Hyperbolic functions – exam questions

Question 1: June 2006

The curve C has equation

$$y = \cosh x - 3\sinh x$$

(a) (i) The line $y = -1$ meets C at the point $(k, -1)$.

Show that

$$e^{2k} - e^{k} - 2 = 0 \qquad \textit{(3 marks)}$$

(ii) Hence find k, giving your answer in the form $\ln a$. *(4 marks)*

(b) (i) Find the x-coordinate of the point where the curve C intersects the x-axis, giving your answer in the form $p \ln a$. *(4 marks)*

Question 2: Jan 2006

(a) Use the definitions

$$\sinh\theta = \tfrac{1}{2}(e^{\theta} - e^{-\theta}) \qquad \text{and} \qquad \cosh\theta = \tfrac{1}{2}(e^{\theta} + e^{-\theta})$$

to show that:

(i) $2\sinh\theta\cosh\theta = \sinh 2\theta$; *(2 marks)*

(ii) $\cosh^2\theta + \sinh^2\theta = \cosh 2\theta$. *(3 marks)*

Question 3: Jan 2007

(a) Given that

$$4\cosh^2 x = 7\sinh x + 1$$

find the two possible values of $\sinh x$. *(4 marks)*

(b) Hence obtain the two possible values of x, giving your answers in the form $\ln p$. *(3 marks)*

Question 4: June 2008

(a) Express

$$5\sinh x + \cosh x$$

in the form $Ae^{x} + Be^{-x}$, where A and B are integers. *(2 marks)*

(b) Solve the equation

$$5\sinh x + \cosh x + 5 = 0$$

giving your answer in the form $\ln a$, where a is a rational number. *(4 marks)*

Question 5: Jan 2009

(a) Use the definitions $\sinh\theta = \tfrac{1}{2}(e^{\theta} - e^{-\theta})$ and $\cosh\theta = \tfrac{1}{2}(e^{\theta} + e^{-\theta})$ to show that

$$1 + 2\sinh^2\theta = \cosh 2\theta \qquad \textit{(3 marks)}$$

(b) Solve the equation

$$3\cosh 2\theta = 2\sinh\theta + 11$$

giving each of your answers in the form $\ln p$. *(6 marks)*

Question 6: June 2009

(a) Sketch the graph of $y = \tanh x$. *(2 marks)*

(b) Given that $u = \tanh x$, use the definitions of $\sinh x$ and $\cosh x$ in terms of e^x and e^{-x} to show that

$$x = \frac{1}{2}\ln\left(\frac{1+u}{1-u}\right)$$ *(6 marks)*

(c) (i) Show that the equation

$$3\,\text{sech}^2 x + 7\tanh x = 5$$

can be written as

$$3\tanh^2 x - 7\tanh x + 2 = 0$$ *(2 marks)*

(ii) Show that the equation

$$3\tanh^2 x - 7\tanh x + 2 = 0$$

has only one solution for x.

Find this solution in the form $\frac{1}{2}\ln a$, where a is an integer. *(5 marks)*

Question 7: June 2010

(a) Show that

$$9\sinh x - \cosh x = 4e^x - 5e^{-x}$$ *(2 marks)*

(b) Given that

$$9\sinh x - \cosh x = 8$$

find the exact value of $\tanh x$. *(7 marks)*

Question 8: Jan 2010

(a) Use the definitions $\cosh x = \frac{1}{2}(e^x + e^{-x})$ and $\sinh x = \frac{1}{2}(e^x - e^{-x})$ to show that

$$\cosh^2 x - \sinh^2 x = 1$$ *(3 marks)*

(b) (i) Express

$$5\cosh^2 x + 3\sinh^2 x$$

in terms of $\cosh x$. *(1 mark)*

(ii) Sketch the curve $y = \cosh x$. *(1 mark)*

(iii) Hence solve the equation

$$5\cosh^2 x + 3\sinh^2 x = 9.5$$

giving your answers in logarithmic form. *(4 marks)*

Hyperbolic functions – exam questions-answers

Question 1: June 2006

$a)i)\ y = Cosh\,x - 3Sinh\,x$ *meets* $y = -1$ *at* $(k,-1)$

This gives $-1 = Cosh\,k - 3Sinh\,k$

$$-1 = \frac{1}{2}\left(e^{k} + e^{-k}\right) - \frac{3}{2}\left(e^{k} - e^{-k}\right)$$

$$-1 = \frac{1}{2}e^{k} + \frac{1}{2}e^{-k} - \frac{3}{2}e^{k} + \frac{3}{2}e^{-k}$$

