De Moivre's theorem – exam questions

Question 1: Jan 2007

(b) Find the value of $\left(\cos\frac{\pi}{6} + \mathrm{i}\sin\frac{\pi}{6}\right)^6$. *(2 marks)*

(c) Show that

$$(\cos\theta + \mathrm{i}\sin\theta)(1 + \cos\theta - \mathrm{i}\sin\theta) = 1 + \cos\theta + \mathrm{i}\sin\theta$$ *(3 marks)*

(d) Hence show that

$$\left(1 + \cos\frac{\pi}{6} + \mathrm{i}\sin\frac{\pi}{6}\right)^6 + \left(1 + \cos\frac{\pi}{6} - \mathrm{i}\sin\frac{\pi}{6}\right)^6 = 0$$ *(4 marks)*

Question 2: June 2007

Use De Moivre's Theorem to find the smallest positive angle θ for which

$$(\cos\theta + \mathrm{i}\sin\theta)^{15} = -\mathrm{i}$$ *(5 marks)*

Question 3: Jan 2006

It is given that $z = e^{i\theta}$.

$a)\,i)$ Show that $z + \frac{1}{z} = 2\cos\theta$ (2 *marks*)

$ii)$ Find a similar expression for $z^2 + \frac{1}{z^2}$ (2 *marks*)

$iii)$ Hence show that $z^2 - z + 2 - \frac{1}{z} + \frac{1}{z^2} = 4\cos^2\theta - 2\cos\theta$ (3 *marks*)

$b)$ Hence solve the quartic equation $z^4 - z^3 + 2z^2 - z + 1 = 0$

giving the roots in the form $a + ib$. (5 *marks*)

Question 4: Jan 2010

(a) (i) Show that $\omega = \mathrm{e}^{\frac{2\pi\mathrm{i}}{7}}$ is a root of the equation $z^7 = 1$. *(1 mark)*

(ii) Write down the five other non-real roots in terms of ω. *(2 marks)*

(b) Show that

$$1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 = 0$$ *(2 marks)*

(c) Show that:

(i) $\omega^2 + \omega^5 = 2\cos\frac{4\pi}{7}$; *(3 marks)*

(ii) $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = -\frac{1}{2}$. *(4 marks)*

Question 5: June 2009

Given that $z = 2e^{\frac{\pi i}{12}}$ satisfies the equation

$$z^4 = a(1 + \sqrt{3}\,i)$$

where a is real:

(a) find the value of a; *(3 marks)*

(b) find the other three roots of this equation, giving your answers in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. *(5 marks)*

Question 6: Jan 2009

(a) Show that

$$(z^4 - e^{i\theta})(z^4 - e^{-i\theta}) = z^8 - 2z^4\cos\theta + 1$$ *(2 marks)*

(b) Hence solve the equation

$$z^8 - z^4 + 1 = 0$$

giving your answers in the form $e^{i\phi}$, where $-\pi < \phi \leqslant \pi$. *(6 marks)*

(c) Indicate the roots on an Argand diagram. *(3 marks)*

Question 7: June 2006

(a) Find the six roots of the equation $z^6 = 1$, giving your answers in the form $e^{i\phi}$, where $-\pi < \phi \leqslant \pi$. *(3 marks)*

(b) It is given that $w = e^{i\theta}$, where $\theta \neq n\pi$.

(i) Show that $\dfrac{w^2 - 1}{w} = 2i\sin\theta$. *(2 marks)*

(ii) Show that $\dfrac{w}{w^2 - 1} = -\dfrac{i}{2\sin\theta}$. *(2 marks)*

(iii) Show that $\dfrac{2i}{w^2 - 1} = \cot\theta - i$. *(3 marks)*

(iv) Given that $z = \cot\theta - i$, show that $z + 2i = zw^2$. *(2 marks)*

(c) (i) Explain why the equation

$$(z + 2i)^6 = z^6$$

has five roots. *(1 mark)*

Question 8: Jan 2008

(a) Express $4+4\mathrm{i}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\leqslant\pi$. *(3 marks)*

(b) Solve the equation

$$z^5 = 4+4\mathrm{i}$$

giving your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\leqslant\pi$. *(5 marks)*

