Polar coordinates

Specifications:

Polar Coordinates

Relationship between polar and Cartesian coordinates.

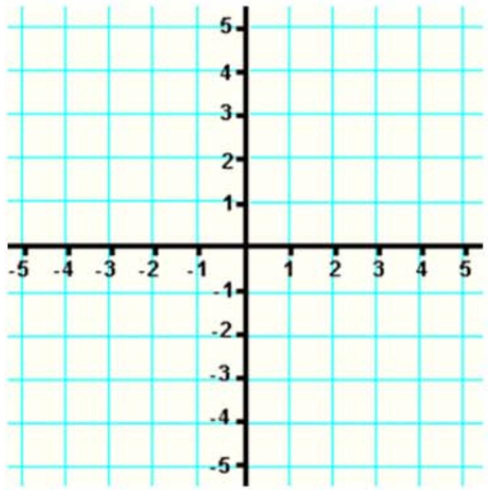
The convention $r > 0$ will be used. The sketching of curves given by equations of the form $r = \mathrm{f}(\theta)$ may be required.

Use of the formula

$$\text{area} = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \mathrm{d}\theta.$$

Principle and definition

So far, in geometry, we have been working with CARTESIAN coordinates:
A(x, y)

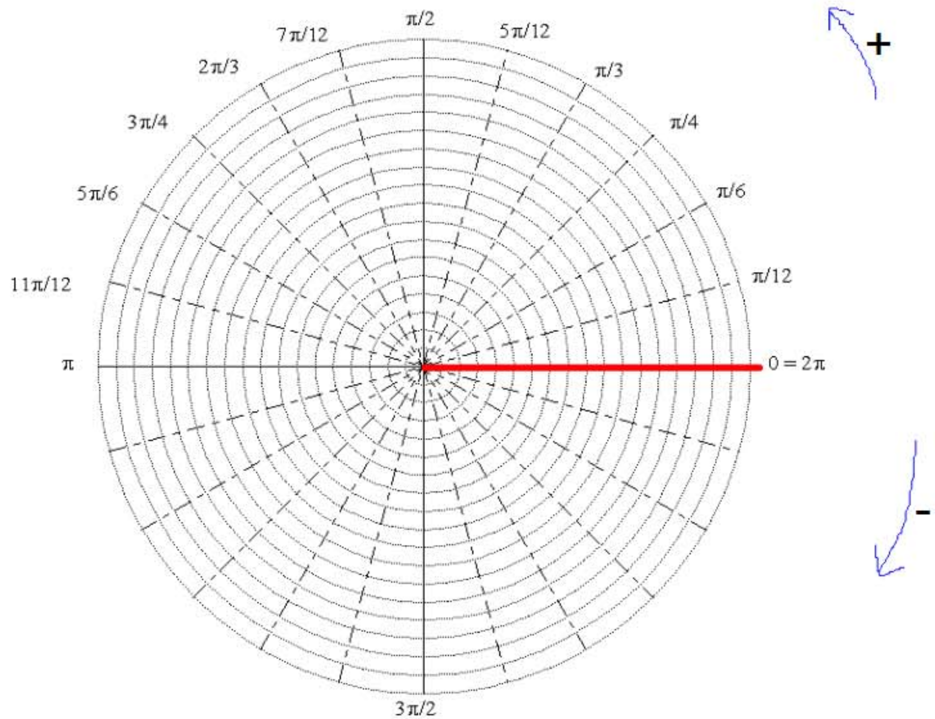


The position of the point A can also be defined with
its distance from O
and
the angle made by OA with the x-axis.

The distance OA is noted " r "
with r > 0.
The angle is noted "θ".

The polar coordinates of A are:
A(r, θ)

To ensure the unicity of the polar coordinates,
we choose θ in a 2π – interval
either $-\pi < \theta \leq \pi$
or $0 < \theta \leq 2\pi$



- The plane in which the points are located with polar coordinates is called the *r - θ plane*
- The point O is called the POLE.
- The line $\theta = 0$, is called the initial line.