$$-1 = -e^{k} + 2e^{-k} \qquad (\times e^{k})$$

$$-e^{k} = -e^{2k} + 2$$

$$e^{2k} - e^{k} - 2 = 0$$

$ii)$ $\quad (e^{k})^{2} - e^{k} - 2 = 0$

$$(e^{k} - 2)(e^{k} + 1) = 0$$

$e^{k} = 2$ *or* $e^{k} = -1$

$k = \ln(2)$ $\qquad (e^{x}$ *is positive for all* $x)$

Question 2: Jan 2006

$a)i)\ Sinh\theta = \frac{1}{2}(e^{\theta} - e^{-\theta})$ *and* $Cosh\theta = \frac{1}{2}(e^{\theta} + e^{-\theta})$

$$2\,Sinh\theta\,Cosh\theta = 2 \times \frac{1}{2}(e^{\theta} - e^{-\theta}) \times \frac{1}{2}(e^{\theta} + e^{-\theta})$$

$$= \frac{1}{2}\left(e^{2\theta} + e^{0} - e^{0} - e^{-2\theta}\right)$$

$$= \frac{1}{2}(e^{2\theta} - e^{-2\theta}) = Sinh(2\theta)$$

$$ii)\ Cosh^{2}\theta + Sinh^{2}\theta = \left(\frac{1}{2}(e^{\theta} - e^{-\theta})\right)^{2} + \left(\frac{1}{2}(e^{\theta} + e^{-\theta})\right)^{2}$$

$$= \frac{1}{4}\left(e^{2\theta} - 2e^{0} + e^{-2\theta}\right) + \frac{1}{4}\left(e^{2\theta} + 2e^{0} + e^{-2\theta}\right)$$

$$= \frac{1}{4}\left(2e^{2\theta} + 2e^{2\theta}\right) = \frac{1}{2}(e^{2\theta} + e^{-2\theta}) = Cosh(2\theta)$$

Question 3: Jan 2007

$a)\ 4Cosh^{2}x = 7Sinh\,x + 1$

Using $\text{Cosh}^{2}x - Sinh^{2}x = 1$,

The equation becomes

$$4(\text{Sinh}^{2}x + 1) = 7\,Sinh\,x + 1$$

$$4\,Sinh^{2}x - 7\,Sinh\,x + 3 = 0$$

$$(4Sinh\,x - 3)(Sinh\,x - 1) = 0$$

$Sinh\,x = \frac{3}{4}$ *or* $Sinh\,x = 1$

$$b)\ x = Sinh^{-1}(\tfrac{3}{4}) = \ln\left(\frac{3}{4} + \sqrt{\left(\frac{3}{4}\right)^{2} + 1}\right)$$

$$= \ln\left(\frac{3 + \sqrt{25}}{4}\right) = \ln(2)$$

or $x = Sinh^{-1}(1) = \ln\left(1 + \sqrt{1^{2} + 1}\right) = \ln\left(1 + \sqrt{2}\right)$

Question 4: June 2008

$$a)\ 5\sinh x + \cosh x = \frac{5}{2}\left(e^{x} - e^{-x}\right) + \frac{1}{2}\left(e^{x} + e^{-x}\right)$$

$$= \frac{5}{2}e^{x} + \frac{1}{2}e^{x} - \frac{5}{2}e^{-x} + \frac{1}{2}e^{-x}$$

$5\sinh x + \cosh x = 3e^{x} - 2e^{-x}$

$b)\ 5\sinh x + \cosh x + 5 = 0$ *becomes*

$$3e^{x} - 2e^{-x} + 5 = 0 \qquad (\times e^{x})$$

$$3e^{2x} - 2 + 5e^{x} = 0$$

$$3e^{2x} + 5e^{x} - 2 = 0$$

$$(3e^{x} - 1)(e^{x} + 2) = 0$$

$e^{x} = \frac{1}{3}$ *or* $e^{x} = -2$ *(no solution)*

$x = \ln\left(\frac{1}{3}\right)$

Question 5: Jan 2009

$$a)\ 1 + 2\sinh^{2}\theta = 1 + 2 \times \frac{1}{4}\left(e^{\theta} - e^{-\theta}\right)^{2} = 1 + \frac{1}{2}\left(e^{2\theta} + e^{-2\theta} - 2\right)$$

$$1 + 2\sinh^{2}\theta = \frac{1}{2}\left(e^{2\theta} + e^{-2\theta}\right) = \cosh 2\theta$$

