Matrix algebra – exam questions

Question 1: June 2010 – Q2

Let $\mathbf{A} = \begin{bmatrix} 1 & x \\ 2 & 3 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 4-4x & 8 \\ 8x-4 & 4 \end{bmatrix}$.

(a) Find $\mathbf{AB}$ in terms of x. *(2 marks)*

(b) Show that $\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}} = \mathbf{C}$ for some value of x. *(5 marks)*

Question 2: June 2008 – Q3

The matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 4 & 3 & k \end{bmatrix}$, where k is a constant.

Determine, in terms of k where appropriate:

(a) det $\mathbf{A}$; *(2 marks)*

(b) $\mathbf{A}^{-1}$. *(5 marks)*

Question 3: June 2007 – Q6

The matrices $\mathbf{A}$ and $\mathbf{B}$ are given by

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & t \end{bmatrix}$$

(a) Find, in terms of t, the matrices:

(i) $\mathbf{AB}$; *(3 marks)*

(ii) $\mathbf{BA}$. *(2 marks)*

(b) Explain why $\mathbf{AB}$ is singular for all values of t. *(1 mark)*

Question 4: June 2011 – Q1

The matrices $\mathbf{A}$ and $\mathbf{B}$ are given in terms of p by

$$\mathbf{A} = \begin{bmatrix} 1 & p & 4 \\ -3 & 2 & 1 \\ 2 & -1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} p & 1 & 5 \\ 9 & p & -1 \\ 2 & 0 & 1 \end{bmatrix}$$

(a) Find each of det $\mathbf{A}$ and det $\mathbf{B}$ in terms of p. *(3 marks)*

(b) Without finding $\mathbf{AB}$, determine all values of p for which $\mathbf{AB}$ is singular. *(3 marks)*

Question 5: June 2009 – Q1

Let $\mathbf{P} = \begin{bmatrix} 1 & 4 & 2 \\ -1 & 2 & 6 \end{bmatrix}$ and $\mathbf{Q} = \begin{bmatrix} k & 1 \\ 2 & -1 \\ 3 & 1 \end{bmatrix}$, where k is a constant.

(a) Determine the product matrix $\mathbf{PQ}$, giving its elements in terms of k where appropriate. *(3 marks)*

(b) Find the value of k for which $\mathbf{PQ}$ is singular. *(2 marks)*

Question 6: June 2006 – Q6

The matrices $\mathbf{P}$ and $\mathbf{Q}$ are given by

$$\mathbf{P} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & t & -2 \\ 3 & 2 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{Q} = \begin{bmatrix} 1 & 1 & 1 \\ -7 & -1 & 5 \\ 11 & -1 & -7 \end{bmatrix}$$

where t is a real constant.

(a) Find the value of t for which $\mathbf{P}$ is singular. *(2 marks)*

(b) (i) Determine the matrix $\mathbf{R} = \mathbf{PQ}$, giving its elements in terms of t where appropriate. *(3 marks)*

(ii) Find the value of t for which $\mathbf{R} = k\mathbf{I}$, for some integer k. *(2 marks)*

(iii) Hence find the matrix $\mathbf{Q}^{-1}$. *(1 mark)*

Question 7: January 2009 – Q2

The 2×2 matrices $\mathbf{A}$ and $\mathbf{B}$ are such that

$$\mathbf{AB} = \begin{bmatrix} 9 & 1 \\ 7 & 13 \end{bmatrix} \quad \text{and} \quad \mathbf{BA} = \begin{bmatrix} 14 & 2 \\ 1 & 8 \end{bmatrix}$$

Without finding $\mathbf{A}$ and $\mathbf{B}$:

(a) find the value of $\det \mathbf{B}$, given that $\det \mathbf{A} = 10$; *(3 marks)*

(b) determine the 2×2 matrices $\mathbf{C}$ and $\mathbf{D}$ given by

$$\mathbf{C} = (\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}) \quad \text{and} \quad \mathbf{D} = (\mathbf{A}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}})^{\mathrm{T}}$$

where $\mathbf{M}^{\mathrm{T}}$ denotes the transpose of matrix $\mathbf{M}$. *(3 marks)*

Question 8: January 2006 – Q2

The matrices **P** and **Q** are defined in terms of the constant k by

$$\mathbf{P} = \begin{bmatrix} 3 & 2 & 1 \\ 1 & -1 & k \\ 5 & 3 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{Q} = \begin{bmatrix} 5 & 4 & 1 \\ 3 & k & -1 \\ 7 & 3 & 2 \end{bmatrix}$$

(a) Express $\det \mathbf{P}$ and $\det \mathbf{Q}$ in terms of k. *(3 marks)*

(b) Given that $\det(\mathbf{PQ}) = 16$, find the two possible values of k. *(4 marks)*

Question 9: January 2007 – Q8

The matrix $\mathbf{P} = \begin{bmatrix} 4 & -1 & 2 \\ 1 & 1 & 3 \\ -2 & 0 & a \end{bmatrix}$, where a is constant.

