Projectiles - part 1

Further Projectiles

Elimination of time from equations to derive the equation of the trajectory of a projectile

Candidates will **not** be required to know the formula $y = x\tan\alpha - \frac{gx^2}{2v^2}\left(1+\tan^2\alpha\right)$, but should be able to derive it when needed. The identity $1+\tan^2\theta = \sec^2\theta$ will be required.

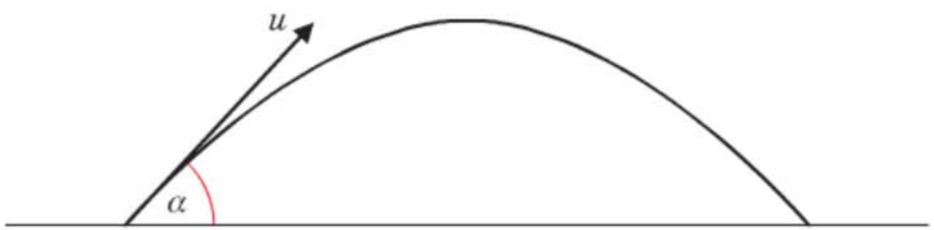
To remember: $\frac{1}{Cos^2\theta} = 1 + Tan^2\theta = Sec^2\theta$

The model and the vocabulary

Consider a particle which is projected with an initial speed u at an angle α .

- We consider the resistance of the air negligeable and no force apart from the gravitational force is applied to the particle.

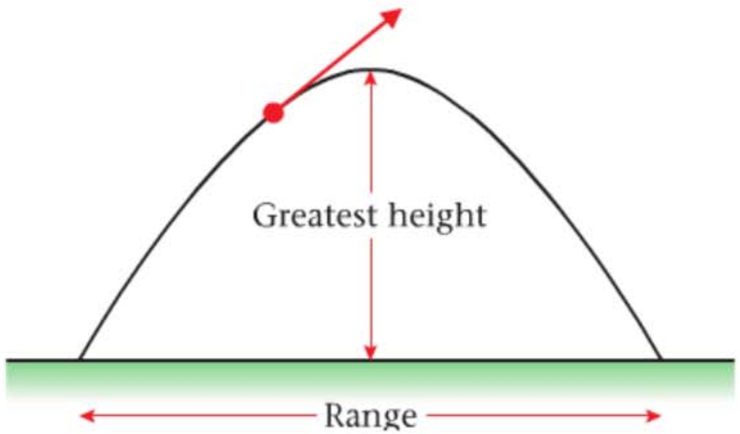
- When a particle is projected with speed u, at an angle α to the horizontal, it will move along a symmetric curve.



The initial speed u is called the **speed of projection** of the particle.
The angle α is called the **angle of projection** or **angle of elevation** of the particle.

- The distance from the point from which the particle was projected to the point where it strikes the horizontal plane is called the **range**.

- The time the particle takes to move from its point of projection to the point where it strikes the horizontal plane is called the **time of flight** of the projectile.

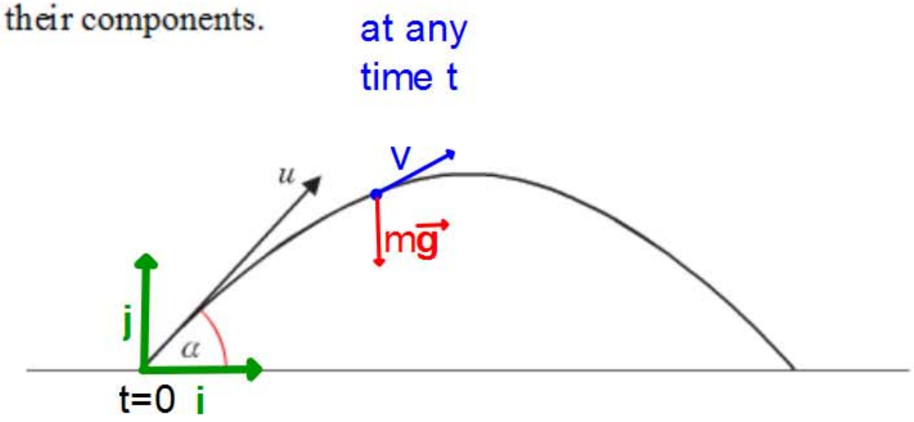


Establishing the equations of movement

We are going to establish the equation of the movement using vectors and their components.

To do so, we introduce two perpendicular unit vectors (**i**, **j**).

i is parallel to the launching plane, **j** is perpendicular to it.



