Differential equations
Numerical methods

Specification

Differential Equations –
First Order

Numerical methods for the solution of differential equations of the form $\frac{dy}{dx} = f(x, y)$.

Euler's formula and extensions to second order methods for this first order differential equation.

Formulae to be used will be stated explicitly in questions, but candidates should be familiar with standard notation such as used in Euler's formula $y_{r+1} = y_r + h\,f(x_r, y_r)$,

the formula $y_{r+1} = y_{r-1} + 2h\,f(x_r, y_r)$,

and the formula $y_{r+1} = y_r + \frac{1}{2}(k_1 + k_2)$

where $k_1 = h\,f(x_r, y_r)$ and $k_2 = h\,f(x_r + h, y_r + k_1)$.

In the formulae book

Numerical solution of differential equations

For $\frac{dy}{dx} = f(x)$ and small h, recurrence relations are:

Euler's method: $y_{n+1} = y_n + h\,f(x_n)$; $x_{n+1} = x_n + h$

For $\frac{dy}{dx} = f(x, y)$:

Euler's method: $y_{r+1} = y_r + h\,f(x_r, y_r)$

Improved Euler method: $y_{r+1} = y_r + \frac{1}{2}(k_1 + k_2)$, where $k_1 = h\,f(x_r, y_r)$, $k_2 = h\,f(x_r + h, y_r + k_1)$

Euler formula

Consider the differential equation : $\frac{dy}{dx} = x + y$

a) Find the general solution of this equation.

b) Work out the solution with the condition $y(0) = 1$.

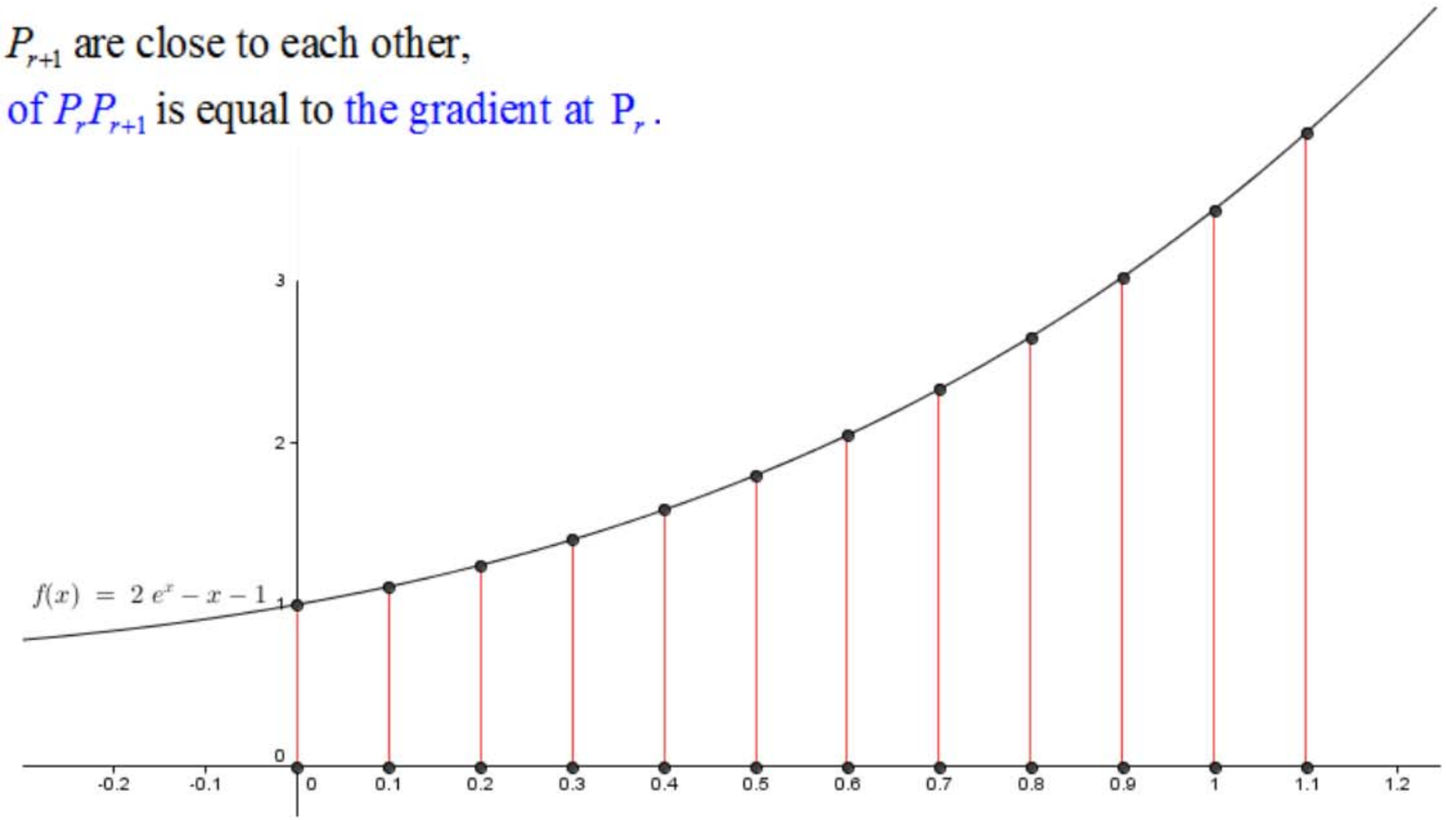
We are now going to consider a numerical method to solve this equation