Question 9: June 2008

(a) (i) Expand

$$\left(z+\frac{1}{z}\right)\left(z-\frac{1}{z}\right)$$ *(1 mark)*

(ii) Hence, or otherwise, expand

$$\left(z+\frac{1}{z}\right)^4\left(z-\frac{1}{z}\right)^2$$ *(3 marks)*

(b) (i) Use De Moivre's theorem to show that if $z=\cos\theta+\mathrm{i}\sin\theta$ then

$$z^n+\frac{1}{z^n}=2\cos n\theta$$ *(3 marks)*

(ii) Write down a corresponding result for $z^n-\dfrac{1}{z^n}$. *(1 mark)*

(c) Hence express $\cos^4\theta\sin^2\theta$ in the form

$$A\cos 6\theta + B\cos 4\theta + C\cos 2\theta + D$$

where A, B, C and D are rational numbers. *(4 marks)*

(d) Find $\displaystyle\int \cos^4\theta\sin^2\theta\ \mathrm{d}\theta$. *(2 marks)*

De Moivre's theorem – exam questions - answers

Question 1: Jan 2007

$$b)\left(Cos\frac{\pi}{6}+iSin\frac{\pi}{6}\right)^6=Cos\frac{6\pi}{6}+iSin\frac{6\pi}{6}=Cos\pi+iSin\pi=-1$$

$$c)(Cos\theta+i\sin\theta)(1+Cos\theta-i\sin\theta)=(Cos\theta+iSin\theta)+(Cos\theta+iSin\theta)(Cos\theta-i\sin\theta)=$$

$$(Cos\theta+iSin\theta)+Cos^2\theta+Sin^2\theta=Cos\theta+iSin\theta+1$$

$$d)(1+Cos\frac{\pi}{6}+iSin\frac{\pi}{6})^6+(1+Cos\frac{\pi}{6}-iSin\frac{\pi}{6})^6=\left((Cos\frac{\pi}{6}+i\sin\frac{\pi}{6})(1+Cos\frac{\pi}{6}-i\sin\frac{\pi}{6})\right)^6+(1+Cos\frac{\pi}{6}-iSin\frac{\pi}{6})^6=$$

$$(Cos\frac{\pi}{6}+iSin\frac{\pi}{6})^6\left(1+Cos\frac{\pi}{6}-iSin\frac{\pi}{6}\right)^6+(1+Cos\frac{\pi}{6}-iSin\frac{\pi}{6})^6=-1\times\left(1+Cos\frac{\pi}{6}-iSin\frac{\pi}{6}\right)^6+(1+Cos\frac{\pi}{6}-iSin\frac{\pi}{6})^6=0$$

Question 2: June 2007

$$\left(Cos\theta+iSin\theta\right)^{15}=Cos\left(15\theta\right)+iSin\left(15\theta\right)=0-i$$

$$Cos(15\theta)=0\;\; and\;\; Sin(15\theta)=-1$$

$$so\;\; 15\theta=\frac{3\pi}{2}\qquad\qquad \theta=\frac{3\pi}{30}=\frac{\pi}{10}$$

Question 3: Jan 2006

$$z=e^{i\theta}$$

$$a)\,i)\; z+\frac{1}{z}=e^{i\theta}+\frac{1}{e^{i\theta}}=e^{i\theta}+e^{-i\theta}$$

$$=Cos\theta+iSin\theta+Cos\theta-iSin\theta\qquad z+\frac{1}{z}=2Cos\theta$$

$$ii)\; z^2+\frac{1}{z^2}=e^{i2\theta}+\frac{1}{e^{i2\theta}}=e^{i2\theta}+e^{-i2\theta}$$

$$=Cos2\theta+iSin2\theta+Cos2\theta-iSin2\theta\qquad z^2+\frac{1}{z^2}=2Cos2\theta$$

$$iii)\; z^2-z+2-\frac{1}{z}+\frac{1}{z^2}=2Cos2\theta-2Cos\theta+2$$

$$we\,know\,that\;\; Cos2\theta=2Cos^2\theta-1\;\; so$$

$$z^2-z+2-\frac{1}{z}+\frac{1}{z^2}=2(2Cos^2\theta-1)-2Cos\theta+2$$

$$z^2-z+2-\frac{1}{z}+\frac{1}{z^2}=4Cos^2\theta-2Cos\theta$$

$$b)\; z^4-z^3+2z^2-z+1=0\qquad \text{factorise by } z^2$$

$$(z=0\,is\,not\,a\,solution)$$

$$z^2(z^2-z+2-\frac{1}{z}+\frac{1}{z^2})=0\;\; This\;\; gives\quad z^2-z+2-\frac{1}{z}+\frac{1}{z^2}=0$$