Notation:

a is acceleration vector

v is the speed vector at any time *t*.

u is the initial speed vector (at t = 0)

s is the displacement vector.

$\mathbf{s}_0$ is the initial position vector (it is usually set to **0**)

- According to Newton's law : $\mathbf{F} = m\mathbf{a}$

  The only force applied to the particle is gravity

  so $\mathbf{F} = m\mathbf{g} = m\mathbf{a}$ *therefore* $\mathbf{a} = \mathbf{g}$

  We know that $\mathbf{a} = \dfrac{d\mathbf{v}}{dt}$ *therefore* $\mathbf{v} = \mathbf{a}t + \mathbf{u}$

  and $\mathbf{v} = \dfrac{d\mathbf{s}}{dt}$ *therefore* $\mathbf{s} = \dfrac{1}{2}\mathbf{a}t^2 + \mathbf{u}t + \mathbf{s}_0$

Now using the vectors' components:

$$\mathbf{a} = \mathbf{g} = \begin{pmatrix} 0 \\ -g \end{pmatrix}$$

This is the starting point for any projectile problem.

Now integrate with respect to t :

once to work out the velocity

twice to work out the displacement.

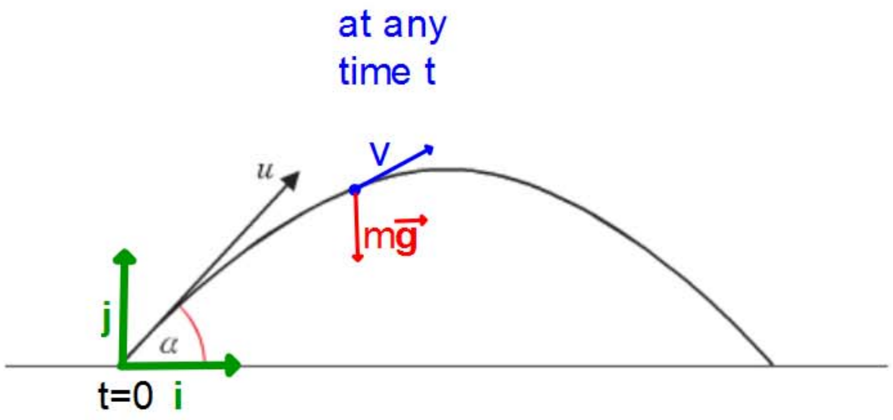
We have establish that $\mathbf{v} = \mathbf{a}t + \mathbf{u} = \begin{pmatrix} 0 \\ -gt \end{pmatrix} + \begin{pmatrix} uCos(\alpha) \\ uSin(\alpha) \end{pmatrix}$

$$\mathbf{v} = \begin{pmatrix} uCos(\alpha) \\ -gt + uSin(\alpha) \end{pmatrix} \quad and$$

$$\mathbf{s} = \frac{1}{2}\mathbf{a}t^2 + \mathbf{u}t + \mathbf{s}_0 = \begin{pmatrix} 0 \\ -\frac{1}{2}gt^2 \end{pmatrix} + \begin{pmatrix} uCos(\alpha)t \\ uSin(\alpha)t \end{pmatrix} + \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$\mathbf{s} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} uCos(\alpha)t + x_0 \\ -\frac{1}{2}gt^2 + uSin(\alpha)t + y_0 \end{pmatrix}$$

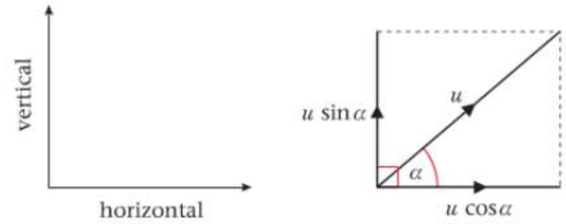
Usually $x_0 = y_0 = 0$ so $\mathbf{s} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} uCos(\alpha)t \\ -\frac{1}{2}gt^2 + uSin(\alpha)t \end{pmatrix}$



Reminder:

"Horizontal" and "vertical" components of a vector

The initial velocity of the projectile can be resolved into two components.



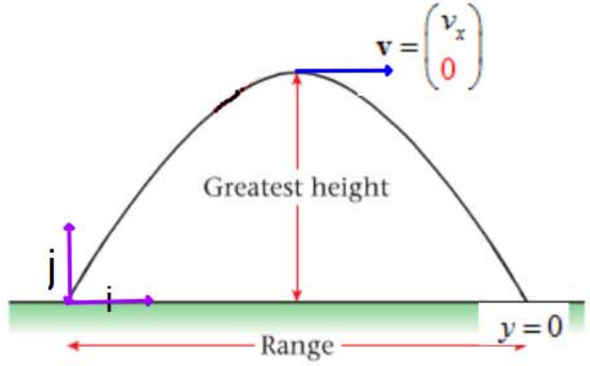
The horizontal component of the initial velocity is $u \cos \alpha$.