Notation: $\frac{dy}{dx} = x + y$ *is noted* $\frac{dy}{dx} = f(x, y)$ *with* $f(x, y) = x + y$

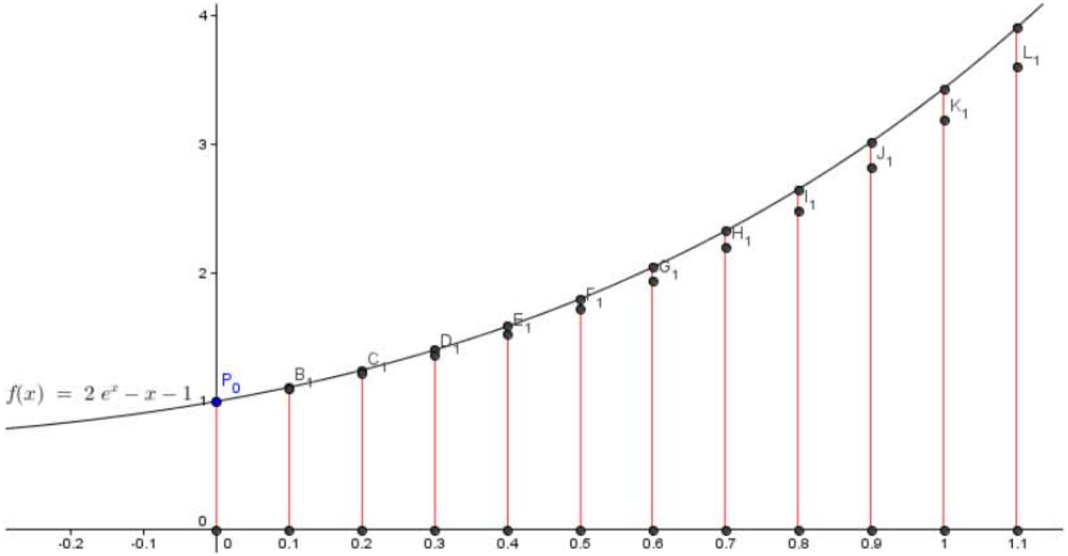
The principle:

Assuming that the two P_r *and* P_{r+1} are close to each other,

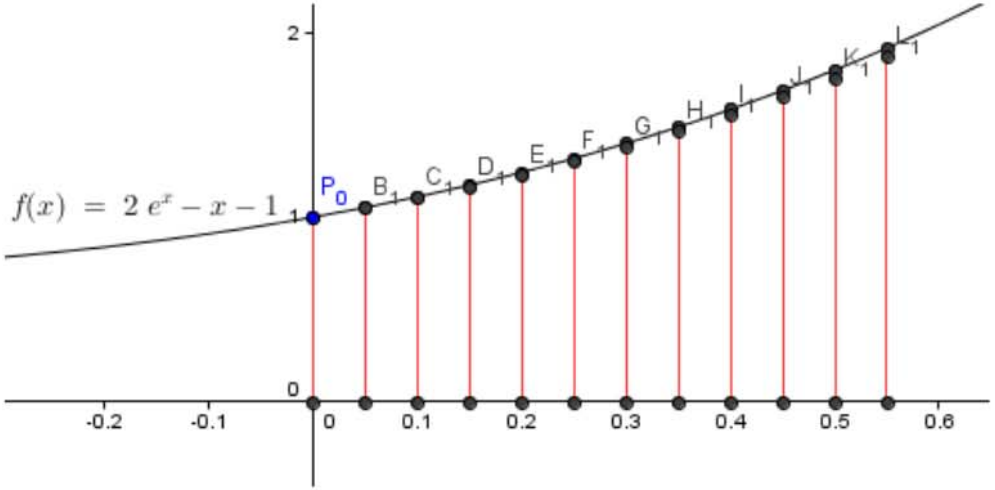
We consider that the gradient of $P_r P_{r+1}$ is equal to the gradient at P_r .



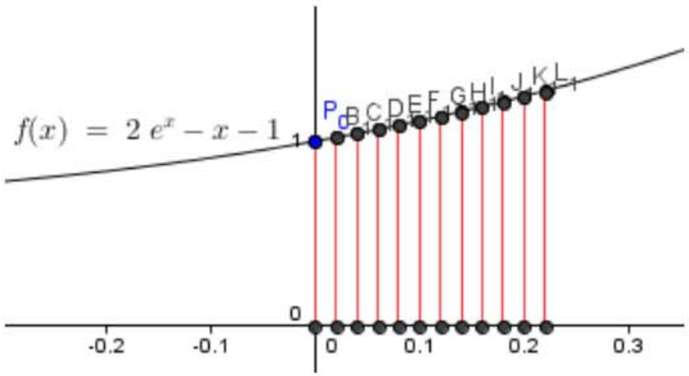
This is the set of the first 12 points obtained with h = 0.1



This is the set of the first 12 points obtained with h = 0.05



This is the set of the first 12 points obtained with h = 0.02



Summary

For $\dfrac{dy}{dx} = f(x, y)$:

Euler's method: $y_{r+1} = y_r + h\,f(x_r, y_r)$

Have a go...

The function $y(x)$ satisfies the differential equation

$$\frac{dy}{dx} = Cos(2x + y)$$

and the condition $y(1) = 0$

Use Euler's formula : $y_{r+1} = y_r + hf(x_r, y_r)$

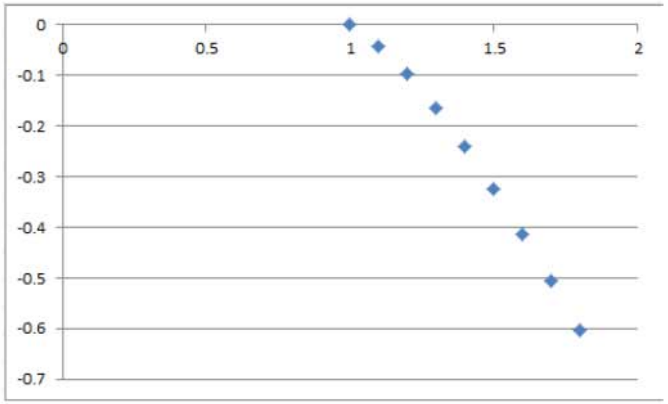
to work out $y(1.1)$, $y(1.2)$, $y(1.3)$ giving your answers to 4 decimal places.

$y_1 = -0.042$

$y_2 = -0.097$

$y_3 = -0.1639$

Done with MSExcel



Exercises:

1. The function $y(x)$ satisfies the differential equation

$$\frac{dy}{dx} = \ln(x+y)$$

and the condition $y(1)=1$.

