Invariant lines and points – Exam questions

Question 1: Jan 2006 – Q5

The transformation T maps (x, y) to (x', y'), where

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

(a) Describe the difference between *an invariant line* and *a line of invariant points* of T. *(1 mark)*

(b) Evaluate the determinant of the matrix $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ and describe the geometrical significance of the result in relation to T. *(2 marks)*

(c) Show that T has a line of invariant points, and find a cartesian equation for this line. *(2 marks)*

(d) (i) Find the image of the point $(x, -x + c)$ under T. *(2 marks)*

(ii) Hence show that all lines of the form $y = -x + c$, where c is an arbitrary constant, are invariant lines of T. *(2 marks)*

(e) Describe the transformation T geometrically. *(3 marks)*

Question 2: June 2006 – Q2

A transformation is represented by the matrix $\mathbf{A} = \begin{bmatrix} 0.28 & -0.96 & 0 \\ 0.96 & 0.28 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(a) Evaluate det **A**. *(1 mark)*

(b) State the invariant line of the transformation. *(1 mark)*

(c) Give a full geometrical description of this transformation. *(3 marks)*

Question 3: June 2006 – Q4

The plane transformation T maps points (x, y) to points (x', y') such that

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \mathbf{A} \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{where } \mathbf{A} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

(a) (i) State the line of invariant points of T. *(1 mark)*

(ii) Give a full geometrical description of T. *(2 marks)*

(b) Find $\mathbf{A}^2$, and hence give a full geometrical description of the single plane transformation given by the matrix $\mathbf{A}^2$. *(3 marks)*

Question 4: Jan 2007 – Q7

The transformation S is a shear with matrix $\mathbf{M} = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}$. Points (x, y) are mapped under S to image points (x', y') such that

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \mathbf{M} \begin{bmatrix} x \\ y \end{bmatrix}$$

(a) Find the equation of the line of invariant points of S. *(2 marks)*

(b) Show that all lines of the form $y = x + c$, where c is a constant, are invariant lines of S. *(3 marks)*

(c) Evaluate $\det \mathbf{M}$, and state the property of shears which is indicated by this result. *(2 marks)*

(d) Calculate, to the nearest degree, the acute angle between the line $y = -x$ and its image under S. *(3 marks)*

Question 5: June 2009 – Q4

(a) Show that the system of equations

$$\begin{aligned} 3x - y + 3z &= 11 \\ 4x + y - 5z &= 17 \\ 5x - 4y + 14z &= 16 \end{aligned}$$

does not have a unique solution and is consistent.

(You are not required to find any solutions to this system of equations.) *(4 marks)*

(b) A transformation T of three-dimensional space maps points (x, y, z) onto image points (x', y', z') such that

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} x - y + 3z - 2 \\ 2x + 6y - 4z + 12 \\ 4x + 11y + 4z - 30 \end{bmatrix}$$

Find the coordinates of the invariant point of T. *(8 marks)*

Question 6: Jan 2011 – Q8

The plane transformation T is represented by the matrix $\mathbf{M} = \begin{bmatrix} -3 & 8 \\ -1 & 3 \end{bmatrix}$.

(a) The quadrilateral $ABCD$ has image $A'B'C'D'$ under T.

Evaluate $\det \mathbf{M}$ and describe the geometrical significance of both its sign and its magnitude in relation to $ABCD$ and $A'B'C'D'$. *(3 marks)*

(b) The line $y = px$ is a line of invariant points of T, and the line $y = qx$ is an invariant line of T.

Show that $p = \frac{1}{2}$ and determine the value of q. *(5 marks)*

(c) (i) Find the 2×2 matrix $\mathbf{R}$ which represents a reflection in the line $y = \frac{1}{2}x$. *(2 marks)*

(ii) Given that T is the composition of a shear, with matrix $\mathbf{S}$, followed by a reflection in the line $y = \frac{1}{2}x$, determine the matrix $\mathbf{S}$ and describe the shear as fully as possible.

(5 marks)

Question 7: Jun 2011 – Q6

(a) The transformation U of three-dimensional space is represented by the matrix

$$\begin{bmatrix} 1 & 4 & -3 \\ 2 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$$

(i) Write down a vector equation for the line L with cartesian equation

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{6}$$ *(2 marks)*

(ii) Find a vector equation for the image of L under U, and deduce that it is a line through the origin. *(4 marks)*

(b) The plane transformation V is represented by the matrix $\begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}$.