The **vertical component** of the initial velocity is $u \sin \alpha$.

Notes about the maximum height and the range:

Let's call $\mathbf{v}=\begin{pmatrix} v_x \\ v_y \end{pmatrix}$ *and* $\mathbf{s}=\begin{pmatrix} x \\ y \end{pmatrix}$

- At the vertex of the trajectory, $v_y = 0$
- At the furthest point of the trajectory (from O), $y = 0$



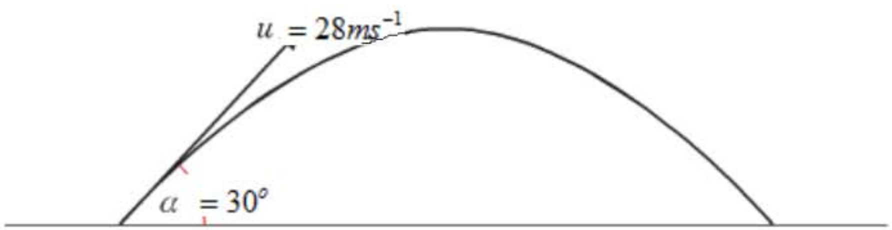
Numerical example:

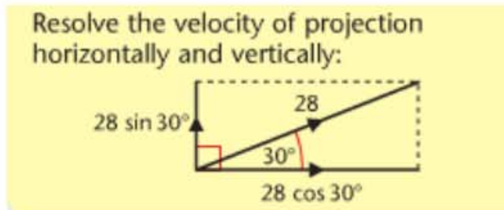
A particle P is projected from a point O on a horizontal plane with speed $28\,\text{m s}^{-1}$ and with angle of elevation 30°. After projection, the particle moves freely under gravity until it strikes the plane at a point A. Find

a the greatest height above the plane reached by P,

b the time of flight of P,

c the distance OA.





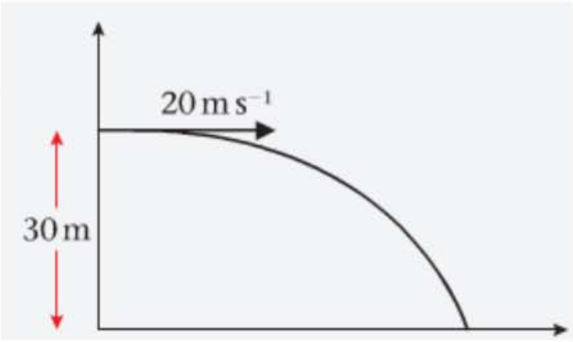
a) 10m
b) 2.86 to 2d.p.
c) 69.28 to 2d.p

Exercises:

1) A ball is thrown horizontally, with speed $20\,\text{m s}^{-1}$, from the top of a building which is 30 m high. Find

a the time the ball takes to reach the ground,

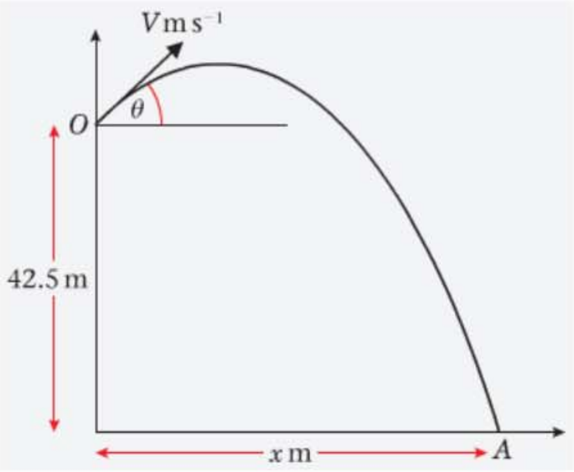
b the distance between the bottom of the building and the point where the ball strikes the ground.



2) A particle is projected from a point O with speed $V\,\text{m s}^{-1}$ and at an angle of elevation of θ, where $\tan\theta = \frac{4}{3}$. The point O is 42.5 m above a horizontal plane. The particle strikes the plane, at a point A, 5 s after it is projected.

a Show that $V = 20$.

b Find the distance between O and A.

