Eigenvalues - Eigenvectors

Specifications

Matrix Algebra

Invariant points and invariant lines.	
Eigenvalues and eigenvectors of 2×2 and 3×3 matrices.	Characteristic equations. Real eigenvalues only. Repeated eigenvalues may be included.
Diagonalisation of 2×2 and 3×3 matrices.	$\mathbf{M} = \mathbf{UDU}^{-1}$ where $\mathbf{D}$ is a diagonal matrix featuring the eigenvalues and $\mathbf{U}$ is a matrix whose columns are the eigenvectors. Use of the result $\mathrm{M}^n = \mathrm{UD}^n\mathrm{U}^{-1}$

Transformations

Invariant points - Invariant lines

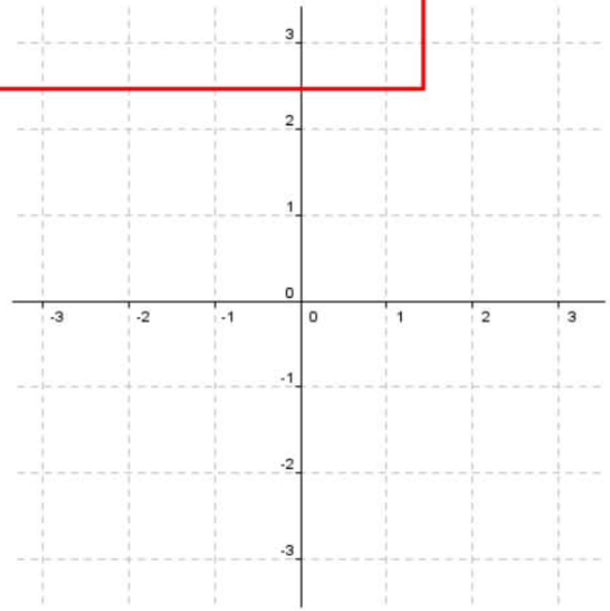
Consider a transformation represented by the matrix **M**

- A point P, with position vector $\mathbf{p}\begin{pmatrix} x \\ y \end{pmatrix}$ is invariant through the transformation when $\mathbf{M} \times \mathbf{p} = \mathbf{p}$

For example: $\mathbf{M} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *and* $\mathbf{p} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

a) Work out the image of P.

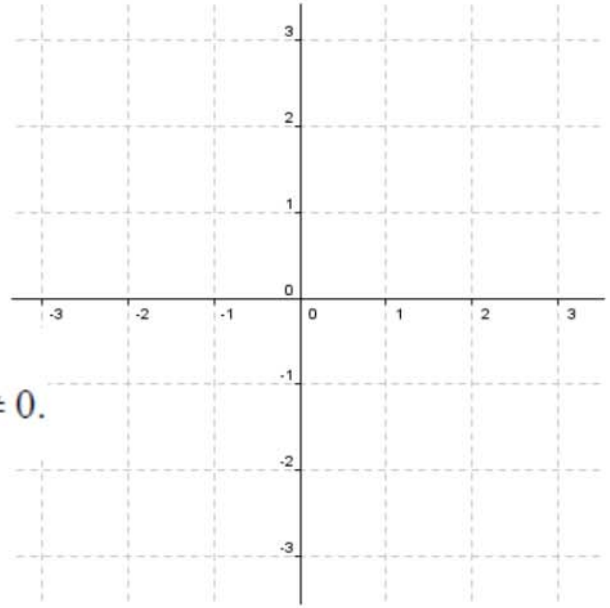
b) Describe the transformation represented by **M**.



An invariant line of a transformation is a line which is left fixed by the transformation.
In some cases, this line will consist of invariant points.
In others, it will consist of points which are moved around on the line.

Be careful, an invariant line may not be a line of invariant points.

How do we find the invariant points and lines?



Invariant points:

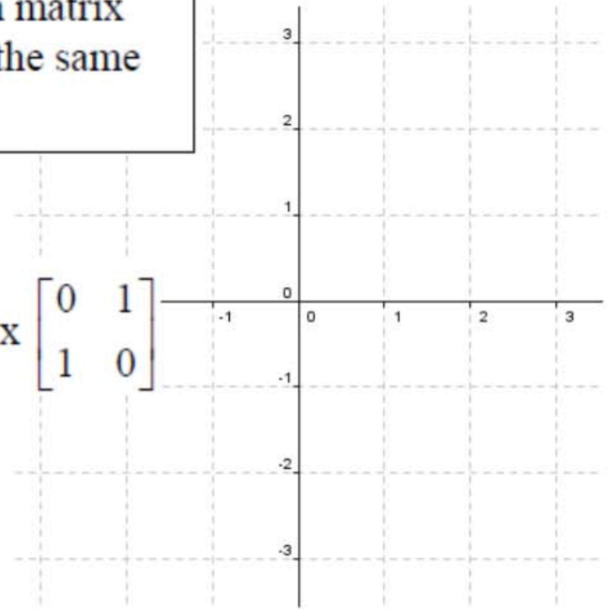
Solve $\boxed{\mathbf{M}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}}$ *or* $\mathbf{M}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Example:

Find the invariant points of the transformation with matrix $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$, where $k \neq 0$.

