^2+\omega^3+\omega^4+\omega^5+\omega^6=0$ $\omega+\omega^2+\omega^3+\omega^{-3}+\omega^{-2}+\omega^{-1}=-1$ $\omega+\omega^{-1}+\omega^2+\omega^{-2}+\omega^3+\omega^{-3}=-1$ $2\cos\frac{2\pi}{7}+2\cos\frac{4\pi}{7}+2\cos\frac{6\pi}{7}=-1$ $\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7}=-\frac{1}{2}$	Few, also, were able to complete part (b). The relation 1+ω +ω^2 = 0 appeared with regularity. Part (c) was poorly answered. Although correct answers were written down as they were given in the question, few responses were convincing and as has already been stated earlier, if answers are given, it is the responsibility of the candidates to supply sufficient working to convince the examiner that they understand the methods involved

Grade boundaries

Component Code	Component Title	Maximum Scaled Mark	Scaled Mark Grade Boundaries A	B	C	D	E
MFP2	GCE MATHEMATICS UNIT FP2	75	61	53	45	38	31

General Certificate of Education
Advanced Level Examination
June 2010

Mathematics MFP2

Unit Further Pure 2

Wednesday 9 June 2010 1.30 pm to 3.00 pm

For this paper you must have:
- the blue AQA booklet of formulae and statistical tables.

You may use a graphics calculator.

Time allowed
- 1 hour 30 minutes

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Fill in the boxes at the top of this page.
- Answer **all** questions.
- Write the question part reference (eg (a), (b)(i) etc) in the left-hand margin.
- You must answer the questions in the spaces provided. Do not write outside the box around each page.
- Show all necessary working; otherwise marks for method may be lost.
- Do all rough work in this book. Cross through any work that you do not want to be marked.

Information
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 75.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.

1 (a) Show that

$$9\sinh x - \cosh x = 4e^{x} - 5e^{-x}$$

(2 marks)

(b) Given that

$$9\sinh x - \cosh x = 8$$

find the exact value of $\tanh x$. *(7 marks)*

---

2 (a) Express $\dfrac{1}{r(r+2)}$ in partial fractions. *(3 marks)*

(b) Use the method of differences to find

$$\sum_{r=1}^{48} \frac{1}{r(r+2)}$$

giving your answer as a rational number. *(5 marks)*

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3 Two loci, L_1 and L_2, in an Argand diagram are given by

$$L_1 : |z+1+3i| = |z-5-7i|$$

$$L_2 : \arg z = \frac{\pi}{4}$$

(a) Verify that the point represented by the complex number $2+2i$ is a point of intersection of L_1 and L_2. *(2 marks)*

(b) Sketch L_1 and L_2 on one Argand diagram. *(5 marks)*

(c) Shade on your Argand diagram the region satisfying

both $|z+1+3i| \leqslant |z-5-7i|$

and $\dfrac{\pi}{4} \leqslant \arg z \leqslant \dfrac{\pi}{2}$ *(2 marks)*

4 The roots of the cubic equation

$$z^3 - 2z^2 + pz + 10 = 0$$

are α, β and γ.

It is given that $\alpha^3 + \beta^3 + \gamma^3 = -4$.

(a) Write down the value of $\alpha + \beta + \gamma$. *(1 mark)*

(b) (i) Explain why $\alpha^3 - 2\alpha^2 + p\alpha + 10 = 0$. *(1 mark)*

(ii) Hence show that

$$\alpha^2 + \beta^2 + \gamma^2 = p + 13$$ *(4 marks)*

(iii) Deduce that $p = -3$. *(2 marks)*

(c) (i) Find the real root α of the cubic equation $z^3 - 2z^2 - 3z + 10 = 0$. *(2 marks)*

(ii) Find the values of β and γ. *(3 marks)*

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5 (a) Using the identities

$$\cosh^2 t - \sinh^2 t = 1\,, \quad \tanh t = \frac{\sinh t}{\cosh t} \quad \text{and} \quad \operatorname{sech} t = \frac{1}{\cosh t}$$

show that:

(i) $\tanh^2 t + \operatorname{sech}^2 t = 1$; *(2 marks)*

(ii) $\dfrac{\mathrm{d}}{\mathrm{d}t}(\tanh t) = \operatorname{sech}^2 t$; *(3 marks)*

(iii) $\dfrac{\mathrm{d}}{\mathrm{d}t}(\operatorname{sech} t) = -\operatorname{sech} t \tanh t$. *(3 marks)*

(b) A curve C is given parametrically by

$$x = \operatorname{sech} t\,, \quad y = 4 - \tanh t$$

(i) Show that the arc length, s, of C between the points where $t = 0$ and $t = \frac{1}{2}\ln 3$ is given by

$$s = \int_0^{\frac{1}{2}\ln 3} \operatorname{sech} t \,\mathrm{d}t$$ *(4 marks)*

(ii) Using the substitution $u = \mathrm{e}^t$, find the exact value of s. *(6 marks)*

---

6 (a) Show that $\dfrac{1}{(k+2)!} - \dfrac{k+1}{(k+3)!} = \dfrac{2}{(k+3)!}$. *(2 marks)*

(b) Prove by induction that, for all positive integers n,

$$\sum_{r=1}^{n} \frac{r \times 2^r}{(r+2)!} = 1 - \frac{2^{n+1}}{(n+2)!}$$ *(6 marks)*

---

7 (a) (i) Express each of the numbers $1 + \sqrt{3}\,\mathrm{i}$ and $1 - \mathrm{i}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$. *(3 marks)*

(ii) Hence express

$$(1 + \sqrt{3}\,\mathrm{i})^8 (1 - \mathrm{i})^5$$

in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$. *(3 marks)*

(b) Solve the equation

$$z^3 = (1 + \sqrt{3}\,\mathrm{i})^8 (1 - \mathrm{i})^5$$

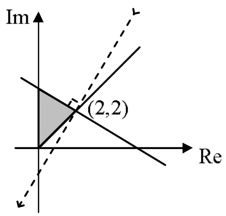
giving your answers in the form $a\sqrt{2}\,\mathrm{e}^{\mathrm{i}\theta}$, where a is a positive integer and $-\pi < \theta \leqslant \pi$. *(4 marks)*

END OF QUESTIONS

MFP2

Q	Solution	Marks	Total	Comments
1(a)	$\frac{9\left(e^x - e^{-x}\right)}{2} - \frac{e^x + e^{-x}}{2}$	M1		M0 if $\cosh x$ mixed up with $\sinh x$
	$= 4e^x - 5e^{-x}$	A1	2	AG
(b)	Attempt to multiply by e^x	M1		
	$4e^{2x} - 8e^x - 5 = 0$	A1		
	$\left(2e^x - 5\right)\left(2e^x + 1\right) = 0$	M1		ft provided quadratic factorises (or use of formula)
	$e^x \neq -\frac{1}{2}$	E1F		PI but not ignored
	$e^x = \frac{5}{2}$	A1F		
	$\tanh x = \frac{\frac{5}{2} - \frac{2}{5}}{\frac{5}{2} + \frac{2}{5}} = \frac{21}{29}$	M1 A1F	7	M1 PI for attempt to use $\tanh x = \frac{\sinh x}{\cosh x}$ or equivalent fraction
	Total		9	
2(a)	$\frac{1}{r(r+2)} = \frac{A}{r} + \frac{B}{r+2}$	M1		
	$A = \frac{1}{2},\ B = -\frac{1}{2}$	A1, A1F	3	ft incorrect A
(b)	$r = 1 \quad \frac{1}{1.3} = \frac{1}{2}\left(\frac{1}{1} - \frac{1}{3}\right)$			
	$r = 2 \quad \frac{1}{2.4} = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{4}\right)$			
	$r = 3 \quad \frac{1}{3.5} = \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right)$	M1		3 rows (PI) numerical values only
	$r = 48 \quad \frac{1}{48.50} = \frac{1}{2}\left(\frac{1}{48} - \frac{1}{50}\right)$	A1F		Last row – could be implied
	Cancelling appropriate pairs	M1		
	$\text{Sum} = \frac{1}{2}\left(\frac{1}{1} + \frac{1}{2} - \frac{1}{49} - \frac{1}{50}\right)$	A1F		Allow if the $\frac{1}{2}$ is missing only
	$= \frac{894}{1225}$	A1	5	CAO (or equivalent fraction)
	Total		8	