(a) (i) Determine $\det \mathbf{P}$ as a linear expression in a. *(2 marks)*

(ii) Evaluate $\det \mathbf{P}$ in the case when $a = 3$. *(1 mark)*

(iii) Find the value of a for which **P** is singular. *(2 marks)*

(b) The 3×3 matrix **Q** is such that $\mathbf{PQ} = 25\mathbf{I}$.

Without finding **Q**:

(i) write down an expression for $\mathbf{P}^{-1}$ in terms of **Q**; *(1 mark)*

(ii) find the value of the constant k such that $(\mathbf{PQ})^{-1} = k\mathbf{I}$; *(2 marks)*

(iii) determine the numerical value of $\det \mathbf{Q}$ in the case when $a = 3$. *(4 marks)*

Question 10: January 2008 – Q7

The non-singular matrix $\mathbf{M} = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix}$.

(a) (i) Show that

$$\mathbf{M}^2 + 2\mathbf{I} = k\mathbf{M}$$

for some integer k to be determined. *(3 marks)*

(ii) By multiplying the equation in part (a)(i) by $\mathbf{M}^{-1}$, show that

$$\mathbf{M}^{-1} = a\mathbf{M} + b\mathbf{I}$$

for constants a and b to be found. *(3 marks)*

Matrix algebra – exam questions

Question 1: June 2010 – Q2

(a)	$\mathbf{AB} = \begin{bmatrix} 2x+1 & 2x-1 \\ 8 & 4 \end{bmatrix}$	M1 A1	2
(b)	$\mathbf{B}^T\mathbf{A}^T = (\mathbf{AB})^T$ **Or** $\begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ x & 3 \end{bmatrix}$	M1	
	$= \begin{bmatrix} 2x+1 & 8 \\ 2x-1 & 4 \end{bmatrix}$	A1√	
	$2x+1 = 4-4x$ **Or** $2x-1 = 8x-4$	M1	
	$x = \frac{1}{2}$	A1	
	Checking/noting $x = \frac{1}{2}$ in other eqn.	B1	5
	Total		**7**

Question 2: June 2008 – Q3

(a)	Det $\mathbf{A} = k + 3 + 12 - 4 - 9 - k = 2$	M1 A1	2
(b)	$\mathbf{A}^{-1} = \frac{1}{\text{Det A}}(\text{adj } \mathbf{A})$	B1 M1	
	$= \frac{1}{2}\begin{bmatrix} k-9 & 3-k & 2 \\ 12-k & k-4 & -2 \\ -1 & 1 & 0 \end{bmatrix}$	M1 A1 A1	5
	Total		**7**

Question 3: June 2007 – Q6

(a)(i)	$\mathbf{AB}$ = a 3×3 matrix	M1	
	$= \begin{pmatrix} 3 & 2 & t+1 \\ 1 & 2 & t-1 \\ 3 & 2 & t+1 \end{pmatrix}$	A1 A1	3
(ii)	$\mathbf{BA}$ = a 2×2 matrix	M1	
	$= \begin{pmatrix} 2 & 2 \\ t & t+4 \end{pmatrix}$	A1	2
(b)	$R_1 = R_3 \; (\Rightarrow \det \mathbf{AB} = 0)$	B1	1

Question 4: June 2011 – Q1

(a)	det $\mathbf{A} = 5p - 1$	B1	
	det $\mathbf{B} = p^2 - 10p - 11$	M1A1	**3**
(b)	Use of det($\mathbf{AB}$) = det $\mathbf{A}$ det $\mathbf{B}$	B1	
	Finding three values of p	M1	
	$p = \frac{1}{5}, 11, -1$	A1F	**3**
	Total		**6**

Question 5: June 2009 – Q1

(a)	$\begin{bmatrix} 1 & 4 & 2 \\ -1 & 2 & 6 \end{bmatrix}\begin{bmatrix} k & 1 \\ 2 & -1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} k+14 & -1 \\ 22-k & 3 \end{bmatrix}$	M1 A1 A1	3
(b)	Det(**PQ**) $= 3k + 42 + 22 - k$	M1	
	$= 2k + 64 = 0$		
	$k = -32$	A1	2
	Total		**5**