$b)\ 3\cosh 2\theta = 2\sinh\theta + 11$

$$3(1 + 2\sinh^{2}\theta) = 2\sinh\theta + 11$$

$$3 + 6\sinh^{2}\theta - 2\sinh\theta - 11 = 0$$

$$6\sinh^{2}\theta - 2\sinh\theta - 8 = 0$$

$$3\sinh^{2}\theta - \sinh\theta - 4 = 0$$

$$(3\sinh\theta - 4)(\sinh\theta + 1) = 0$$

$\sinh\theta = \frac{4}{3}$ *or* $\sinh\theta = -1$

$$\theta = \sinh^{-1}\left(\frac{4}{3}\right) = \ln\left(\frac{4}{3} + \sqrt{1 + \left(\frac{4}{3}\right)^{2}}\right) = \ln 3$$

or $\theta = \sinh^{-1}(-1) = \ln\left(-1 + \sqrt{1 + (-1)^{2}}\right) = \ln\left(-1 + \sqrt{2}\right)$

Question 6: June 2009

$a)\ y = \tanh x$

$$b)\ u = \tanh x = \frac{\sinh x}{\cosh x} = \frac{\frac{1}{2}\left(e^{x} - e^{-x}\right)}{\frac{1}{2}\left(e^{x} + e^{-x}\right)} = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$$

Factorise num. and den. by e^{-x}

$$u = \tanh x = \frac{e^{-x}(e^{2x} - 1)}{e^{-x}(e^{2x} + 1)} = \frac{e^{2x} - 1}{e^{2x} + 1} = u$$

Now make e^{2x} the subject of the expression

$$e^{2x} - 1 = u(e^{2x} + 1)$$

$$e^{2x} - ue^{2x} = u + 1$$

$$e^{2x}(1 - u) = 1 + u$$

$e^{2x} = \frac{1+u}{1-u}$ $\qquad$ *so* $2x = \ln\left(\frac{1+u}{1-u}\right)$

$x = \frac{1}{2}\ln\left(\frac{1+u}{1-u}\right)$

$c)\,i)\,3\operatorname{sech}^2 x+7\tanh x=5$

$$\operatorname{sech}^2 x=\frac{1}{\cosh^2 x}=1-\tanh^2 x$$

$$3(1-\tanh^2 x)+7\tanh x=5$$

$$3-3\tanh^2 x+7\tanh x-5=0$$

$$3\tanh^2 x-7\tanh x+2=0$$

$ii)\,3\tanh^2 x-7\tanh x+2=0$

$$(3\tanh x-1)(\tanh x-2)=0$$

$$\tanh x=\frac{1}{3}\ or\ \tanh x=2$$

$(no\ solution\ for\ all\ x,-1\le\tanh\le 1)$

$$x=\frac{1}{2}\ln\left(\frac{1+\frac{1}{3}}{1-\frac{1}{3}}\right)=x=\frac{1}{2}\ln(2)$$

Question 7: June 2010

$$a)\,9\sinh x-\cosh x=\frac{9}{2}\left(e^x-e^{-x}\right)-\frac{1}{2}\left(e^x+e^{-x}\right)$$

$$=\frac{8}{2}e^x-\frac{10}{2}e^{-x}$$

$$=4e^x-5e^{-x}$$

$b)\,9\sinh x-\cosh x=8\ \ is\ equivalent\ to$

$$4e^x-5e^{-x}=8\qquad(\times e^x)$$

$$4e^{2x}-5-8e^x=0$$

$$4\left(e^x\right)^2-8e^x-5=0$$

$$(2e^x-5)\left(2e^x+1\right)=0$$

$$e^x=\frac{5}{2}\ or\ e^x=-\frac{1}{2}(no\ solution)$$

$$\tanh x=\frac{e^x-e^{-x}}{e^x+e^{-x}}=\frac{\frac{5}{2}-\frac{2}{5}}{\frac{5}{2}+\frac{2}{5}}=\frac{25-4}{25+4}=\frac{21}{29}$$

$$\tanh x=\frac{21}{29}$$

Question 8: Jan 2010

$$a)\,\cosh^2 x-\sinh^2 x=\frac{1}{4}\left(e^x+e^{-x}\right)^2-\frac{1}{4}\left(e^x-e^{-x}\right)^2$$

$$=\frac{1}{4}\left(e^{2x}+e^{-2x}+2\right)-\frac{1}{4}\left(e^{2x}+e^{-2x}-2\right)$$

$$=\frac{2}{4}+\frac{2}{4}=1$$

$b)\,i)\,5\cosh^2 x+3\sinh^2 x=5\cosh^2 x+3(\cosh^2 x-1)$

$$=8\cosh^2 x-3$$

$ii)\,y=\cosh x\ graph$

$iii)\,5\cosh^2 x+3\sinh^2 x=9.5$

$$8\cosh^2 x-3=9.5$$

$$\cosh^2 x=1.5625$$

$$\cosh x=1.25\ or\ \cosh x=-1.25\ \ (no\ solution)$$

$$x=\cosh^{-1}(1.25)\ or\ x=-\cosh^{-1}(1.25)$$

$$x=\ln\left(1.25+\sqrt{1.25^2-1}\right)=\ln 2\ or\ -\ln 2$$