Invariant lines:

The invariant lines of the transformation with matrix **M** can be found by substituting **x** and **Mx** in the same equation of a line



Example:

Find, in the form $y = mx + c$, the invariant lines of the transformation with matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Exercises:

1. Find all invariant lines, of the form $y = mx$, for the matrix transformations

   (a) $\begin{bmatrix} 5 & 15 \\ -2 & -8 \end{bmatrix}$, (b) $\begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}$.

2. The plane transformation T is defined by

$$x' = 4x + 2y - 7$$
$$y' = -3x - y + 7.$$

   (a) Show that T has a line of invariant points and find its equation.

   (b) Show that there is an infinite number of invariant lines and find the general equation of such lines.

[AQA]

1. (a) $y = -\frac{1}{5}x$; $y = -\frac{2}{3}x$ (b) $y = x$; $y = -\frac{4}{5}x$

2. (a) $3x + 2y = 7$ (b) $y = -\frac{3}{2}x + \frac{7}{2}$ or $y = c - x$

Eigenvectors - Eigenvalues

- If a square matrix **M** and vector **v** satisfy the equation

$$\mathbf{Mv} = \lambda \mathbf{v}, \qquad \mathbf{v} \neq \mathbf{0},$$

then **v** is called an **eigenvector** of the transformation and λ is called an **eigenvalue**.

Example: A transformation is represented by $\mathbf{M} = \begin{pmatrix} 7 & 6 \\ -4 & 18 \end{pmatrix}$

Show that $\mathbf{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ *and* $\mathbf{v} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ are eigenvectors

Find out the corresponding eigenvalues.

- If **v** is an eigenvector, then the line $\mathbf{r} = t\mathbf{v}$ is an invariant line through the origin. In particular, any multiple of **v** is itself an eigenvector.

- Note: If $\lambda = 1$, the eigenvector is an INVARIANT vector through the transformation.

  The line $r = t\mathbf{v}$ is a line of invariant points.

Proof:

The eigenvectors of a transformation determine the directions of all invariant lines which pass through the origin

How do we work out the eigenvalues and eigenvectors?

A special method has been developed to solve the important equation

$$\mathbf{Mv} = \lambda\mathbf{v}, \qquad \mathbf{v} \neq \mathbf{0}.$$

First, it is written as $(\mathbf{M}-\lambda\mathbf{I})\,\mathbf{v}=\mathbf{0}$. If the matrix $\mathbf{M}-\lambda\mathbf{I}$ had an inverse then you could multiply by this inverse, obtaining $\mathbf{v}=\mathbf{0}$. Since $\mathbf{v}\neq\mathbf{0}$, the conclusion must be that $\mathbf{M}-\lambda\mathbf{I}$ is singular.

This leads to the following method for finding eigenvalues:

> The eigenvectors for a matrix **M** can be found by solving $|\mathbf{M}-\lambda\mathbf{I}|=0$

Explanation

Consider a square matrix **M**.

v is an eigenvector of the matrix **M** with eigenvalue λ:

$$\mathbf{Mv}=\lambda\mathbf{v} \text{ with } \mathbf{v}\neq\mathbf{0}$$

Let's re-arrange this equation:

$$\mathbf{Mv}-\lambda\mathbf{v}=\mathbf{0}$$

$$(\mathbf{M}-\lambda\mathbf{I})\mathbf{v}=\mathbf{0}$$

Suppose the matrix $(\mathbf{M}-\lambda\mathbf{I})$ has an inverse, then

$$\mathbf{v}=(\mathbf{M}-\lambda\mathbf{I})^{-1}\times\mathbf{0}=\mathbf{0}$$

but an eigenvector is not the vector **0**.

Conclusion: The matrix ($\mathbf{M}$-$\lambda\mathbf{I}$) is SINGULAR (has no inverse)

$$\Leftrightarrow \det(\mathbf{M}-\lambda\mathbf{I})=0$$

The characteristic equation

The equation $|\mathbf{M}-\lambda\mathbf{I}|=0$ is called the **characteristic equation** for the matrix **M**.

Example:

Find the eigenvalues and eigenvectors of the matrix $\mathbf{M}=\begin{bmatrix}3 & 2 & 2\\ -1 & 0 & 1\\ 1 & 2 & 1\end{bmatrix}$.

Method:

a) Work out $\det(\mathbf{M}\text{-}\lambda\mathbf{I})$ in terms of λ.

b) By solving $\det(\mathbf{M}\text{-}\lambda\mathbf{I})=0$, work out the three eigenvalues λ_1, λ_2 *and* λ_3.

c) Let's call $\mathbf{v}_1=\begin{pmatrix}x\\ y\\ z\end{pmatrix}$ an eigenvector with the eigenvalue λ_1.