$$4Cos^2\theta-2Cos\theta=0$$

$$2Cos\theta(2Cos\theta-1)=0$$

$$Cos\theta=0\;\; or\;\; Cos\theta=\frac{1}{2}$$

$$\theta=\frac{\pi}{2}\; or\;\; \theta=-\frac{\pi}{2}\; or\;\; \theta=\frac{\pi}{3}\; or\; \theta=-\frac{\pi}{3}$$

$$z=e^{\pm i\frac{\pi}{2}}=\pm i\;\; or\;\; z=e^{\pm i\frac{\pi}{3}}=\frac{1}{2}\pm i\frac{\sqrt{3}}{2}$$

Question 4: Jan 2010

$a)\, i)\ \omega = e^{i\frac{2\pi}{7}}$

$$\text{so } \omega^7 = \left(e^{i\frac{2\pi}{7}}\right)^7 = e^{2i\pi} = \cos 2\pi + i\sin 2\pi = 1$$

ω is a solution of $z^7 = 1$

$ii)\ 7\theta = k\times 2\pi \qquad \theta = k\times\frac{2\pi}{7}$

the other non-real solutions are

$$\text{for } k=2,\ e^{i\frac{4\pi}{7}} = \omega^2 \qquad \text{for } k=3,\ e^{i\frac{6\pi}{7}} = \omega^3$$

$$\text{for } k=4,\ e^{i\frac{8\pi}{7}} = \omega^4 \qquad \text{for } k=5,\ e^{i\frac{10\pi}{7}} = \omega^5$$

$$\text{for } k=6,\ e^{i\frac{12\pi}{7}} = \omega^6$$

$b)\ 1+\omega+\omega^2+\omega^3+\omega^4+\omega^5+\omega^6$

is a geometric series with common ration ω

$$1+\omega+\omega^2+\omega^3+\omega^4+\omega^5+\omega^6 = \frac{1-\omega^7}{1-\omega} = \frac{1-1}{1-\omega} = 0$$

$c)\, i)\ \omega^2+\omega^5 = \omega^2+\omega^{-2} = e^{i\frac{4\pi}{7}} + e^{-\frac{4\pi}{7}} = 2\cos\frac{4\pi}{7}$

$ii)\ 1+\omega+\omega^2+\omega^3+\omega^4+\omega^5+\omega^6 = 0$

$$\omega+\omega^2+\omega^3+\omega^{-3}+\omega^{-2}+\omega^{-1} = -1$$

$$\omega+\omega^{-1}+\omega^2+\omega^{-2}+\omega^3+\omega^{-3} = -1$$

$$2\cos\frac{2\pi}{7}+2\cos\frac{4\pi}{7}+2\cos\frac{6\pi}{7} = -1$$

$$\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7} = -\frac{1}{2}$$

Question 5: June 2009

$a)\ z = 2e^{i\frac{\pi}{12}} \text{ so } z^4 = \left(2e^{i\frac{\pi}{12}}\right)^4 = 16e^{i\frac{\pi}{3}} \qquad z^4 = 16\left(\cos\frac{\pi}{3}+iSin\frac{\pi}{3}\right) = 16\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)$

$z^4 = 8(1+i\sqrt{3}) \qquad a = 8$

$b)$ let's write $z = re^{i\theta}$, $z^4 = r^4e^{i4\theta}$

z is a solution of this equation when

$$r^4 = 16 \text{ and } 4\theta = \frac{\pi}{3}+k2\pi$$

$$r = 2 \text{ and } \theta = \frac{\pi}{12}+k\frac{\pi}{2} \quad k = -2,-1,0,1$$

Solutions are : $2e^{i\frac{\pi}{12}}$, $2e^{-i\frac{5\pi}{12}}$, $2e^{-i\frac{11\pi}{12}}$, $2e^{i\frac{7\pi}{12}}$

Question 6: Jan 2009

$$a)\ \left(z^4-e^{i\theta}\right)\left(z^4-e^{-i\theta}\right) = z^8 - z^4e^{-i\theta} - z^4e^{i\theta}+1 = z^8 - z^4\left(e^{i\theta}+e^{-i\theta}\right)+1$$

$$= z^8 - z^4\times 2\cos\theta + 1 = z^8 - 2z^4\cos\theta + 1$$

$b)$ for $\cos\theta = \frac{1}{2}$ $(\theta = \frac{\pi}{3})$, $z^8 - 2z^4\cos\theta+1$ becomes $z^8 - z^4 + 1 = 0$