Use Euler's formula with a step length of 0.1 to obtain approximations to the values of $y(1.1), y(1.2), y(1.3), y(1.4)$ and $y(1.5)$. Give your answers to three decimal places.

2. The function $y(x)$ satisfies the differential equation

$$y\frac{dy}{dx} = x+y$$

and the condition $y(0)=1$.

Use Euler's method with a step length of 0.25 to obtain an estimate of the value of $y(1)$.

Give your answer to three decimal places.

3. The function $y(x)$ satisfies the differential equation

$$\frac{dy}{dx} = \mathrm{f}(x,y)$$

where $$\mathrm{f}(x,y) = \left(e^{x}+e^{y}\right)^{2}.$$

The value of $y(0)$ is zero.

(a) Use of the Euler formula gave $y_1 = 0.05$. Determine the value that was used for the step length h.

(b) Show that, with this value for h, the Euler formula gives $y_2 = 0.103$ to three decimal places.

(c) Calculate the value of y_3 to three decimal places.

1. 1.069, 1.147, 1.232, 1.325, 1.425

2. $y(1) = 2.230$

3. (a) $h = 0.0125$ (c) $y_3 = 0.160$

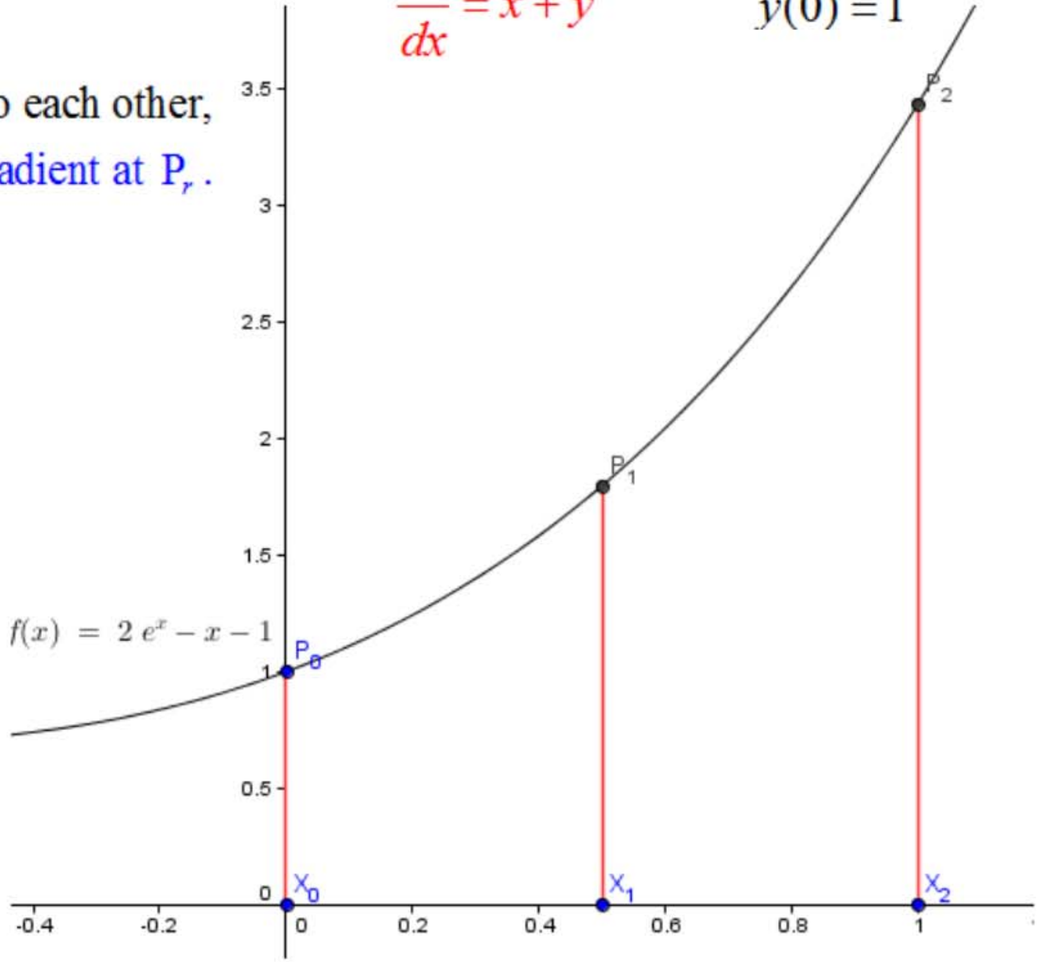
The mid-point formula

The principal:

Assuming that the three points P_{r-1}, P_r *and* P_{r+1} are close to each other,

We consider that the gradient of $P_{r-1}P_{r+1}$ is equal to the gradient at P_r.

$$\frac{dy}{dx} = x + y \qquad y(0) = 1$$



Summary: The mid-point formula

the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x, y)$

$$y_{r+1} = y_{r-1} + 2h\,\mathrm{f}(x_r, y_r)$$

As with Euler's formula, the mid-point formula can be used to calculate values of y successively, but at each stage the previous **two** values of y are required. In particular, putting $r = 1$ gives

$$y_2 = y_0 + 2h\,\mathrm{f}(x_1, y_1).$$

Since only the value of y_0 is known initially, it is necessary to calculate y_1 by some other method before the application of the mid-point formula can begin. Euler's formula may be used for this purpose.

for a given step length h, the mid-point formula is more accurate than Euler's formula.

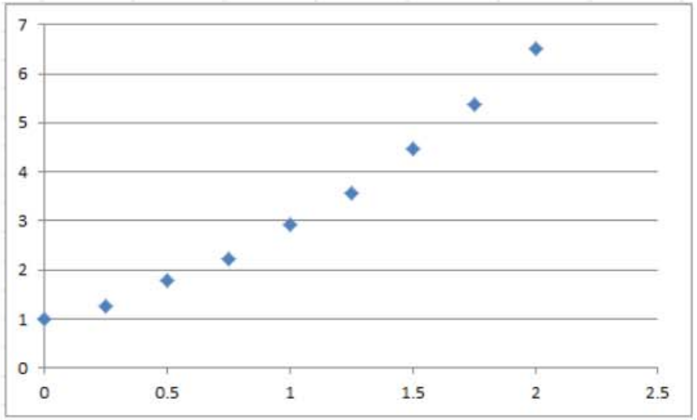
Let's try the formula:

The function $y(x)$ satisfies the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + \sqrt{xy}$$

and the condition $y(0) = 1$.

Use the mid-point formula with a step length of 0.25 to obtain an approximate value for $y(1)$ to three decimal places. Take $y(0.25)$ to be the value given by Euler's formula.



$y(0.25) = 1.25$

$y(0.5) = 1.77951$

$y(0.75) = 2.22163$

$y(1) = 2.925$

Exercises:

1. The function $y(x)$ satisfies the differential equation

$$\frac{dy}{dx} = x^3 + y^3$$

and the condition $y(1) = 1$.

(a) Using a step length of 0.1 in the Euler formula

$$y_{r+1} = y_r + h f(x_r, y_r),$$

obtain estimates of the values of y when $x = 1.1$ and $x = 1.2$.

(b) Use the mid-point formula,

$$y_{r+1} = y_{r-1} + 2h f(x_r, y_r),$$

with $h = 0.1$ and your value of y_1 obtained in part (a), to calculate an improved estimate of the value of y when $x = 1.2$.

2. The function $y(x)$ satisfies the differential equation

$$\frac{dy}{dx} = \cos xy$$

and the condition $y(1) = 2$.

(a) Verify that the Euler formula with a step length of 0.25 gives $y(1.25) = 1.89596$ to five decimal places.

(b) Use the mid-point formula with a step length of 0.25 to obtain an estimate of the value of $y(2)$. Give your answer to three decimal places.

3. The function $y(x)$ satisfies the differential equation

$$\frac{dy}{dx} = \left(1 + x^2 + y^2\right)^{\frac{1}{2}}$$

and the condition $y(0) = 0$.