MFP2 (cont)

Q	Solution	Marks	Total	Comments
3				
(a)	$\lvert 2+2i+1+3i \rvert = \lvert 2+2i-5-7i \rvert$	B1		Clearly shown do not allow $\lvert 3+5i \rvert = \lvert -3-5i \rvert$ without comment
	$\arg(2+2i) = \frac{\pi}{4}$	B1	2	Clearly shown
(b)	L_1 : straight line with negative gradient perpendicular to line joining	B1		
	$(-1,-3)$ to $(5,7)$	B1		
	through $(2,2)$	B1		The point (2,2) must be shown either by (2,2) or 2+2i or with numbered axes
	L_2 : half line through O	B1		
	through $(2,2)$	B1	5	
(c)	Shading between $\frac{\pi}{4}$ and $\frac{\pi}{2}$	B1		No marks for shading if circles drawn in (b)
	Below L_1	B1	2	
	Total		9	
4(a)	$\alpha + \beta + \gamma = 2$	B1	1	
(b)(i)	α is a root and so satisfies the equation	E1	1	
(ii)	$\sum\alpha^3 - 2\sum\alpha^2 + p\sum\alpha + 30 = 0$	M1A1		
	Substitution for $\sum\alpha^3$ and $\sum\alpha$	ml		
	$\sum\alpha^2 = p + 13$	A1	4	AG
(iii)	$\left(\sum\alpha\right)^2 = \sum\alpha^2 + 2\sum\alpha\beta$ used	M1		do not allow this M mark if used in (b)(ii)
	$p = -3$	A1	2	AG
(c)(i)	$f(-2) = 0$	M1		
	$\alpha = -2$	A1	2	
(ii)	$(z+2)\left(z^2 - 4z + 5\right) = 0$	M1		For attempting to find quadratic factor
	$z = \frac{4 \pm \sqrt{-4}}{2}$	m1		Use of formula or completing the square m0 if roots are not complex
	$= 2 \pm i$	A1	3	CAO
	Total		13	

MFP2 (cont)