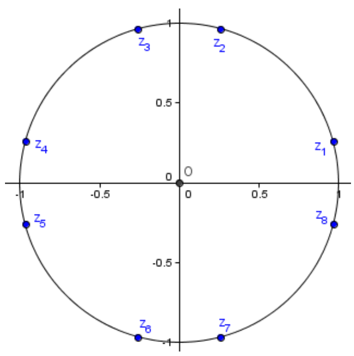
We can factorise as $\left(z^4 - e^{i\frac{\pi}{3}}\right)\left(z^4 - e^{-i\frac{\pi}{3}}\right) = 0$

We need to solve $z^4 = e^{\pm i\frac{\pi}{3}}$

$$(re^{i\phi})^4 = e^{\pm i\frac{\pi}{3}} \qquad r^4e^{4i\phi} = e^{\pm i\frac{\pi}{3}}$$

$$r = 1 \text{ or } 4\phi = \pm\frac{\pi}{3}+k\times 2\pi$$

$$\phi = \pm\frac{\pi}{12}+\frac{k\pi}{2}$$



This gives $z = e^{\pm i\frac{11\pi}{12}}$, $e^{\pm i\frac{5\pi}{12}}$, $e^{\pm i\frac{\pi}{12}}$, $e^{\pm i\frac{9\pi}{12}}$

$c)$

Question 7: June 2006

a) Let note $z = re^{i\phi}$ then $z^6 = r^6 \times e^{i6\phi}$ and $1 = 1e^{i0}$

The equation $z^6 = 1$ is equivalent to $r^6 \times e^{i6\phi} = 1 \times e^{i0}$

This gives $r^6 = 1 \quad r = 1$

$$6\phi = 0 + k2\pi \qquad -3 < k \le 3$$

$$\phi = k\frac{\pi}{3} \qquad -2 \le k \le 3$$

so $z = e^{-i\frac{2\pi}{3}}$ or $e^{-i\frac{\pi}{3}}$ or e^{i0} or $e^{i\frac{\pi}{3}}$ or $e^{i\frac{2\pi}{3}}$ or $e^{i\pi}$

b) i) $\dfrac{w^2-1}{w} = \dfrac{e^{i2\theta}-1}{e^{i\theta}} = \dfrac{e^{i\theta}(e^{i\theta}-e^{-i\theta})}{e^{i\theta}} = e^{i\theta} - e^{-i\theta} = 2iSin\theta$

ii) $\dfrac{w}{w^2-1} = \dfrac{1}{2iSin\theta} = \dfrac{1}{2iSin\theta} \times \dfrac{i}{i} = -\dfrac{i}{2Sin\theta}$

iii) $\dfrac{2i}{w^2-1} = \dfrac{2i}{2iwSin\theta} = \dfrac{1}{e^{i\theta} \times Sin\theta} = \dfrac{e^{-i\theta}}{Sin\theta} = \dfrac{Cos\theta - iSin\theta}{Sin\theta}$

$$= \frac{Cos\theta}{Sin\theta} - i\frac{Sin\theta}{Sin\theta} = Cot\theta - i$$

iv) $z = Cot\theta - i$ so $\dfrac{2i}{w^2-1} = z$

$$2i = zw^2 - z$$

$$z + 2i = zw^2$$

c) i) $(z+2i)^6 = z^6$ is equivalent to order 5 polynomial=0

(the term in z^6 cancel out)

ii) $(z+2i)^6 = z^6 \qquad \left(\dfrac{z+2i}{z}\right)^6 = 1 \qquad (w^2)^6 = 1$

So $w^2 = e^{i\frac{\pi k}{3}}$ (question a) $\qquad w = e^{i\frac{\pi k}{6}}$

This gives $z = \cot 0 - i$, $\cot\dfrac{\pi}{6} - i$, $\cot\dfrac{\pi}{3} - i$, $\cot\dfrac{2\pi}{3} - i$, $\cot\dfrac{5\pi}{6} - i$

$z = -i$, $\sqrt{3} - i$, $\dfrac{\sqrt{3}}{3} - i$, $-\dfrac{\sqrt{3}}{3} - i, -\sqrt{3} - i$

