Lines and planes – exam questions

Question 1: Jan 2006 Q3

(a) The plane Π has equation $\mathbf{r} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} + \mu \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}$.

(i) Find a vector which is perpendicular to both $\begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}$. *(2 marks)*

(ii) Hence find an equation for Π in the form $\mathbf{r}.\mathbf{n} = d$. *(2 marks)*

(b) The line L has equation $\left(\mathbf{r} - \begin{bmatrix} -1 \\ 2 \\ 6 \end{bmatrix}\right) \times \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} = \mathbf{0}$.

Verify that $\mathbf{r} = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix} + t \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$ is also an equation for L. *(2 marks)*

(c) Determine the position vector of the point of intersection of Π and L. *(4 marks)*

Question 2: Jun 2006 Q1

Two planes, Π_1 and Π_2, have equations $\mathbf{r} \,.\, \begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix} = 0$ and $\mathbf{r} \,.\, \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} = 0$ respectively.

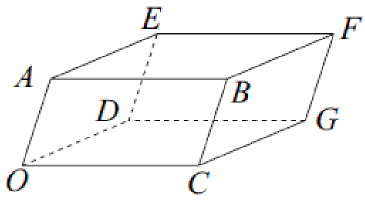
(a) Determine the cosine of the acute angle between Π_1 and Π_2. *(4 marks)*

(b) (i) Find $\begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix} \times \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$. *(2 marks)*

(ii) Find a vector equation for the line of intersection of Π_1 and Π_2. *(2 marks)*

Question 3: Jun 2006 Q7

The diagram shows the parallelepiped $OABCDEFG$.



Points A, B, C and D have position vectors

$$\mathbf{a} = \begin{bmatrix} 4 \\ -1 \\ 7 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 6 \\ 1 \\ 6 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \text{ and } \mathbf{d} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$$

respectively, relative to the origin O.

(a) Show that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are linearly dependent. *(1 mark)*

(b) Determine the volume of the parallelepiped. *(3 marks)*

(c) Determine a vector equation for the plane $ABDG$:

(i) in the form $\mathbf{r} = \mathbf{u} + \lambda\mathbf{v} + \mu\mathbf{w}$; *(2 marks)*

(ii) in the form $\mathbf{r} \cdot \mathbf{n} = d$. *(4 marks)*

(d) Find cartesian equations for the line OF, and hence find the direction cosines of this line. *(4 marks)*

Question 4: Jan 2008 Q6

(a) The line l has equation $\mathbf{r} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ 2 \\ 6 \end{bmatrix}$.

(i) Write down a vector equation for l in the form $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \mathbf{0}$. *(1 mark)*

(ii) Write down cartesian equations for l. *(2 marks)*

(iii) Find the direction cosines of l and explain, geometrically, what these represent. *(3 marks)*

(b) The plane Π has equation $\mathbf{r} = \begin{bmatrix} 7 \\ 5 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$.

(i) Find an equation for Π in the form $\mathbf{r} \cdot \mathbf{n} = d$. *(4 marks)*

(ii) State the geometrical significance of the value of d in this case. *(1 mark)*

(c) Determine, to the nearest 0.1°, the angle between l and Π. *(4 marks)*

Question 5: Jun 2008 Q4

Two planes have equations

$$\mathbf{r} \cdot \begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix} = 12 \quad \text{and} \quad \mathbf{r} \cdot \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} = 7$$

(a) Find, to the nearest $0.1°$, the acute angle between the two planes. *(4 marks)*

(b) (i) The point $P(0, a, b)$ lies in both planes. Find the value of a and the value of b. *(3 marks)*

(ii) By using a vector product, or otherwise, find a vector which is parallel to both planes. *(2 marks)*

(iii) Find a vector equation for the line of intersection of the two planes. *(2 marks)*

Question 6: Jan 2009 Q1

The line l has equation $\mathbf{r} = (1 + 4t)\mathbf{i} + (-2 + 12t)\mathbf{j} + (1 - 3t)\mathbf{k}$.