Q	Solution	Marks	Total	Comments
5(a)(i)	Divide $\cosh^2 t - \sinh^2 t = 1$ by $\cosh^2 t$	M1		Or $\frac{\sinh^2 t}{\cosh^2 t} + \frac{1}{\cosh^2 t}$
	Rearrange	A1	2	AG If solved back to front with no conclusion ending $\cosh^2 t - \sinh^2 t = 1$ B1 only
(ii)	$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\sinh t}{\cosh t}\right) = \frac{\cosh^2 t - \sinh^2 t}{\cosh^2 t}$	M1A1		
	$= \operatorname{sech}^2 t$	A1	3	AG
(iii)	$\frac{\mathrm{d}}{\mathrm{d}t}(\operatorname{sech} t) = -(\cosh t)^{-2} \sinh t$	M1A1		Allow A1 if negative sign missing
	$= -\operatorname{sech} t \tanh t$	A1	3	AG
(b)(i)	$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = \operatorname{sech}^4 t + \operatorname{sech}^2 t \tanh^2 t$	M1		Allow slips of sign before squaring for this M1
	Use of $\tanh^2 t + \operatorname{sech}^2 t = 1$	m1		Correct formula only for m1
	$= \operatorname{sech}^2 t$	A1		
	$\therefore s = \int_0^{\frac{1}{2}\ln 3} \operatorname{sech} t \ \mathrm{d}t$	A1	4	AG (including limits)
(ii)	$u = \mathrm{e}^t \quad \mathrm{d}u = \mathrm{e}^t \mathrm{d}t$	B1		
	$\int \operatorname{sech} t \ \mathrm{d}t = \int \frac{2}{u^2 + 1} \mathrm{d}u$	M1A1		CAO M1 for putting integrand in terms of u (no sech (ln$u$))
	$\left[2\tan^{-1} u\right]$	A1		Or $2\tan^{-1}\mathrm{e}^t$
	Change limits correctly or change back to t	m1		At some stage
	$= \frac{2\pi}{3} - \frac{2\pi}{4} = \frac{\pi}{6}$	A1	6	CAO
	Total		**18**	
6(a)	$\frac{1}{(k+2)!} = \frac{k+3}{(k+3)!}$	M1		
	Result	A1	2	
(b)	Assume true for $n = k$ For $n - k + 1$			
	$\sum_{r=1}^{k+1} \frac{r \times 2^r}{(r+2)!} = 1 - \frac{2^{k+1}}{(k+2)!} + \frac{(k+1)2^{k+1}}{(k+3)!}$	M1A1		If no LHS of equation, M1A0
	$= 1 - 2^{k+1}\left(\frac{1}{(k+2)!} - \frac{k+1}{(k+3)!}\right)$	m1		m1 for a suitable combination clearly shown
	$= 1 - \frac{2^{k+2}}{(k+3)!}$	A1		clearly shown or stated true for $n - k + 1$
	True for $n = 1$	B1		Shown
	Method of induction set out properly	E1	6	Provided previous 5 marks all earned
	Total		**8**	

MFP2 (cont)

Q	Solution	Marks	Total	Comments
7(a)(i)	$1 + \sqrt{3}\mathrm{i} = 2\mathrm{e}^{\frac{\pi \mathrm{i}}{3}}$	B1		B1 both correct
	$1 - \mathrm{i} = \sqrt{2}\,\mathrm{e}^{-\frac{\pi \mathrm{i}}{4}}$	B1B1	3	OE
(ii)	$2^{\frac{21}{2}}$ or equivalent single expression	B1F		No decimals; must include fractional powers
	Raising and adding powers of e	M1		
	$\frac{17\pi}{12}$ or equivalent angle	AIF	3	Denominators of angles must be different
(b)	$z = \sqrt[3]{2^{10}\sqrt{2}}\,\mathrm{e}^{\frac{17\pi \mathrm{i}}{36} + \frac{2k\pi \mathrm{i}}{3}}$	M1		
	$\sqrt[3]{2^{10}\sqrt{2}} = 8\sqrt{2}$	B1		CAO
	$\theta = \frac{17\pi}{36}, -\frac{7\pi}{36}, -\frac{31\pi}{36}$	A2,1F	4	Correct answers outside range: deduct 1 mark only
	Total		**10**	
	TOTAL		**75**	

AQA – Further pure 2 – Jun 2010 – Answers

Question 1:

$$a)\,9\sinh x-\cosh x=\frac{9}{2}\left(e^{x}-e^{-x}\right)-\frac{1}{2}\left(e^{x}+e^{-x}\right)$$

$$=\frac{8}{2}e^{x}-\frac{10}{2}e^{-x}$$

$$=4e^{x}-5e^{-x}$$

$$b)\,9\sinh x-\cosh x=8\ \ is\ \ equivalent\ to$$

$$4e^{x}-5e^{-x}=8\qquad(\times e^{x})$$

$$4e^{2x}-5-8e^{x}=0$$

$$4\left(e^{x}\right)^{2}-8e^{x}-5=0$$

$$(2e^{x}-5)\left(2e^{x}+1\right)=0$$

$$e^{x}=\frac{5}{2}\ \ or\ \ e^{x}=-\frac{1}{2}(no\ solution)$$

$$\tanh x=\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}=\frac{\frac{5}{2}-\frac{2}{5}}{\frac{5}{2}+\frac{2}{5}}=\frac{25-4}{25+4}=\frac{21}{29}$$