Sketching a curve

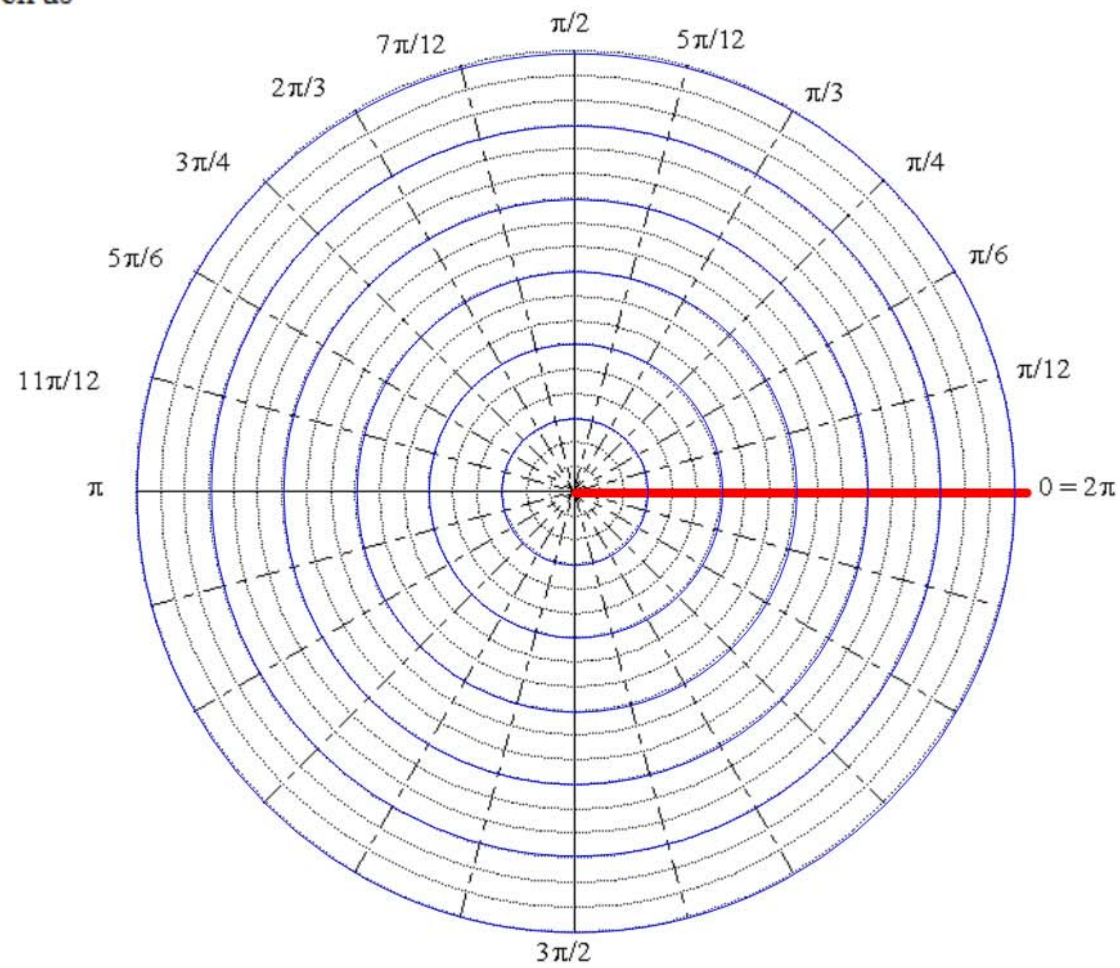
In cartesian coordinates, the equation of the curve will be given as

$$y = f(x)$$

In polar coordinates, the equation of a curve is given by

$$r = f(\theta)$$

(The radius depends on the angle with ox)