(a) Write down a direction vector for l. *(1 mark)*

(b) (i) Find direction cosines for l. *(2 marks)*

(ii) Explain the geometrical significance of the direction cosines in relation to l. *(1 mark)*

(c) Write down a vector equation for l in the form $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \mathbf{0}$. *(2 marks)*

Question 7: Jan 2009 Q6

The line L and the plane Π are, respectively, given by the equations

$$\mathbf{r} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \quad \text{and} \quad \mathbf{r} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 20$$

(a) Determine the size of the acute angle between L and Π. *(4 marks)*

(b) The point P has coordinates $(10, -5, 37)$.

(i) Show that P lies on L. *(1 mark)*

(ii) Find the coordinates of the point Q where L meets Π. *(4 marks)*

(iii) Deduce the distance PQ and the shortest distance from P to Π. *(3 marks)*

Question 8: Jun 2009 Q3

The plane Π has equation $\mathbf{r} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} + \mu \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}$.

(a) Find an equation for Π in the form $\mathbf{r} \cdot \mathbf{n} = d$. *(4 marks)*

(b) Show that the line with equation $\mathbf{r} = \begin{bmatrix} 7 \\ 1 \\ 4 \end{bmatrix} + t \begin{bmatrix} 10 \\ 1 \\ 5 \end{bmatrix}$ does not intersect Π, and explain the geometrical significance of this result. *(4 marks)*

Question 9: Jan 20120 Q6

(a) Find the value of p for which the planes with equations

$$\mathbf{r} \cdot \begin{bmatrix} 6 \\ -3 \\ 2 \end{bmatrix} = 42 \quad \text{and} \quad \mathbf{r} \cdot \begin{bmatrix} 4p+1 \\ p-2 \\ 1 \end{bmatrix} = -7$$

(i) are perpendicular; *(3 marks)*

(ii) are parallel. *(3 marks)*

(b) In the case when $p = 4$:

(i) write down a cartesian equation for each plane; *(2 marks)*

(ii) find, in the form $\mathbf{r} = \mathbf{a} + \lambda \mathbf{d}$, an equation for l, the line of intersection of the planes. *(6 marks)*

(c) Determine a vector equation, in the form $\mathbf{r} = \mathbf{u} + \beta \mathbf{v} + \gamma \mathbf{w}$, for the plane which contains l and which passes through the point (30, 7, 30). *(2 marks)*

Question 10: Jun 2010 Q3

The plane Π_1 is perpendicular to the vector $9\mathbf{i} - 8\mathbf{j} + 72\mathbf{k}$ and passes through the point $A(2, 10, 1)$.

(a) Find, in the form $\mathbf{r} . \mathbf{n} = d$, a vector equation for Π_1. *(3 marks)*

(b) Determine the exact value of the cosine of the acute angle between Π_1 and the plane Π_2 with equation $\mathbf{r} . (\mathbf{i} + \mathbf{j} + \mathbf{k}) = 11$. *(4 marks)*

Lines and planes – exam questions

Question 1: Jan 2006 Q3

(a)	(i) $\mathbf{n} = (3\mathbf{j} + \mathbf{k}) \times (4\mathbf{i} - \mathbf{j}) = \mathbf{i} + 4\mathbf{j} - 12\mathbf{k}$	M1A1	2
	(ii) $d = (2\mathbf{i} + 5\mathbf{j} + \mathbf{k}) \cdot (\mathbf{i} + 4\mathbf{j} - 12\mathbf{k}) = 10$	M1 A1√	2
(b)	Either substituting. $\mathbf{r} = \begin{bmatrix} 2+3t \\ 2 \\ 5-t \end{bmatrix}$ into $\left(\mathbf{r} - \begin{bmatrix} -1 \\ 2 \\ 6 \end{bmatrix}\right) \times \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$	M1	
	And showing result is **0**	A1	2
	Or		
	Explaining p.v. and d.v. of line	M1 A1	2
(c)	Subst^{g}. $\mathbf{r} = \begin{bmatrix} 2+3t \\ 2 \\ 5-t \end{bmatrix}$ into their plane eqn.	M1	
	Solving a linear eqn. in t	dM1	
	t = 4	A1√	
	Pt. of i/sctn. at $14\mathbf{i} + 2\mathbf{j} + \mathbf{k}$	B1	4