Question 6: June 2006 – Q6

6(a)	Setting det $\mathbf{P}$ $(2t - 6 + 2 - 3t + 8 - 1 = 3 - t) = 0$	M1	
	$\Rightarrow t = 3$	A1	2
(b)(i)	$\begin{bmatrix} 6 & 0 & 0 \\ -7t-21 & 3-t & 15+5t \\ 0 & 0 & 6 \end{bmatrix}$	M1 A1 A1	3
(ii)	When $t = -3$, $\mathbf{PQ} = 6\mathbf{I}$	B1 B1	2
(iii)	$\mathbf{Q}^{-1} = \frac{1}{6}\mathbf{P} = \frac{1}{6}\begin{bmatrix} 2 & 1 & 1 \\ 1 & -3 & -2 \\ 3 & 2 & 1 \end{bmatrix}$	B1	1

Question 7: January 2009 – Q2

(a)	det $\mathbf{AB} = 110$	B1	
	Use of det $\mathbf{AB}$ = det $\mathbf{A}$ det $\mathbf{B}$	M1	
	det $\mathbf{B} = 11$	A1F	3
(b)	$\mathbf{C} = (\mathbf{AB})^T = \begin{bmatrix} 9 & 7 \\ 1 & 13 \end{bmatrix}$	M1 A1	
	$\mathbf{D} = [\,(\mathbf{BA})^T\,]^T = \mathbf{BA} = \begin{bmatrix} 14 & 2 \\ 1 & 8 \end{bmatrix}$	B1	3
	Total		**6**

Question 8: January 2006 – Q2

(a)	Attempt at either determinant	M1	
	det $\mathbf{P} = k - 2$ det $\mathbf{Q} = 3k - 28$	A1 A1	3
(b)	Use of det ($\mathbf{PQ}$) = (det $\mathbf{P}$)(det $\mathbf{Q}$)	M1	
	Creating a quadratic	dM1	
	$3k^2 - 34k + 40 = 0$	A1√	
	$k = \frac{4}{3}$ or 10	B1	4

Question 9: January 2007 – Q8

(a)(i)	det $\mathbf{P} = 4a + 6 + 4 + a = 5a + 10$	M1 A1	2
(ii)	When $a = 3$, det $\mathbf{P} = 25$	B1F	1
(iii)	Setting their det $\mathbf{P} = 0 \Rightarrow a = -2$	M1 A1F	2
(b)(i)	$\mathbf{P}^{-1} = \frac{1}{25}\mathbf{Q}$	B1	1
(ii)	$(\mathbf{PQ})^{-1} = (25\,\mathbf{I})^{-1} = \frac{1}{25}\mathbf{I}$	M1 A1	2
	Or $(\mathbf{PQ})^{-1} = \mathbf{Q}^{-1}\mathbf{P}^{-1}$	(M1)	
	$= \mathbf{Q}^{-1}.\frac{1}{25}\mathbf{Q} = \frac{1}{25}\mathbf{I}$	(A1)	(2)
(iii)	det $\mathbf{PQ}$ = det $(25\,\mathbf{I}) = 25^3$ or 15625	M1 A1	
	det $\mathbf{PQ}$ = det $\mathbf{P}$. det $\mathbf{Q}$	M1	
	$\Rightarrow 25^3 = 25$ det $\mathbf{Q}$		
	$\Rightarrow$ det $\mathbf{Q} = 25^2$ or 625	A1	4
	Total		**12**

Question 10: January 2008 – Q7

(a)(i)	$\mathbf{M}^2 = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix}$	M1	
	$= \begin{bmatrix} 4 & -3 & 3 \\ 3 & -2 & 3 \\ 3 & -3 & 4 \end{bmatrix}$	A1	
	$\mathbf{M}^2 + 2\mathbf{I} = \begin{bmatrix} 4 & -3 & 3 \\ 3 & -2 & 3 \\ 3 & -3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$		
	$= \begin{bmatrix} 6 & -3 & 3 \\ 3 & 0 & 3 \\ 3 & -3 & 6 \end{bmatrix} = 3\mathbf{M}$	A1	3
(ii)	Multiplying by $\mathbf{M}^{-1}$	M1	
	to get $\mathbf{M} + 2\mathbf{M}^{-1} = 3\mathbf{I}$	A1	
	so that $\mathbf{M}^{-1} = \frac{3}{2}\mathbf{I} - \frac{1}{2}\mathbf{M}$	A1	3