Complete the table of values
and plot the curve with equation

$r = 4 - 2Cos\theta \qquad -\pi < \theta \leq \pi$

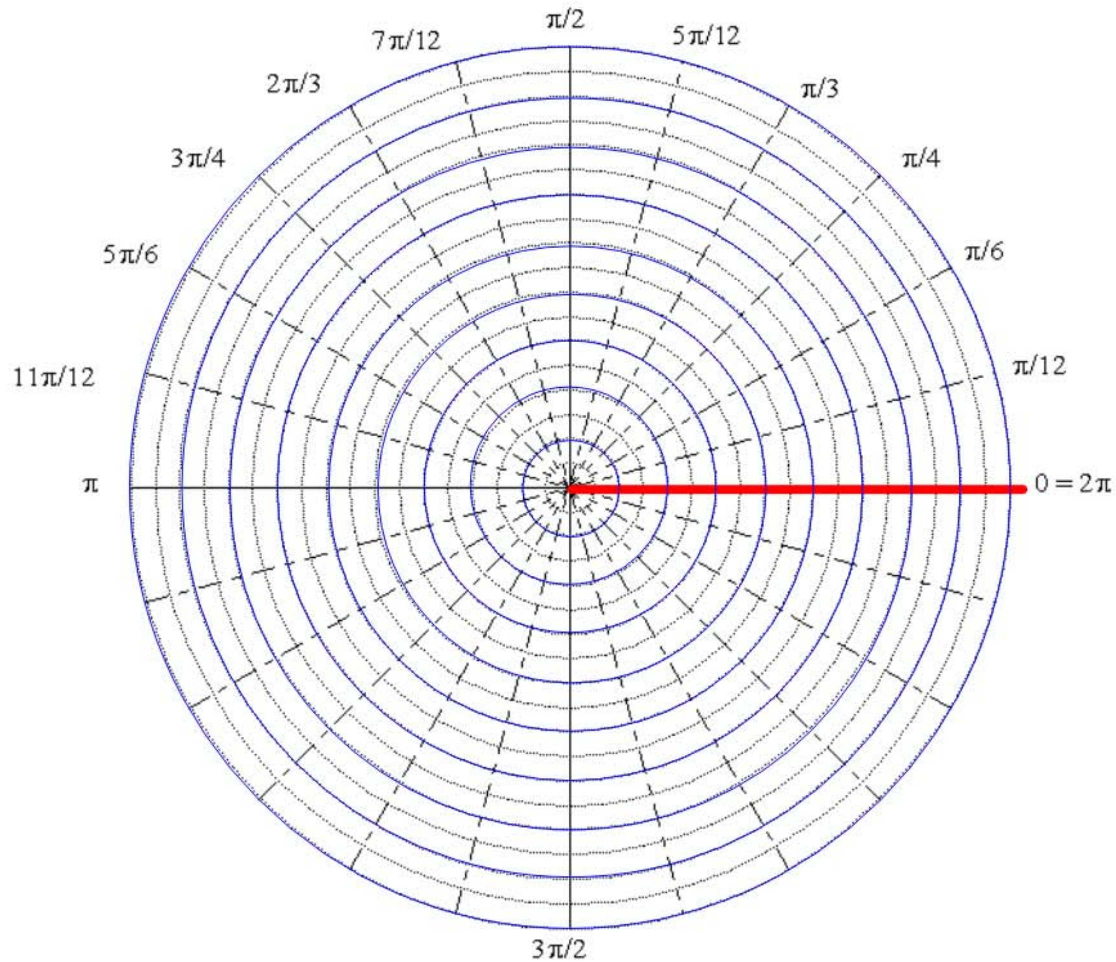
θ										
r										

Complete the table of values

and plot the curve with equation

$r = 6Cos\theta + 8Sin\theta \qquad 0 \le \theta \le \pi$

θ										
r										

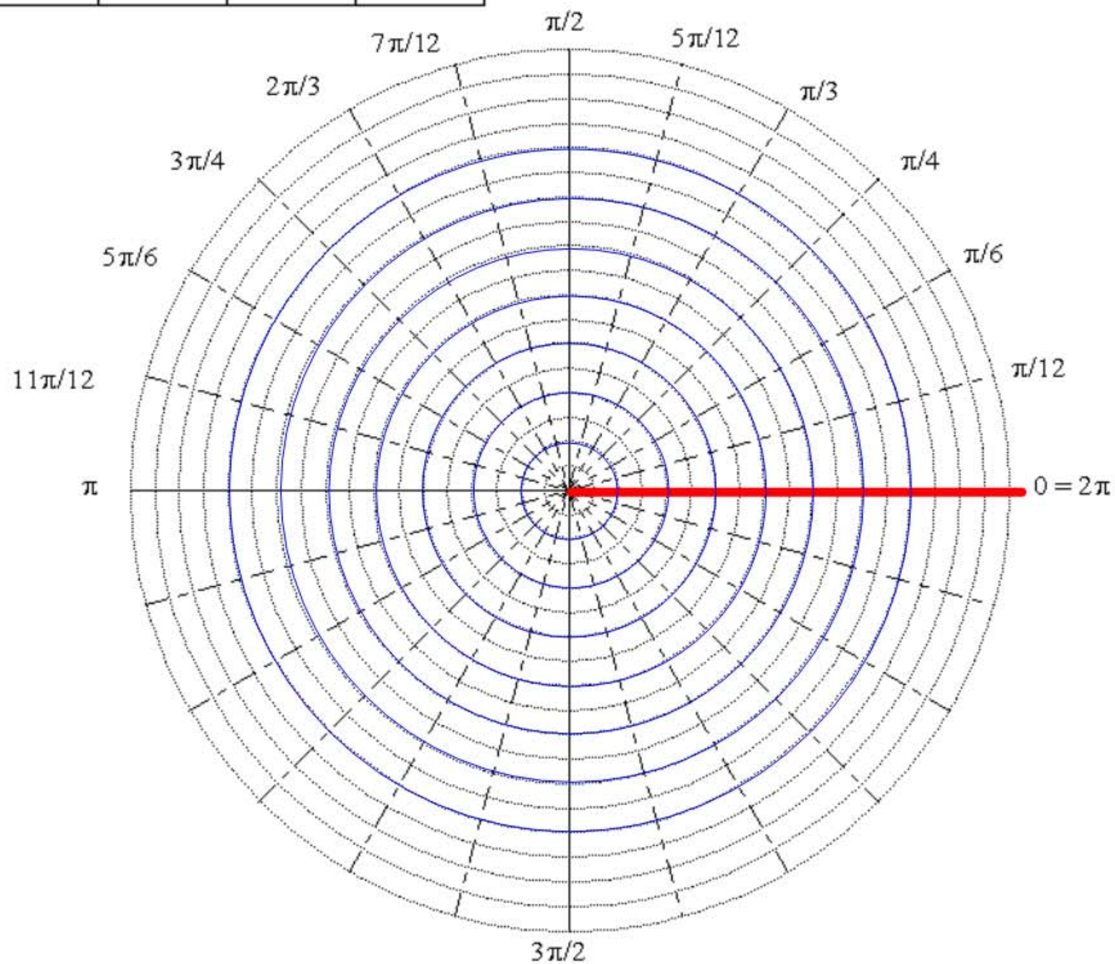


Complete the table of values