$$\tanh x=\frac{21}{29}$$

Exam report

Almost all candidates scored the two available marks in part (a). However in part (b) a number of candidates did not draw on the hint of part (a) but instead tried to manipulate the equation given in part (b). A few of these candidates expressed the given equation in tanh *x* and sech *x* and then squared, obtaining a quadratic in tanh *x*. When factorised these candidates obtained two values for tanh *x*, only one of which was the correct one. However, virtually no one rejected the incorrect solution so that it was almost impossible to award full marks when this method was used.

Question 2:

$$a)\frac{1}{r(r+2)}=\frac{A}{r}+\frac{B}{(r+2)}\qquad\times r(r+2)$$

$$1=A(r+2)+Br$$

$$r=0\ gives\ 2A=1\qquad A=\frac{1}{2}$$

$$r=-2\ gives\ -2B=1\qquad B=-\frac{1}{2}$$

$$\frac{1}{r(r+2)}=\frac{1}{2r}-\frac{1}{2(r+2)}$$

$$b)\sum_{r=1}^{48}\frac{1}{r(r+2)}=\sum_{r=1}^{48}\frac{1}{2r}-\frac{1}{2(r+2)}$$

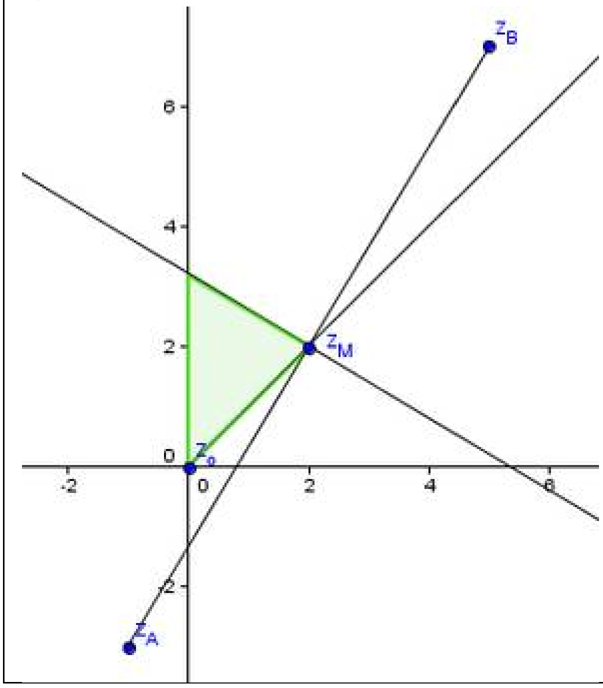
$$=\frac{1}{2}-\cancel{\frac{1}{6}}+\frac{1}{4}-\cancel{\frac{1}{8}}+\cancel{\frac{1}{6}}-\cancel{\frac{1}{10}}+\ldots$$

$$\cancel{\frac{1}{92}}-\cancel{\frac{1}{96}}+\cancel{\frac{1}{94}}-\frac{1}{98}+\cancel{\frac{1}{96}}-\frac{1}{100}$$

$$\sum_{r=1}^{48}\frac{1}{r(r+2)}=\frac{1}{2}+\frac{1}{4}-\frac{1}{98}-\frac{1}{100}=\frac{894}{1225}$$

Exam report

This question was generally well done, with many candidates scoring full marks. When errors did occur they were usually in the omission of one of the four fractions that made up the sum, notably $\frac{1}{98}$