Question 2: Jun 2006 Q1

1(a)	$\cos\theta = \dfrac{\text{scalar product}}{\text{product of moduli}}$	M1	
	Scalar product in numerator = 24	B1	
	Both moduli in denominator: $\sqrt{50}$ and $\sqrt{18}$	B1	
	$\cos\theta = \frac{4}{5}$ or 0.8	A1	4
b)(i)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 5 & 3 \\ 4 & 1 & 1 \end{vmatrix} = 2\mathbf{i} + 8\mathbf{j} - 16\mathbf{k}$	M1 A1	2
(ii)	$\mathbf{r} = \lambda\begin{bmatrix} 1 \\ 4 \\ -8 \end{bmatrix}$ or $\mathbf{r} \times \begin{bmatrix} 1 \\ 4 \\ -8 \end{bmatrix} = 0$	B1√ B1	2
	Total		**8**

Question 3: Jun 2006 Q7

(a)	$\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = \begin{vmatrix} 4 & -1 & 7 \\ 6 & 1 & 6 \\ 2 & 2 & -1 \end{vmatrix} = 0$	B1	1
(b)	$\mathbf{a} \cdot \mathbf{c} \times \mathbf{d} = \begin{vmatrix} 4 & -1 & 7 \\ 2 & 2 & -1 \\ 1 & 3 & -2 \end{vmatrix} = 21$	M1 m1 A1	3
(i)	$\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a}) + \mu(\mathbf{d} - \mathbf{a})$	M1	
	$\mathbf{r} = \begin{bmatrix} 4 \\ -1 \\ 7 \end{bmatrix} + \lambda\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} + \mu\begin{bmatrix} -3 \\ 4 \\ -9 \end{bmatrix}$	A1	2
(ii)	$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & -1 \\ -3 & 4 & -9 \end{vmatrix} = -14\mathbf{i} + 21\mathbf{j} + 14\mathbf{k}$	M1 A1√	
	$d = \begin{bmatrix} 4 \\ -1 \\ 7 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix} = 3$	M1 A1√	4
(d)	$\mathbf{r} = \nu(\mathbf{a} + \mathbf{c} + \mathbf{d}) = \nu(7\mathbf{i} + 4\mathbf{j} + 4\mathbf{k})$	B1	
	Cartesian equations are $\frac{x}{7} = \frac{y}{4} = \frac{z}{4}$	B1√	
	$\sqrt{7^2 + 4^2 + 4^2} = 9$	M1	
	giving d.c.s $\frac{7}{9}$, $\frac{4}{9}$ and $\frac{4}{9}$	A1√	4
	Total		**14**

Question 4: Jan 2008 Q6

(a)(i)	$\mathbf{a} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$	B1	1
(ii)	Equating for λ: $\frac{x-1}{3} = \frac{y-1}{2} = \frac{z-2}{6}$	M1 A1	2
(iii)	$\sqrt{3^2 + 2^2 + 6^2} = 7$	B1	
	Direction cosines are $\frac{3}{7}, \frac{2}{7}$ and $\frac{6}{7}$	B1	
	These are the cosines of the angles between the line and the x- , y- and z-axes (respectively)	B1	3
(b)(i)	$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 3 & 2 \\ 1 & 1 & 3 \end{vmatrix} = 7\mathbf{i} - 10\mathbf{j} + \mathbf{k}$	M1A1	
	$d = \begin{bmatrix} 7 \\ 5 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ -10 \\ 1 \end{bmatrix} = 0$	M1 A1	4
(ii)	$d = 0 \Rightarrow$ plane through / contains the origin	B1	1
(c)	$\sin\theta / \cos\theta = \dfrac{\text{scalar product}}{\text{product of moduli}}$	M1	
	Numerator = 21 – 20 + 6 = 7	B1	
	Denominator = $7.\sqrt{150}$	B1	
	$\theta = 4.7°$	A1	4
	Total		**15**