and plot the curve with equation

$r = 1 + 6e^{-\frac{\theta}{\pi}} \qquad 0 \le \theta \le 2\pi$

θ									
r									



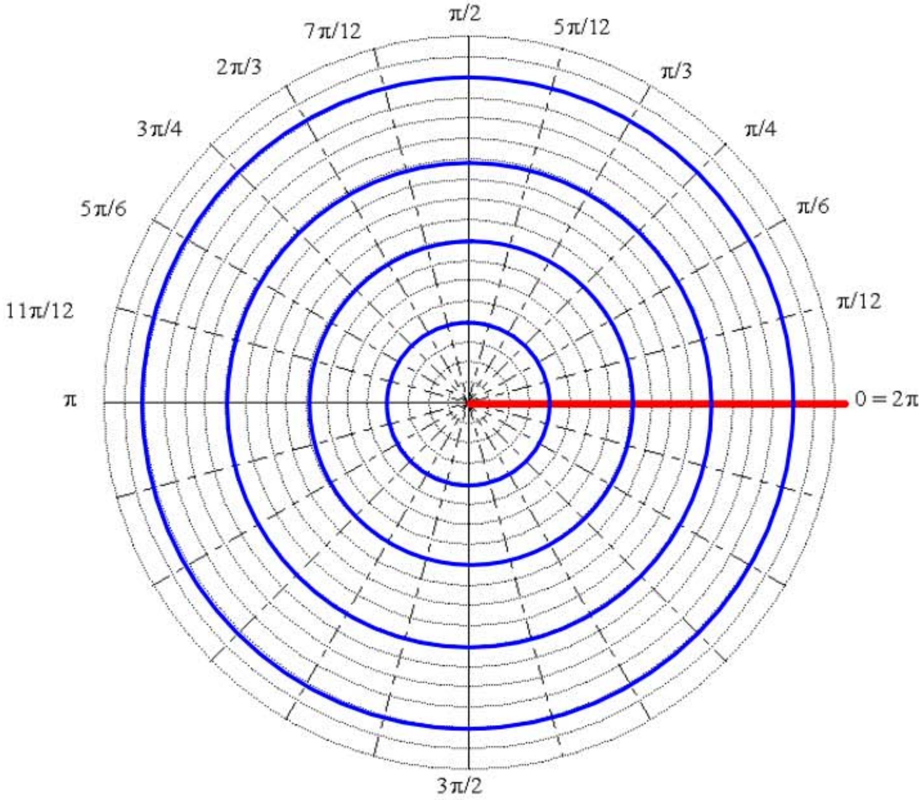
Distance between two points

The polar coordinates of the point A and B are:

$$A(2,\frac{\pi}{6}) \ \ and \ \ B(3,-\frac{\pi}{2})$$

$a)$ Find the angle between OA and OB, where O is the pole.

$b)$ Work out the length AB is the triangle OAB.