(a) Given that in this case $y_{-1} = -y_1$, show that the mid-point formula gives $y_1 = h$, where h is the step length.

(b) Use the mid-point formula with $h = 0.1$ to obtain an approximation to the value of $y(0.5)$. Give your answer to three decimal places.

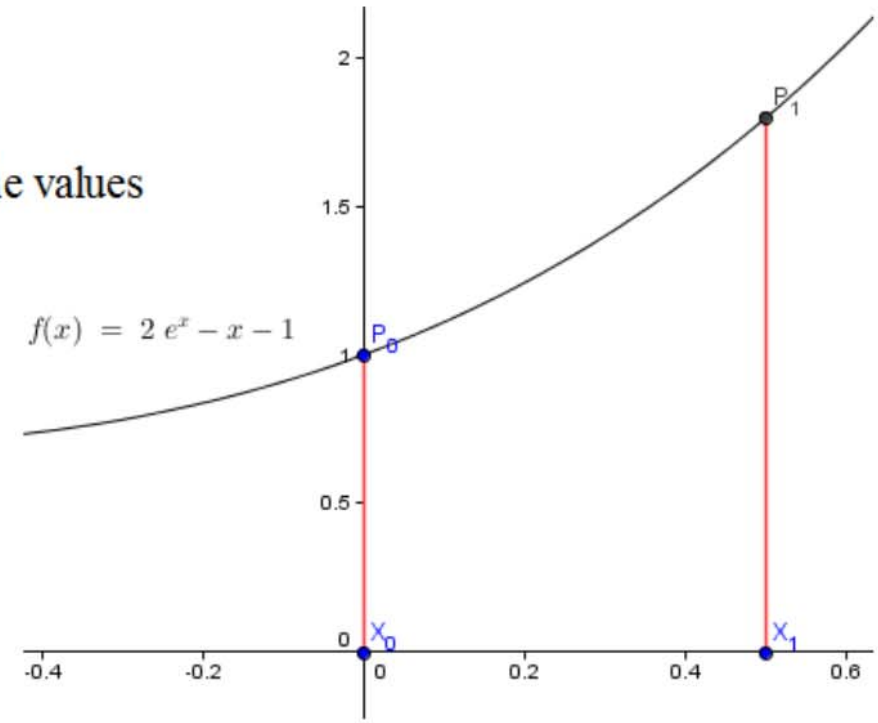
1. (a) 1.2, 1.5059 (b) 1.6118
2. (b) $y(2) = 1.204$
3. (b) $y(0.5) = 0.539$. (The intermediate values are $y_1 = 0.1$, $y_2 = 0.20199$, $y_3 = 0.30792$ and $y_4 = 0.41969$)

The improved Euler Formula

The principle:

This formula is more accurate than Euler's formula:

We assume that the gradient of P_rP_{r+1} is equal to the mean of the values of the gradient at P_r *and* the gradient at P_{r+1}.



Summary: Improved Euler formula

$$y_{r+1} = y_r + \frac{h}{2}\left[f(x_r, y_r) + f\left(x_{r+1}, y^*_{r+1}\right)\right].$$

with $y^*_{r+1} = y_r + h f(x_r, y_r)$.

Other notation (in the exam):

the improved Euler formula

$$y_{r+1} = y_r + \tfrac{1}{2}(k_1 + k_2)$$

where $k_1 = hf(x_r, y_r)$ and $k_2 = hf(x_r + h, y_r + k_1)$

These two formulae are equivalent.

Example:

The function $y(x)$ satisfies the differential equation

$$\frac{dy}{dx} = \ln(x+y), \qquad x \geq 1.$$

The boundary condition is $y(1) = 1$. Use the improved Euler method with a step length of 0.25 to estimate the value of $y(1.5)$ to three decimal places.

Possible layout for your answers:

r	x_r	y_r	k_1	$x_r + h$	$y_r + k_1$	k_2	y_{r+1}
0							
1							
2							

y(1.5)=1.443

Exercises:

1. The function $y(x)$ satisfies the differential equation

$$\frac{dy}{dx} = \ln(xy)$$

and the condition $y(2) = 1$. An estimate of the value of $y(3)$ to three decimal places is required.