L_1 is the line with equation $y = \frac{1}{2}x + k$, and L_2 is the image of L_1 under V.

(i) Find, in the form $y = mx + c$, the cartesian equation for L_2. *(4 marks)*

(ii) Deduce that L_2 is parallel to L_1 and find, in terms of k, the distance between these two lines. *(3 marks)*

Invariant lines and points – Exam questions

Question 1: Jan 2006 – Q5

5(a)	For an *invariant line*, all points on the line have image points also on the line. For a *line of invariant points*, all points on the line map onto themselves.	B1	1
(b)	Det = 3	B1	
	The s.f.. of **area** enlargement under T	B1	2
5(c)	Setting $x' = x$ and $y' = y$ and solving	M1	
	$\Rightarrow y = x$	A1	
	Or		
	Char. Eqn. is $\lambda^2 - 4\lambda + 3 = 0$	M1	
	$\lambda = 1$ gives l.o.i.p.s and $y = x$	A1	2
d)(i)	$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ -x+c \end{bmatrix}$	M1	
	$= \begin{bmatrix} 3x - c \\ -3x + 2c \end{bmatrix}$	A1	2
(ii)	$y' = -x' + c$ also	M1	
	$\Rightarrow y = -x + c$ invariant for all c	A1	2
(e)	Stretch, s.f. 3,	M1 A1	
	perp^r. to $y = x$ (or $\parallel$ to $y = -x$)	A1	3
	Total		**12**

Question 2: June 2006 – Q2

(a)	det **A** = 1	B1	1
(b)	The z-axis (i.e $x = y = 0$)	B1	1
(c)	Rotation	M1	
	about the z-axis	A1√	
	through $\cos^{-1} 0.28$	A1	3
	Total		**5**

Question 3: June 2006 – Q4

(a)(i)	The x-axis	B1	1
(ii)	Shear (parallel to the x-axis)	M1	
	mapping e.g. $(0, 1) \to (3, 1)$ or $(1, 1) \to (4, 1)$	A1	2
(b)	$\mathbf{A}^2 = \begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix}$	B1	
	Shear (parallel to the x-axis)	M1	
	mapping e.g. $(0, 1) \to (6, 1)$ or $(1, 1) \to (7, 1)$	A1	3
	Total		**6**

Question 4: Jan 2007 – Q7

(a)	Setting $x' = x$ and $y' = y$	M1	
	$x = -x + 2y$ and $y = -2x + 3y$ gives $y = x$	A1	2
b)	$\begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ x+c \end{bmatrix} = \begin{bmatrix} x+2c \\ x+3c \end{bmatrix}$	M1A1	
	And $y' = x' + c$ also	B1	3
(c)	det **M** = 1 $\Rightarrow$ Areas of shapes invariant	B1 B1	2
(d)	$\begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} a \\ -a \end{bmatrix} = \begin{bmatrix} -3a \\ -5a \end{bmatrix}$	M1	
	$\Rightarrow$ Image of $y = -x$ under S is $y = \frac{5}{3}x$	A1	
	Angle is $135° - \tan^{-1}\frac{5}{3} = 76°$	B1F	3
	N.B. Final angle can be gained via scalar product: $\cos\theta = \left\|\frac{(\mathbf{i} - \mathbf{j})\cdot(-3\mathbf{i} - 5\mathbf{j})}{\sqrt{2}\sqrt{34}}\right\|$ $\Rightarrow \theta = \cos^{-1}(1/\sqrt{17}) = 76°$		
	Total		**10**