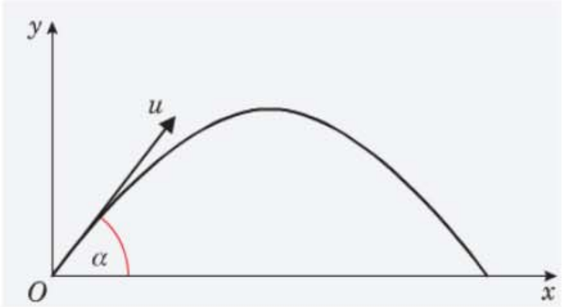


These answers should be rounded to an approriate degree of accuracy in the exam.

1) a) t = 2.474 s
b) 49.487 m
2) b) 73.527 m

Eliminating "t" in the equations

A particle is projected from a point with speed u at an angle of elevation α and moves freely under gravity. When the particle has moved a horizontal distance x, its height above the point of projection is y.

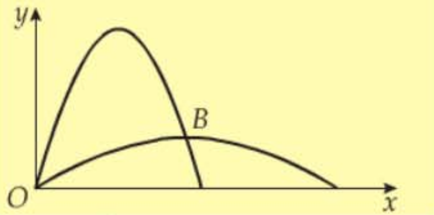


a Show that $y = x\tan\alpha - \dfrac{gx^2}{2u^2}(1 + \tan^2\alpha)$

A particle is projected from a point A on a horizontal plane, with speed $28\,\text{m s}^{-1}$ at an angle of elevation α. The particle passes through a point B, which is at a horizontal distance of 32 m from A and at a height of 8 m above the plane.

b Find the two possible values of α, giving your answers to the nearest degree.

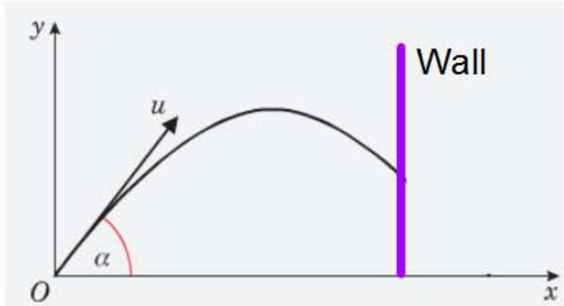
There are two possible angles of elevation for which the particle will pass through B. This sketch illustrates the two paths.



$\alpha = 27°$ and $77°$, to the nearest degree

Exam question

A ball is hit from a point O on a horizontal surface. It initially moves with speed 14 m s^{-1} at an angle α above the horizontal. At time t the horizontal displacement of the ball from O is x metres and the vertical displacement is y metres. Assume that the only force acting on the ball after it has been thrown is gravity.



(a) Show that $y = x\tan\alpha - \frac{x^2}{40}\left(1+\tan^2\alpha\right)$. *(5 marks)*

(b) A vertical wall is 10 metres from O. The ball hits the wall at a height of 4 metres. Find the two possible values of α. *(6 marks)*

More practice:

Whenever a numerical value of g is required, take $g = 9.8\,\text{m s}^{-2}$.

1 A particle is projected with speed $35\,\text{m s}^{-1}$ at an angle of elevation of 60°. Find the time the particle takes to reach its greatest height.

2 A ball is projected from a point 5 m above horizontal ground with speed $18\,\text{m s}^{-1}$ at an angle of elevation of 40°. Find the height of the ball above the ground 2 s after projection.

3 A stone is projected horizontally from a point above horizontal ground with speed $32\,\text{m s}^{-1}$. The stone takes 2.5 s to reach the ground. Find

a the height of the point of projection above the ground,

b the distance from the point on the ground vertically below the point of projection to the point where the stone reached the ground.

4 A projectile is launched from a point on horizontal ground with speed $150\,\text{m s}^{-1}$ at an angle of 10° to the horizontal. Find

a the time the projectile takes to reach its highest point above the ground,

b the range of the projectile.

5 A particle is projected from a point O on a horizontal plane with speed $20\,\text{m s}^{-1}$ at an angle of elevation of 45°. The particle moves freely under gravity until it strikes the ground at a point X. Find

a the greatest height above the plane reached by the particle,

b the distance OX.

6 A ball is projected from a point A on level ground with speed $24\,\text{m s}^{-1}$. The ball is projected at an angle θ to the horizontal where $\sin\theta = \frac{4}{5}$. The ball moves freely under gravity until it strikes the ground at a point B. Find

a the time of flight of the ball,

b the distance from A to B.

1 3.1 (2 s.f.)
2 8.5 m (2 s.f.)
3 **a** 31 m (2 s.f.) **b** 80 m
4 **a** 2.7 s (2 s.f.) **b** 790 m (2 s.f.)
5 **a** 10 m (2 s.f.) **b** 41 m (2 s.f.)
6 **a** 3.9 s (2 s.f.) **b** 56 m (2 s.f.)