Use the improved Euler method with a step length of 0.25 to obtain this estimate

r	x_r	y_r	k_1	$x_r + h$	$y_r + k_1$	k_2	y_{r+1}
0							
1							
2							
3							
4							

y(3)=2.305

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2. The function $y(x)$ satisfies the differential equation

$$\frac{dy}{dx} = e^{x^2 + y^2}$$

and the condition $y(0) = 0$.

Use the improved Euler method with a step length of 0.2 to obtain an approximate value of $y(0.4)$. Give your answer to three decimal places.

r	x_r	y_r	k_1	$x_r + h$	$y_r + k_1$	k_2	y_{r+1}
0							
1							
2							

y(0.4) = 0.458

More exercises:

1. The function $y(x)$ satisfies the differential equation

$$\frac{dy}{dx} = \left(x^3 + y^3\right)^{\frac{1}{2}}$$

and the condition $\quad y(1) = 0.5.$

(a) Use the Euler formula,

$$y_{r+1} = y_r + h\,f(x_r, y_r),$$

with $h = 0.1$, to show that $y(1.1) = 0.606$ to three decimal places.

(b) Use the mid-point formula,

$$y_{r+1} = y_{r-1} + 2h\,f(x_r, y_r),$$

with $h = 0.1$, to find an approximate value for $y(1.3)$ giving your answer to three decimal places.

[AQA, 2001]

2. The function $y(x)$ satisfies the differential equation

$$\frac{dy}{dx} = (x + y - 4)\,y$$

and the condition $\quad y(1) = 2.$

(a) Use the Euler formula,

$$y_{r+1} = y_r + h\,f(x_r, y_r),$$

to show that

$$y(1+h) \approx 2(1-h).$$

(b) (i) Denoting the value for $y(1+h)$ obtained in part (a) by y_1^*, show that

$$f\left(x_1, y_1^*\right) = -2\left(1-h^2\right).$$

(ii) Hence obtain an improved estimate for the value of $y(1+h)$ using the formula

$$y_{r+1} = y_r + \tfrac{1}{2}h\left[f(x_r, y_r) + f\left(x_{r+1}, y_{r+1}^*\right)\right]$$

giving your answer in the form

$$a(1-h) + bh^3,$$

where a and b are numbers to be found.

[AQA, 2000]

3. (a) (i) Show that the integrating factor of the differential equation

$$\frac{dy}{dx} - \frac{x}{1-x^2}\,y = \frac{1}{1-x^2}, \qquad |x| < 1,$$

is $\sqrt{1-x^2}$.

(ii) Hence, or otherwise, solve the differential equation given that $x = 0$ when $y = 0$.

(iii) Show that when $x = 0.5$, $y = \dfrac{\pi}{3\sqrt{3}}$.

(b) The above differential equation may be written as

$$\frac{dy}{dx} = f(x, y),$$

where

$$f(x, y) = \frac{1+xy}{1-x^2}.$$

The table below shows approximate values of y obtained using the improved Euler formula

$$y_{r+1} = y_r + \tfrac{1}{2}(k_1 + k_2),$$

where $\quad k_1 = h\,f(x_r, y_r)$

and $\quad k_2 = h\,f(x_r + h, y_r + k_1).$

x	0	0.1	0.2	0.3	0.4
y	0	0.1010101	0.2062236	0.3205873	0.4508417

Use the improved Euler formula, with $h = 0.1$, to calculate an approximate value of y at $x = 0.5$ giving your answer to five decimal places.

(c) Use the answer to part (a)(iii) to find the percentage error in your answer to part (b).

[AQA, 2001]

1. (b) $y(1.3) = 0.899$

2. (b)(ii) $a = 2,\ b = 1$

3. (a)(ii) $y = \dfrac{\sin^{-1} x}{\sqrt{1-x^2}}$ (b) 0.60748 (c) 0.48% (approximately)