Question 3:	**Exam report**
$z=2+2i\ and\ M(z)$ Does M belong to L_1 ? $\lvert z+1+3i\rvert=\lvert 2+2i+1+3i\rvert=\lvert 3+5i\rvert=\sqrt{9+25}=\sqrt{34}$ $\lvert z-5-7i\rvert=\lvert 2+2i-5-7i\rvert=\lvert -3-5i\rvert=\sqrt{9+25}=\sqrt{34}$ $M(z=2+2i)$ belongs to L_1 Does M belong to L_2 ? $\arg(z)=\arg(2+2i)=\tan^{-1}\left(\frac{2}{2}\right)=\frac{\pi}{4}$ $M(z=2+2i)$ belongs to L_2 $M(z)$ is a point of the intersection between L_1 *and* L_2 $b)$ L_1 is the perpendicular bisector of the line AB with $A(z_A=-1-3i)$ *and* $B(z_B=5+7i)$ L_2 is the half line from O with gradient $\tan\frac{\pi}{4}=1.$ $c)$ 	The verifications in part (a) were not always convincing, especially the verification that the point representing the complex number 2 + 2i lay on the line L_1. The sketches in part (b) varied considerably. Those candidates who made a reasonably careful drawing generally scored higher marks as they were able to clearly show that the point representing 2 + 2i lay on both L_1 and L_2. Careful sketches also improved a candidate's chance of scoring full marks in part (c).

Question 4:	**Exam report**
a) $z^3-2z^2+pz+10=0$ *has roots* α,β,γ $\alpha^3+\beta^3+\gamma^3=-4$ a) $\alpha+\beta+\gamma=2$ b) i) α *is a root so it satifies the equation* $\alpha^3-2\alpha^2+p\alpha+10=0$ ii) The same applies to β and γ $\alpha^3-2\alpha^2+p\alpha+10=0$ $\beta^3-2\beta^2+p\beta+10=0$ $\gamma^3-2\gamma^2+p\gamma+10=0$ *by adding* $(\alpha^3+\beta^3+\gamma^3)-2(\alpha^2+\beta^2+\gamma^2)+p(\alpha+\beta+\gamma)+30=0$ $-4-2(\alpha^2+\beta^2+\gamma^2)+2p+30=0$ $-2(\alpha^2+\beta^2+\gamma^2)=-2p-26$ $\alpha^2+\beta^2+\gamma^2=p+13$ iii) $\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\gamma\beta)$ $=2^2-2p=4-2p$ *so* $4-2p=p+13$ $-3p=9$ $p=-3$ c) i) Test the values $-2,-1,0,1,$or 2 for α. here, $\alpha=-2$ *indeed* $(-2)^3-2\times(-2)^2-3\times(-2)+10=$ $-8-8+6+10=0$ ii) $z^3-2z^2-3z+10=(z+2)(z^2-4z+5)$ z^2-4z+5 discriminant $=(-4)^2-4\times1\times5=-4=(2i)^2$ $\beta=\frac{4+2i}{2}=\beta=2+i$ *and* $\gamma=2-i$	Whilst parts (a) and (b)(i) were well done, few candidates were able to complete part (b)(ii) correctly through not taking note of the hint given in part (b)(i). Those candidates attempting to work out $(\alpha+\beta+\gamma)^3$ were inevitably doomed to failure. Part (b)(iii) was usually attempted by assuming the result of part (b)(ii). There were many correct solutions to part(c) although slips of sign often led to a solution with three real roots, contrary to the statement of part (c)(i).

Question 5:	**Exam report**
i) $\tanh^2 t+\text{sech}^2 t=\frac{\sinh^2 t}{\cosh^2 t}+\frac{1}{\cosh^2 t}=\frac{\sinh^2 t+1}{\cosh^2 t}=\frac{\cosh^2 t}{\cosh^2 t}=1$ ii) $\frac{d}{dt}(\tanh t)=\frac{d}{dt}\left(\frac{\sinh t}{\cosh t}=\frac{u}{v}\right)=(\frac{u'v-uv'}{v^2})=\frac{\cosh t\times\cosh t-\sinh t\times\sinh t}{\cosh^2 t}$ $=\frac{\cosh^2 t-\sinh^2 t}{\cosh^2 t}=\frac{1}{\cosh^2 t}=\text{sech}^2 t$ iii) $\frac{d}{dt}(\text{sech t})=\frac{d}{dt}\left(\frac{1}{\cosh t}=\frac{1}{u}\right)=\left(-\frac{u'}{u^2}\right)=-\frac{\sinh t}{\cosh^2 t}$ $=-\frac{\sinh t}{\cosh t}\times\frac{1}{\cosh t}=-\text{sech}\,t\tanh t$	