Question 5: June 2009 – Q4

4(a)	$3 \times [1] - [2] \Rightarrow 5x - 4y + 14z = 16$	M2 A1	
	Giving no unique soln. *and* consistent	E1	
	For those who just show $\Delta = 0$ to conclude that there is no unique soln.	(M1) (A1)	
	OR		
	Solving e.g. in [1] & [2]: $\frac{x-4}{2} = \frac{y-1}{27} = \frac{z}{7} = \lambda$	(M1) (A1)	
	Subst^g. in [3] for x, y, z in terms of λ	(M1)	
	Showing LHS = RHS = 16	(A1)	
	OR		
	$\left(\begin{array}{ccc\|c} 3 & -1 & 3 & 11 \\ 4 & 1 & -5 & 17 \\ 5 & -4 & 14 & 16 \end{array}\right) \to \left(\begin{array}{ccc\|c} 3 & -1 & 3 & 1 \\ 1 & 2 & -8 & 6 \\ -1 & -2 & 8 & -6 \end{array}\right)$	(M1) (A1) (A1)	
	$R_2' = -R_3' \Rightarrow$ no unique soln. and consistency	(E1)	
	OR		
	Showing $\Delta = 0 \Rightarrow$ no unique soln.	(M1) (A1)	
	Attempt at each of $\Delta_x = \begin{vmatrix} 11 & -1 & 3 \\ 17 & 1 & -5 \\ 16 & -4 & 14 \end{vmatrix}$, $\Delta_y = \begin{vmatrix} 3 & 11 & 3 \\ 4 & 17 & -5 \\ 5 & 16 & 14 \end{vmatrix}$ and $\Delta_z = \begin{vmatrix} 3 & -1 & 11 \\ 4 & 1 & 17 \\ 5 & -4 & 16 \end{vmatrix}$	(M1)	
	Each shown = 0 and this $\Rightarrow$ consistency	(A1)	4
(b)	Setting $x' = x,\ y' = y,\ z' = z$	M1	
	$2 = -y + 3z$; $-12 = 2x + 5y - 4z$; $30 = 4x + 11y + 3z$	A1	
	E.g. $\left.\begin{array}{l} 2 = 3z - y \\ 54 = 11z + y \end{array}\right\}$ by (3) – 2 × (2)	M1 A1	
	$z = 4,\ y = 10$	M1 A1	
	$x = -23$	M1 A1	8
	OR Other methods for solving a 3×3 system will be constructed should they arise		
	Total		**12**

Question 6: Jan 2011 – Q8

8(a)	Det(M) = – 1	B1	
	Magnitude = 1 $\Rightarrow$ area invariant	B1√	
	– ve sign $\Rightarrow$ cyclic order of vertices is reversed OR "reflection" involved	B1	3
(b)	Method 1		
	Char. Eqn.: $\lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$	M1 A1	
	Subst^g. back: $\lambda = 1 \Rightarrow y = \frac{1}{2}x$	M1 A1	
	and $\lambda = -1 \Rightarrow y = \frac{1}{4}x$	A1	5
	Method 2		
	$\begin{bmatrix} -3 & 8 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} x \\ mx \end{bmatrix} = \begin{bmatrix} (8m-3)x \\ (3m-1)x \end{bmatrix}$	(M1)	
	Use of $y' = mx'$: $3m - 1 = 8m^2 - 3m$	(M1)	
	Solving a quadratic eqn. in $m = \frac{1}{4}, \frac{1}{2}$	(M1A1)	
	$p = \frac{1}{2}$ and $q = \frac{1}{4}$	(A1)	
(c)	(i) $p = \frac{1}{2} = \tan\theta$ $\Rightarrow \cos 2\theta = \frac{3}{5}$ and $\sin 2\theta = \frac{4}{5}$	M1	
	$\mathbf{R} = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix}$	A1	2
	(ii) Use $\begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix}\mathbf{S} = \begin{bmatrix} -3 & 8 \\ -1 & 3 \end{bmatrix}$	M1	
	S found using inverse matrix	M1	
	$= \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix}\begin{bmatrix} -3 & 8 \\ -1 & 3 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -13 & 36 \\ -9 & 23 \end{bmatrix}$	A1	
	Shear, parallel to $y = \frac{1}{2}x$	B1	
	mapping (e.g.) (1, 1) $\to$ (4.6, 2.8)	B1√	5
	Total		15

Question 7: Jun 2011 – Q6

6 (a)(i)	$\mathbf{r} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \lambda\begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}$	B2,1	2
	(ii) $\mathbf{r} = \begin{bmatrix} 1 & 4 & -3 \\ 2 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1+2\lambda \\ 2+3\lambda \\ 3+6\lambda \end{bmatrix} = \begin{bmatrix} -4\lambda \\ \lambda \\ -\lambda \end{bmatrix}$	M1 A1 A1	
	Clear and valid explanation that this is a line through O	E1	4
(b) (i)	$\begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} p \\ \frac{1}{2}p+k \end{bmatrix} = \begin{bmatrix} 3p+4k \\ \frac{3}{2}p-k \end{bmatrix}$	B1	
		M1A1	
	Answer satisfies $y = \frac{1}{2}x - 3k$	A1	4
	(ii) Equal gradients, hence parallel	E1F	
	Distance = $\lvert k-c\rvert \cos\theta$ with $\tan\theta = \frac{1}{2}$	M1	
	$\ldots = \frac{8k}{\sqrt{5}}$	A1	3
			13