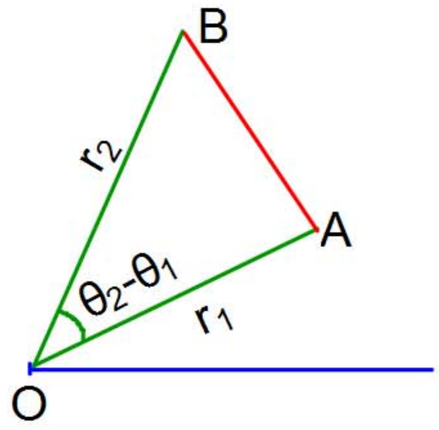


Cos rule

In a triangle ABC,

$$BC^2 = AB^2 + AC^2 - 2AB \times AC \times Cos(\widehat{ABC})$$

General case:



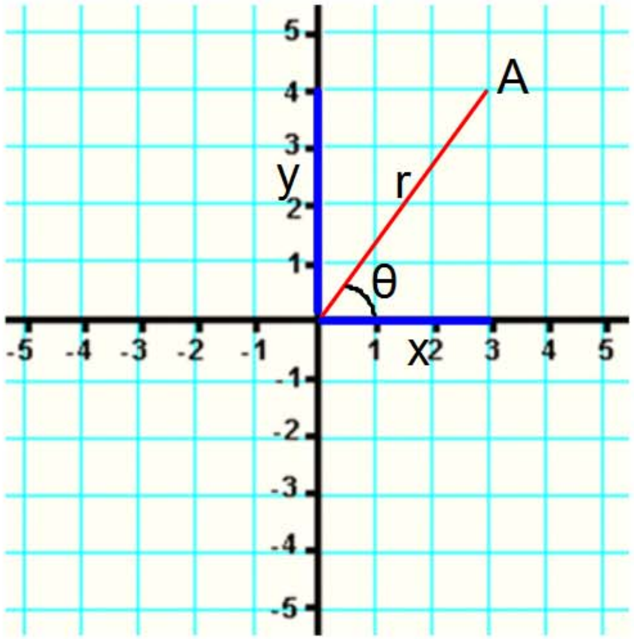
Good to remember :

$A(r_1,\theta_1)$ *and* $B(r_2,\theta_2)$ are two points

$$AB^2 = r_1^{\,2} + r_2^{\,2} - 2r_1r_2Cos(\theta_2 - \theta_1)$$

Changing between Cartesian and Polar coordinates

The point A has cartesian coordinates $A(x, y)$
and polar coordinates $A(r, \theta)$



Relationship between cartesian and polar coordiantes

$$x = r\cos\theta, \quad y = r\sin\theta$$

$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}$$

Be careful :

$\theta = \text{Arctan}\left(\frac{y}{x}\right)$ *if* $x > 0$

but

$\theta = \text{Arctan}\left(\frac{y}{x}\right) \pm \pi$ *if* $x < 0$

Summary:

Relationship between cartesian and polar coordiantes

$$x = r\cos\theta, \quad y = r\sin\theta$$

$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}$$

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but

$\theta = \text{Arctan}\left(\frac{y}{x}\right) \pm \pi$ *if* $x < 0$

Exercises:

1 Find the polar coordinates of the following points

a $(5,12)$ **b** $(-5, 12)$ **c** $(-5, -12)$

d $(2, -3)$ **e** $(\sqrt{3}, -1)$

2 Find Cartesian coordinates of the following points. Angles are measured in radians.

a $\left(6, \frac{\pi}{6}\right)$ **b** $\left(6, -\frac{\pi}{6}\right)$ **c** $\left(6, \frac{3\pi}{4}\right)$

d $\left(10, \frac{5\pi}{4}\right)$ **e** $(2, \pi)$

1) $a)(13, 1.18)$ $b)(13, 1.97)$ $c)(13, -1.97)$ $d)(\sqrt{13}, -0.98)$ $e)\left(2, -\frac{\pi}{6}\right)$

2) $a)(3\sqrt{3}, 3)$ $b)(3\sqrt{3}, -3)$ $c)(-3\sqrt{2}, 3\sqrt{2})$ $d)(-5\sqrt{2}, -5\sqrt{2})$ $e)(-2, 0)$

Equations of curves in polar and cartesian form

Find Cartesian equations of the following curves.

a $r = 5$ **b** $r = 2 + \cos 2\theta$ **c** $r^2 = \sin 2\theta \quad 0 < \theta < \frac{\pi}{2}$

Find polar equations for the following:

a $y^2 = 4x$ **b** $x^2 - y^2 = 5$ **c** $y\sqrt{3} = x + 4$

Exercises:

Find Cartesian equations for the following curves where a is a positive constant.