Question 8: Jan 2008

a) $4 + 4i = 4\sqrt{2}\left(\dfrac{1}{\sqrt{2}} + i\dfrac{1}{\sqrt{2}}\right) = 4\sqrt{2}e^{i\frac{\pi}{4}}$

b) Let's write $z^5 = (re^{i\theta})^5 = r^5 e^{i5\theta}$

$z^5 = 4 + 4i$ becomes

$$r^5 e^{i5\theta} = 4\sqrt{2}e^{i\frac{\pi}{4}}$$

so $r^5 = 4\sqrt{2}$ and $5\theta = \dfrac{\pi}{4} + k \times 2\pi$

$$r = \sqrt{2} \text{ and } \theta = \frac{\pi}{20} + k \times \frac{2\pi}{5} \quad k = -2, -1, 0, 1, 2$$

$$r = \sqrt{2} \text{ and } \theta = -\frac{15\pi}{20}, -\frac{7\pi}{20}, \frac{\pi}{20}, \frac{9\pi}{20}, \frac{17\pi}{20}$$

The 5^{th} roots of $4 + 4i$ are:

$\sqrt{2}e^{-i\frac{3\pi}{4}}$, $\sqrt{2}e^{-i\frac{7\pi}{20}}$, $\sqrt{2}e^{i\frac{\pi}{20}}$, $\sqrt{2}e^{i\frac{9\pi}{20}}$, $\sqrt{2}e^{i\frac{17\pi}{20}}$

Question 9: June 2008

$$a)\,i)\left(z+\frac{1}{z}\right)\left(z-\frac{1}{z}\right)=z^2-1+1-\frac{1}{z^2}=z^2-\frac{1}{z^2}$$

$$ii)\left(z+\frac{1}{z}\right)^4\left(z-\frac{1}{z}\right)^2=\left(z+\frac{1}{z}\right)^2\times\left[\left(z+\frac{1}{z}\right)\left(z-\frac{1}{z}\right)\right]^2$$

$$=\left(z^2+\frac{1}{z^2}+2\right)\left(z^2-\frac{1}{z^2}\right)^2$$

$$=\left(z^2+\frac{1}{z^2}+2\right)\left(z^4+\frac{1}{z^4}-2\right)$$

$$=z^6+\frac{1}{z^2}-2z^2+z^2+\frac{1}{z^6}-\frac{2}{z^2}+2z^4+\frac{2}{z^4}-4$$

$$=\left(z^6-\frac{1}{z^6}\right)-\left(z^2+\frac{1}{z^2}\right)+2\left(z^4+\frac{1}{z^4}\right)-4$$

$$b)\,i)\ z^n+\frac{1}{z^n}=(\cos n\theta+i\sin n\theta)+(\cos n\theta-i\sin n\theta)=2\cos n\theta$$

$$ii)\ z^n-\frac{1}{z^n}=(\cos n\theta+i\sin n\theta)-(\cos n\theta-i\sin n\theta)=2i\sin n\theta$$

$$c)\cos^4\theta\sin^2\theta=\left(\frac{1}{2^4}\left(z+\frac{1}{z}\right)^4\right)\left(\frac{1}{(2i)^2}\left(z-\frac{1}{z}\right)^2\right)$$

$$=-\frac{1}{64}\left(z+\frac{1}{z}\right)^4\left(z-\frac{1}{z}\right)^2$$

$$=-\frac{1}{64}\left[\left(z^6-\frac{1}{z^6}\right)-\left(z^2+\frac{1}{z^2}\right)+2\left(z^4+\frac{1}{z^4}\right)-4\right]$$

$$=-\frac{1}{64}(2\cos 6\theta-2\cos 2\theta+4\cos 4\theta-4)$$

$$=-\frac{1}{32}\cos 6\theta-\frac{1}{16}\cos 4\theta+\frac{1}{32}\cos 2\theta+\frac{1}{16}$$

$$d)\int\cos^4\theta\sin^2\theta\,d\theta=\int-\frac{1}{32}\cos 6\theta-\frac{1}{16}\cos 4\theta+\frac{1}{32}\cos 2\theta+\frac{1}{16}d\theta$$

$$=-\frac{1}{192}\sin 6\theta-\frac{1}{64}\sin 4\theta+\frac{1}{64}\sin 2\theta+\frac{1}{16}\theta+c$$