Initial velocity given as a vector

In some problems, instead of giving the initial speed as a amplitude and angle, it is given with its two components: **u = ai + bj**

Note that in this case,

$u\cos(\alpha) = \text{a}$ and $u\sin(\alpha) = \text{b}$

(it is "easer" in a way!)

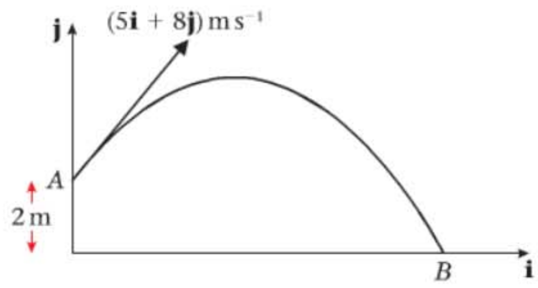
Example:

A ball is struck by a racket at a point A which is 2 m above horizontal ground. Immediately after being struck, the ball has velocity $(5\mathbf{i} + 8\mathbf{j})\ \text{m s}^{-1}$, where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors horizontally and vertically respectively. After being struck, the ball travels freely under gravity until it strikes the ground at the point B, as shown in the diagram above. Find

a the greatest height above the ground reached by the ball,

b the speed of the ball as it reaches B,

c the angle the velocity of the ball makes with the ground as the ball reaches B.



Exercises:

9 A particle P is projected from the origin with velocity $(12\mathbf{i} + 24\mathbf{j})\,\text{m s}^{-1}$, where $\mathbf{i}$ and $\mathbf{j}$ are horizontal and vertical unit vectors respectively. The particle moves freely under gravity. Find

a the position vector of P after 3 s,

b the speed of P after 3 s.

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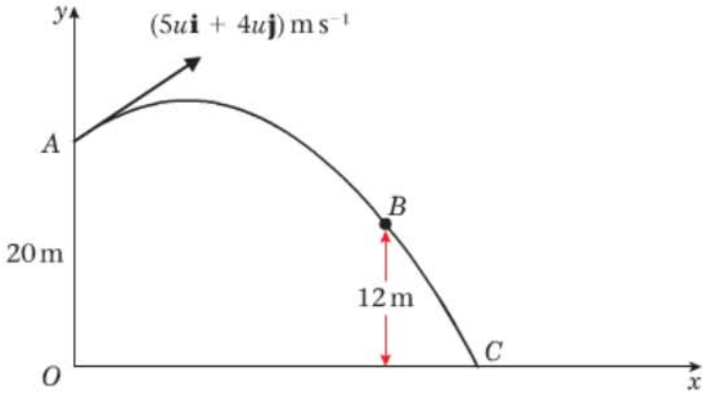

[In this question, the unit vectors $\mathbf{i}$ *and* $\mathbf{j}$ *are in a vertical plane,* $\mathbf{i}$ *being horizontal and* $\mathbf{j}$ *being vertical.]*

A particle P is projected from a point A with position vector $20\mathbf{j}$ m with respect to a fixed origin O. The velocity of projection is $(5u\mathbf{i} + 4u\mathbf{j})\,\text{m s}^{-1}$. The particle moves freely under gravity, passing through a point B, which has position vector $(k\mathbf{i} + 12\mathbf{j})$ m, where k is a constant, before reaching the point C on the x-axis, as shown in the figure above.
The particle takes 4 s to move from A to B. Find

a the value of u,

b the value of k,

c the angle the velocity of P makes with the x-axis as it reaches C.

9 **a** $(36\mathbf{i} + 27.9\mathbf{j})$ m **b** $13\,\text{m s}^{-1}$ (2 s.f.)

12 **a** 4.4 **b** 88 **c** 50° (2 s.f.)

And finally, just for fun...

General case: establishing formulae

A projectile is launched from a point on a horizontal plane with initial speed $u\,\text{m s}^{-1}$ at an angle of elevation α. The particle moves freely under gravity until it strikes the plane. The range of the projectile is R m.

a Show that the time of flight of the particle is $\dfrac{2u\sin\alpha}{g}$ seconds.

b Show that $R = \dfrac{u^2 \sin 2\alpha}{g}$.

c Deduce that, for a fixed u, the greatest possible range is when $\alpha = 45°$.

d Given that $R = \dfrac{2u^2}{5g}$, find the two possible values of the angle of elevation at which the projectile could have been launched.