Question 5:continues	Exam report
$b)\ x=\operatorname{sech} t$ *and* $y=4-\tanh t$ $i)\ \frac{dx}{dt}=-\operatorname{sech} t\times\tanh t$ *and* $\frac{dy}{dx}=-\operatorname{sech}^2 t$ $\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dx}\right)^2=\operatorname{sech}^2 t\times\tanh^2 t+\operatorname{sech}^4 t$ $=\operatorname{sech}^2 t\left(\tanh^2 t+\sec h^2 t\right)=\operatorname{sech}^2 t$ $s=\int_0^{\frac{1}{2}\ln 3}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dx}\right)^2}\,dt=\int_0^{\frac{1}{2}\ln 3}\operatorname{sech} t\,dt$ $ii)\ u=e^t \qquad \frac{du}{dt}=e^t=u \qquad \frac{du}{u}=dt$ *when* $t=0,\ u=1$ $t=\frac{1}{2}\ln 3=\ln\sqrt{3},\quad u=\sqrt{3}$ $s=\int_0^{\frac{1}{2}\ln 3}\operatorname{sech} t\,dt=\int_1^{\sqrt{3}}\frac{2}{e^x+e^{-x}}dx=\int_1^{\sqrt{3}}\frac{2}{u+\frac{1}{u}}\times\frac{du}{u}$ $s=\int_1^{\sqrt{3}}\frac{2}{u^2+1}du=\left[2\tan^{-1}u\right]_1^{\sqrt{3}}=2\tan^{-1}\sqrt{3}-2\tan^{-1}1$ $s=2\times\frac{\pi}{3}-2\times\frac{\pi}{4}=\frac{2\pi}{3}-\frac{\pi}{2}=\frac{\pi}{6}$	Part (a) was a source of good marks for almost all candidates. If errors did occur they were usually errors of sign. Part (b) was also generally well done although it was disappointing to see the square root of $\operatorname{sech}^2 t\tanh^2 t+\operatorname{sech}^4 t$ written as $\operatorname{sech} t\tanh t+\operatorname{sech}^2 t$ a significant number of times. Responses to part (b)(ii) were mixed. Poor algebraic manipulation in the handling of sech *t* when expressed in terms of *u* let many candidates down badly so that they ended up with a polynomial in *u* to integrate.

Question 6:	**Exam report**
$a)\ \frac{1}{(k+2)!}-\frac{k+1}{(k+3)!}=\frac{k+3}{(k+3)!}-\frac{k+1}{(k+3)!}=\frac{2}{(k+3)!}$ $b)$ the proposition P_n :for all $n\ge 1,\sum_{r=1}^{n}\frac{r\times 2^r}{(r+2)!}=1-\frac{2^{n+1}}{(n+2)!}$ is to be proven by induction Base case: $n=1$ $LHS:\sum_{r=1}^{1}\frac{r\times 2^r}{(r+2)!}=\frac{1\times 2^1}{(1+2)!}=\frac{2}{3!}=\frac{1}{3}$ $RHS:1-\frac{2^{1+1}}{(1+2)!}=1-\frac{4}{3!}=1-\frac{2}{3}=\frac{1}{3}$ P_1 *is true* Let's suppose that for $n=k$, the proposition is true. Let's show that it is true for $n=k+1$ *i.e* Let's show that $\sum_{r=1}^{k+1}\frac{r\times 2^r}{(r+2)!}=1-\frac{2^{k+2}}{(k+3)!}$ − $\sum_{r=1}^{k+1}\frac{r\times 2^r}{(r+2)!}=\sum_{r=1}^{k}\frac{r\times 2^r}{(r+2)!}+\frac{(k+1)2^{k+1}}{(k+3)!}=1-\frac{2^{k+1}}{(k+2)!}+\frac{(k+1)2^{k+1}}{(k+3)!}$ $=1-2^{k+1}\left(\frac{1}{(k+2)!}-\frac{(k+1)}{(k+3)!}\right)=1-2^{k+1}\times\frac{2}{(k+3)!}$ *from Qa)* $=1-\frac{2^{k+2}}{(k+3)!}$ *Q.E.D* − Conclusion: If the proposition is true for $n=k$, then it is true for $n=k+1$. Because it is true for $n=1$, we can conclude, according to the induction principal that it is true for all $n\ge 1$.	Again responses to this question were mixed. It was evident that some candidates thought that $(k+2)!$ started at k, and wrote it as $k(k+1)(k+2)$. Others wrote down the result after some rather dubious algebra. Although there has been considerable improvement in the way that solutions by induction have been expressed, in this case what would otherwise have been acceptable solutions were spoilt by errors of sign. The same error occurred frequently. It occurred when a candidate tried to combine, in one bracket, a negative expression followed by a positive expression by placing a negative sign outside the combining bracket and then by forgetting to alter the sign before the positive term to compensate.