Question 5: Jun 2008 Q4

4(a)	Use of $\cos\theta = \dfrac{\text{scalar product}}{\text{product of moduli}}$	M1	
	Numerator = 7 Denominator = $\sqrt{21}\sqrt{27}$	B1,B1	
	$\theta = 72.9°$	A1	4
b)(i)	$a + 4b = 7$ and $a - b = 12$	B1	
		M1	
	$a = 11$ and $b = -1$	A1	3
(ii)	$\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 1 & -1 \\ 2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 5 \\ -22 \\ 3 \end{bmatrix}$	M1	
		A1	2
(iii)	$\mathbf{r} = \begin{bmatrix} 0 \\ 11 \\ -1 \end{bmatrix} + \lambda \begin{bmatrix} 5 \\ -22 \\ 3 \end{bmatrix}$	M1	
	or other equivalent line form eg $(\mathbf{r} - \mathbf{a}) \times \mathbf{d} = \mathbf{0}$	A1	2
	Total		**11**

Question 6: Jan 2009 Q1

1(a)	$4\mathbf{i} + 12\mathbf{j} - 3\mathbf{k}$ or equivalent	B1	1
b)(i)	$\sqrt{4^2 + 12^2 + 3^2} = 13$	M1	
	d.c.'s are $\frac{4}{13}, \frac{12}{13}$ and $-\frac{3}{13}$	A1F	2
(ii)	The cosines of the angles between the line and the coordinate axes	B1	1
(c)	$\mathbf{a} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and	B1	
	$\mathbf{b}$ = their d.v.	B1F	2
	Total		**6**

Question 7: Jan 2009 Q6

6(a)	Use of $\sin\theta$ or $\cos\theta$ = (dot product)/(product of moduli)	M1	
	Num^r. = 3	B1F	
	Denom^r. = $\sqrt{18}\sqrt{2}$	B1F	
	$\theta = 30°$	A1	4
b)(i)	$\lambda = 8$ noted or found	B1	1
(ii)	$\begin{bmatrix} 2+\lambda \\ 3-\lambda \\ 5+4\lambda \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 20$	M1	
	$3 - \lambda + 5 + 4\lambda = 20 \Rightarrow \lambda = 4$	M1 A1	
	giving $Q = (6, -1, 21)$	B1F	4
(iii)	$PQ = \sqrt{4^2 + 4^2 + 16^2} = 12\sqrt{2}$	M1 A1	
	Sh. Dist. = $12\sqrt{2}\,.\sin 30° = 6\sqrt{2}$	B1F	3
	Total		**12**