1 **a** $r = 2$ **b** $r = 3 \sec \theta$ **c** $r = 5 \operatorname{cosec} \theta$

2 **a** $r = 4a \tan \theta \sec \theta$ **b** $r = 2a \cos \theta$ **c** $r = 3a \sin \theta$

3 **a** $r = 4(1 - \cos 2\theta)$ **b** $r = 2 \cos^2 \theta$ **c** $r^2 = 1 + \tan^2 \theta$

Find polar equations for the following curves:

4 **a** $x^2 + y^2 = 16$ **b** $xy = 4$ **c** $(x^2 + y^2)^2 = 2xy$

5 **a** $x^2 + y^2 - 2x = 0$ **b** $(x + y)^2 = 4$ **c** $x - y = 3$

6 **a** $y = 2x$ **b** $y = -\sqrt{3}x + a$ **c** $y = x(x - a)$

1 **a** $x^2 + y^2 = 4$ **b** $x = 3$ **c** $y = 5$

2 **a** $x^2 = 4ay$ or $y = \frac{x^2}{4a}$

b $x^2 + y^2 = 2ax$ or $(x - a)^2 + y^2 = a^2$

c $x^2 + y^2 = 3ay$ or $x^2 + \left(y - \frac{3a}{2}\right)^2 = \frac{9a^2}{4}$

3 **a** $(x^2 + y^2)^{\frac{3}{2}} = 8y^2$ **b** $(x^2 + y^2)^{\frac{3}{2}} = 2x^2$

c $x^2 = 1$ or $x = \pm 1$

4 **a** $r = 4$ **b** $r^2 = 8 \operatorname{cosec} 2\theta$

c $r^2 = \sin 2\theta$

5 **a** $r = 2 \cos \theta$ **b** $r^2 = \frac{4}{1 + \sin 2\theta}$

c $r = \frac{3}{\sqrt{2}} \sec\left(\theta + \frac{\pi}{4}\right)$

6 **a** $\theta = \arctan 2$ **b** $r = \frac{a}{2} \operatorname{cosec}\left(\theta + \frac{\pi}{3}\right)$