Question 7:	Exam report
$a)\ i)\ 1+i\sqrt{3}=2\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)=2e^{i\frac{\pi}{3}}$ $1-i=\sqrt{2}\left(\frac{1}{\sqrt{2}}-i\frac{1}{\sqrt{2}}\right)=\sqrt{2}e^{-i\frac{\pi}{4}}$ $ii)\ \left(1+i\sqrt{3}\right)^8\left(1-i\right)^5=\left(2e^{i\frac{\pi}{3}}\right)^8\times\left(\sqrt{2}e^{-i\frac{\pi}{4}}\right)^5$ $=2^8e^{i\frac{8\pi}{3}}\times\sqrt{2}^5e^{-i\frac{5\pi}{4}}=2^8e^{i\frac{2\pi}{3}}\times\sqrt{2}^5e^{i\frac{3\pi}{4}}$ $=2^8\times2^{\frac{5}{2}}\times e^{i\left(\frac{2\pi}{3}+\frac{3\pi}{4}\right)}=2^{\frac{21}{2}}e^{i\frac{17\pi}{12}}=1024\sqrt{2}\ e^{i\frac{17\pi}{12}}$ $b)\ z^3=\left(re^{i\theta}\right)^3=r^3e^{i3\theta}=2^{\frac{21}{2}}e^{i\frac{17\pi}{12}}$ $so\ r^3=2^{\frac{21}{2}}\quad and\quad 3\theta=\frac{17\pi}{12}+k\times2\pi$ $r=2^{\frac{7}{2}}=8\sqrt{2}\ \ and\ \ \theta=\frac{17\pi}{36}+k\times\frac{2\pi}{3}\quad k=-2,-1,0$ $solutions:\ z=8\sqrt{2}e^{i\frac{17\pi}{36}}\ \ or\ \ z=8\sqrt{2}e^{-i\frac{7\pi}{36}}\ \ or\ \ z=8\sqrt{2}e^{-i\frac{31\pi}{36}}$	Part (a)(i) was generally well done. The less successful candidates usually wrote the argument of 1 – i as 3π/4 instead of – π/4. In part (a)(ii) there was some poor handling of fractions in the argument of the product of the two complex numbers, and also some omission of raising the moduli of the two complex numbers to their respective powers. Many of the candidates who had been successful in part (a) often went on to complete part (b) correctly, although some candidates lost marks either through not giving *z* in the form asked for or by giving values for *ϑ* outside the specified range.

Grade boundaries

Code	Title	Max. Scaled Mark	Scaled Mark Grade Boundaries and A* Conversion Points					
			A*	A	B	C	D	E
MFP2	GCE MATHEMATICS UNIT FP2	75	70	65	57	49	41	33


Good Luck