Question 8: Jun 2009 Q3

3(a)	$\mathbf{n} = (3\mathbf{i} + \mathbf{j} + 2\mathbf{k}) \times (4\mathbf{i} - \mathbf{j} + \mathbf{k})$	M1	
	$= 3\mathbf{i} + 5\mathbf{j} - 7\mathbf{k}$	A1	
	$d = (2\mathbf{i} + \mathbf{j} + \mathbf{k}) \bullet (\text{their } \mathbf{n}) = 4$	M1 A1	4
(b)	$\begin{bmatrix} 7+10t \\ 1+t \\ 4+5t \end{bmatrix}$ subst^d. into their plane eqn.	M1	
	$21 + 30t + 5 + 5t - 28 - 35t = 4$	dM1	
	Since $-2 \neq 4$, no intersection	A1	
	Line parallel to plane	B1	
	OR		
	$\begin{bmatrix} 3 \\ 5 \\ -7 \end{bmatrix} \bullet \begin{bmatrix} 10 \\ 1 \\ 5 \end{bmatrix} = 0$	(M1) (A1) (B1)	
	Line perp^r. to nml. $\Rightarrow$ line // to plane	(B1)	
	OR		
	$\begin{bmatrix} 7+10t \\ 1+t \\ 4+5t \end{bmatrix}$ equated to $\begin{bmatrix} 2+3\lambda+4\mu \\ 1+\lambda-\mu \\ 4+2\lambda+\mu \end{bmatrix}$	(M1)	
	Eliminating λ, μ to get linear eqn. in t	(dM1)	
	Since $-2 \neq 4$, no intersection	(A1)	
	Line parallel to plane	(B1)	4
	Total		**8**

Question 9: Jan 20120 Q6

(a)(i)	$\begin{bmatrix}6\\-3\\2\end{bmatrix}\bullet\begin{bmatrix}4p+1\\p-2\\1\end{bmatrix}=0$	M1	
	$24p+6-3p+6+2=0$	M1	
	$p=-\frac{2}{3}$	A1	3
(ii)	$\begin{bmatrix}6\\-3\\2\end{bmatrix}=m\begin{bmatrix}4p+1\\p-2\\1\end{bmatrix}$	M1	
	$m=2\,,\ p=\frac{1}{2}$	A1,A1	3
	ALT $\begin{bmatrix}6\\-3\\2\end{bmatrix}\times\begin{bmatrix}4p+1\\p-2\\1\end{bmatrix}=\mathbf{0}$ $\quad\begin{bmatrix}1-2p\\8p-4\\18p-9\end{bmatrix}$	(M1A1)	
	$p=\frac{1}{2}$	(A1)	
(b)(i)	$6x-3y+2z=42\,,\ 17x+2y+z=-7$	B1,B1	2
(ii)	eg 2.② – ①: $7[4x+y=-8]$	M1A1	
	$\frac{x+2}{-1}=\frac{y}{4}=\lambda$	M1	
	Substituting back to find 3rd variable	m1	
	$z=9\lambda+27$	A1	
	$\mathbf{r}=\begin{bmatrix}0\\-8\\9\end{bmatrix},\begin{bmatrix}1\\-12\\0\end{bmatrix}$ **or** $\begin{bmatrix}-2\\0\\27\end{bmatrix}+\lambda\begin{bmatrix}-1\\4\\9\end{bmatrix}$	B1	6
	OR method for finding the d.v. $\begin{bmatrix}-1\\4\\9\end{bmatrix}$	(M2A1)	
	Method for finding any pt. on l	(M1A1)	
	Putting them together as a line eqn.	(B1)	
(c)	$\mathbf{r}=\mathbf{a}+u\,\mathbf{d}_1+v\,\mathbf{d}_2$ where **a** is an pt. on plane $\mathbf{d}_1$ = any d.v. $\mathbf{d}_2=\begin{bmatrix}30\\7\\30\end{bmatrix}$ – any **a** in plane		2
	Total		**16**

Question 10: Jun 2010 Q3

3(a)	Clearly identifying $\mathbf{n}=\begin{bmatrix}9\\-8\\72\end{bmatrix}$	B1	
	$d=\begin{bmatrix}9\\-8\\72\end{bmatrix}\bullet\begin{bmatrix}2\\10\\1\end{bmatrix}=10$	M1 A1	3
(b)	Use of $\dfrac{\text{Sc.prod. of normals}}{\text{prod. of their moduli}}$	M1	
	Nr. = 73	B1√	
	Dr. = $73\sqrt{3}$ or $\sqrt{15987}$	B1√	
	$\cos\theta=\dfrac{1}{\sqrt{3}}$	A1	4
	Total		**7**