c $r = \tan \theta \sec \theta + a \sec \theta$

The area bounded by a polar curve

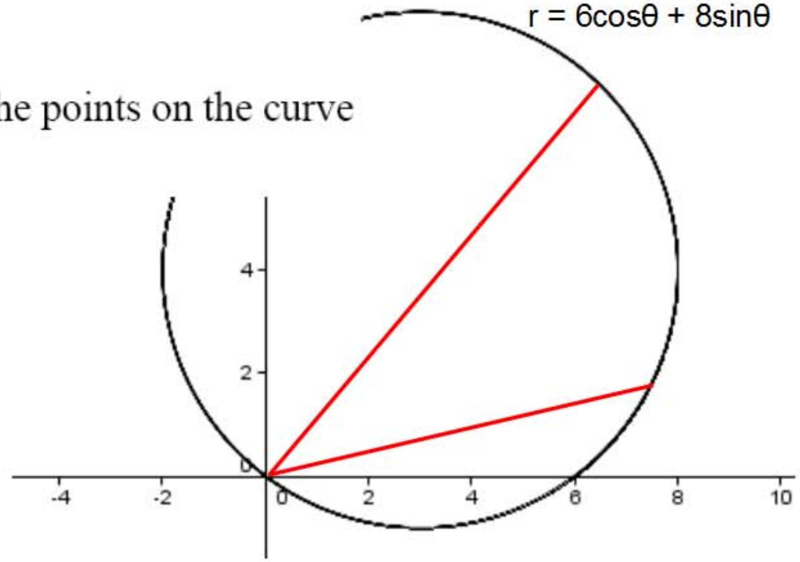
Consider the curve

$$r = \mathrm{f}(\theta), \qquad \alpha \le \theta \le \beta.$$

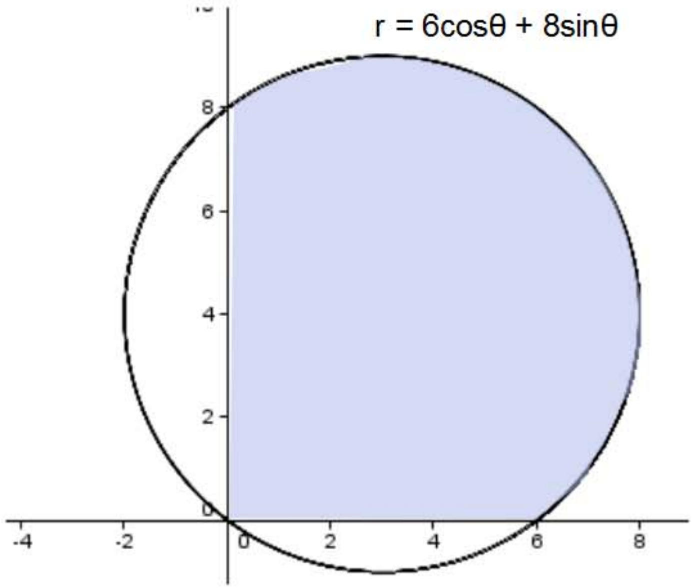
Suppose that $r \ge 0$ throughout the interval $\alpha \le \theta \le \beta$. Let P and Q be the points on the curve at which $\theta = \alpha$ and $\theta = \beta$, respectively.

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, \mathrm{d}\theta$$

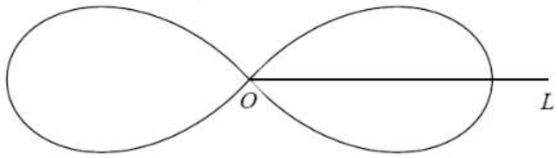
in formulae booklet



Work out the are of the shaded shape



Find the total area of the two loops of the curve $r = a\cos 2\theta$, where $a > 0$ and $-\pi < \theta \le \pi$.



$\frac{\pi a^2}{4}$

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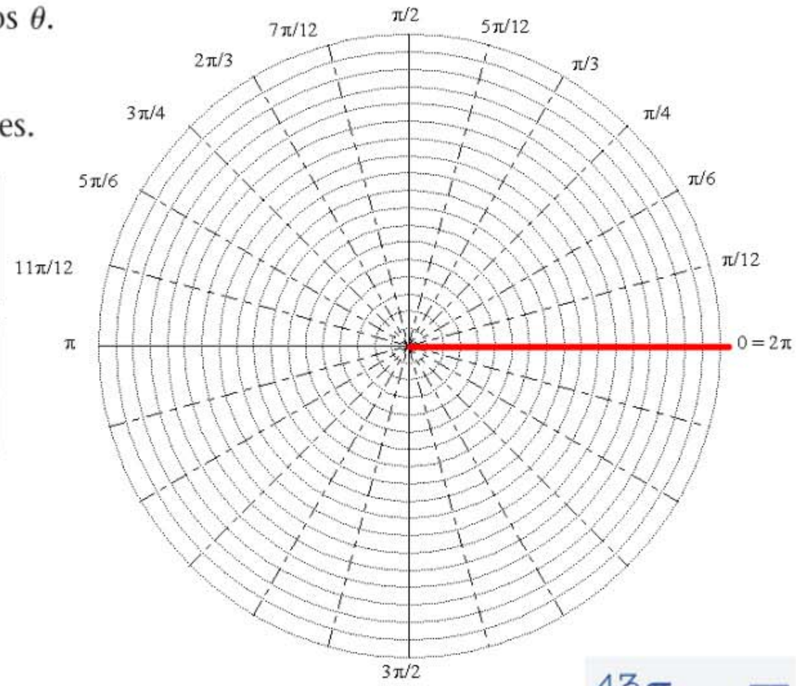
a On the same diagram sketch the curves with equations $r = 2 + \cos\theta$, $r = 5\cos\theta$.

b Find the polar coordinates of the points of intersection of these two curves.

c Find the exact value of the area of the finite region bounded by these two curves.

θ										
r										

θ										
r										



$\frac{43\pi}{12} - \sqrt{3}$

Exercises:

Find the area of the finite region bounded by the curve with the given polar equation and the half-lines $\theta = \alpha$ and $\theta = \beta$.

1 $r = a\cos\theta$, $\alpha = 0, \beta = \frac{\pi}{2}$

2 $r = a(1 + \sin\theta)$, $\alpha = -\frac{\pi}{2}, \beta = \frac{\pi}{2}$

3 $r = a\sin 3\theta$, $\alpha = \frac{\pi}{6}, \beta = \frac{\pi}{4}$

4 $r^2 = a^2\cos 2\theta$, $\alpha = 0, \beta = \frac{\pi}{4}$

5 $r^2 = a^2\tan\theta$, $\alpha = 0, \beta = \frac{\pi}{4}$

6 $r = 2a\theta$, $\alpha = 0, \beta = \pi$

7 $r = a(3 + 2\cos\theta)$, $\alpha = 0, \beta = \frac{\pi}{2}$

8 Show that the area enclosed by the curve with polar equation $r = a(p + q\cos\theta)$ is $\frac{2p^2 + q^2}{2}\pi a^2$.

9 Find the area of a single loop of the curve with equation $r = a\cos 3\theta$.

10 Find the finite area enclosed between $r = a\sin 4\theta$ and $r = a\sin 2\theta$ for $0 \leqslant \theta \leqslant \frac{\pi}{2}$.

11 Find the area of the finite region R enclosed by the curve with equation $r = (1 + \sin\theta)$ that lies entirely within the curve with equation $r = 3\sin\theta$.

Answers

1	$\frac{\pi a^2}{8}$	**2**	$\frac{3\pi a^2}{4}$
3	$\frac{(\pi + 2)a^2}{48}$	**4**	$\frac{a^2}{4}$
5	$\frac{a^2\ln\sqrt{2}}{2}$ or $\frac{a^2\ln 2}{4}$	**6**	$\frac{2a^2\pi^3}{3}$
7	$\frac{a^2}{4}(11\pi + 24)$	**8**	$\frac{a^2(2p^2 + q^2)\pi}{2}$
9	$\frac{\pi a^2}{12}$	**10**	$\frac{a^2}{4}\left[\frac{\pi}{4} - \frac{3\sqrt{3}}{16}\right]$
11	$\frac{5\pi}{4}$		

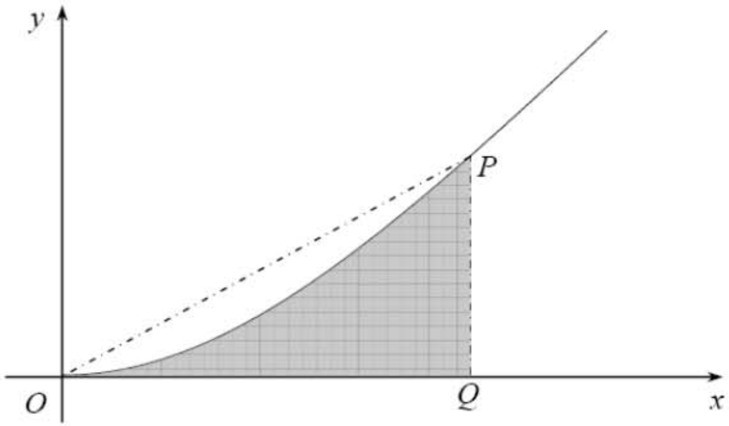
Miscellaneous questions:

In terms of polar coordinates (r,θ), the equation of a curve C is

$$r = \tan 2\theta \quad , \quad 0 \le \theta < \frac{\pi}{4}.$$

(a) Write down expressions in terms of θ for the Cartesian coordinates (x, y) of a general point on C.

(b)



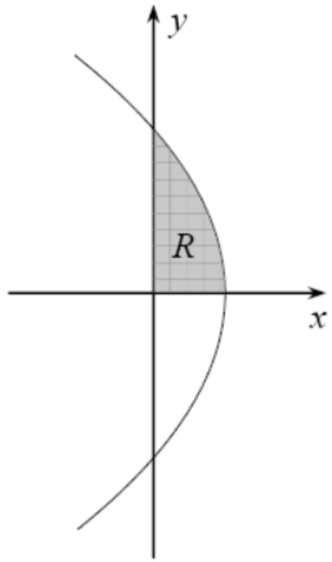
The diagram shows a sketch of part of the curve C. The point P lies on the curve and is such that $\angle POQ = \frac{\pi}{6}$, where Q is the foot of the perpendicular from P to the x-axis.

(i) Find the exact value of the area of the triangle OPQ.

(ii) Show that the area of the shaded region bounded by OQ, PQ and the arc of the curve between O and P is

$$\frac{\pi}{12} + \frac{\sqrt{3}}{8}.$$

The diagram shows a sketch of the curve $y^2 = 4(1-x)$.



(a) Show that the area of region R bounded by the axes and the curve is $\frac{4}{3}$.

(b) (i) Show that the equation of the curve can be expressed as

$$x^2 + y^2 = (2-x)^2 .$$

(ii) Hence, obtain the polar equation of the above curve in the form $r = \mathrm{f}(\theta)$.

(c) Hence, or otherwise, show that

$$\int_0^{\frac{1}{2}\pi} \frac{\mathrm{d}\theta}{(1+\cos\theta)^2} = \frac{2}{3}.$$