Considering the set of equations equivalent to $(\mathbf{M}\text{-}\lambda_1\mathbf{I})\mathbf{v}_1=0$,

work out the component of a possible eigenvector $\mathbf{v}_1$.

d) Do the same wit λ_1 *and* λ_2.

$\lambda=-1, 2 \text{ or } 3$

$\begin{bmatrix}0\\ 1\\ -1\end{bmatrix} \begin{bmatrix}-2\\ 1\\ 0\end{bmatrix} \begin{bmatrix}4\\ -1\\ 1\end{bmatrix}$.

Your turn:

1 Find the eigenvalues and corresponding eigenvectors of the matrices

a $\begin{pmatrix} 2 & 4 \\ 1 & 5 \end{pmatrix}$ **b** $\begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix}$ **c** $\begin{pmatrix} 3 & -2 \\ 0 & 4 \end{pmatrix}$.

2 A transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ is represented by the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & 4 \\ -2 & 9 \end{pmatrix}$$

a Find the eigenvalues of **A**.

b Find Cartesian equations of the two lines passing through the origin which are invariant under T.

3 Find the eigenvalues and corresponding eigenvectors of the matrices

a $\begin{pmatrix} 3 & 0 & 0 \\ 2 & 4 & 2 \\ -2 & 0 & 1 \end{pmatrix}$ **b** $\begin{pmatrix} 4 & -2 & -4 \\ 2 & 3 & 0 \\ 2 & -5 & -4 \end{pmatrix}$.

4 The matrix $\mathbf{A} = \begin{pmatrix} 2 & 2 & -2 \\ -3 & 2 & 0 \\ 1 & 4 & -3 \end{pmatrix}$.

a Show that -1 is the only real eigenvalue of **A**.

b Find an eigenvector corresponding to the eigenvalue -1.

5 The matrix $\mathbf{A} = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 2 & 4 \\ 0 & 2 & 0 \end{pmatrix}$.

a Show that 4 is an eigenvalue of **A** and find the other two eigenvalues of **A**.

b Find an eigenvector corresponding to the eigenvalue 4.

6 The matrix $\mathbf{A} = \begin{pmatrix} 1 & 1 & 3 \\ 2 & 4 & -1 \\ 4 & 4 & 3 \end{pmatrix}$.

Given that 3 is an eigenvalue of **A**,

a find the other two eigenvalues of **A**,

b find eigenvectors corresponding to each of the eigenvalues of **A**.

Answers:

1 **a** The eigenvalues are 1 and 6.
An eigenvector corresponding to the eigenvalue 1 is $\begin{pmatrix} -4 \\ 1 \end{pmatrix}$.
An eigenvector corresponding to the eigenvalue 6 is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

b The eigenvalues are 3 and 5.
An eigenvector corresponding to the eigenvalue 3 is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
An eigenvector corresponding to the eigenvalue 5 is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

c The eigenvalues are 3 and 4.
An eigenvector corresponding to the eigenvalue 3 is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.
An eigenvector corresponding to the eigenvalue 4 is $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

2 **a** 5, 7
b $y = \frac{1}{2}x, y = x$

3 **a** The eigenvalues are 1, 3 and 4.
An eigenvector corresponding to the eigenvalue 1 is $\begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix}$.
An eigenvector corresponding to the eigenvalue 3 is $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.
An eigenvector corresponding to the eigenvalue 4 is $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

b The eigenvalues are −1, 0 and 4.
An eigenvector corresponding to the eigenvalue −1 is $\begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix}$.
An eigenvector corresponding to the eigenvalue 0 is $\begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix}$.
An eigenvector corresponding to the eigenvalue 4 is $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$.

4 **b** $\begin{pmatrix} 1 \\ 1 \\ \frac{5}{2} \end{pmatrix}$

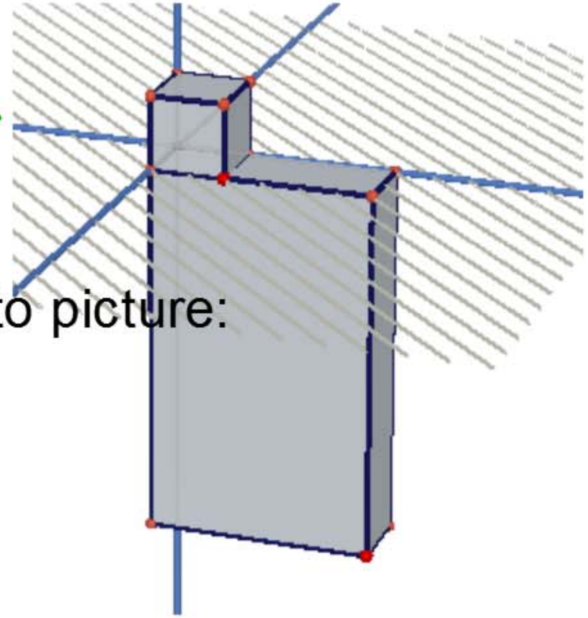
5 **b** $\begin{pmatrix} \frac{1}{2} \\ 2 \\ 1 \end{pmatrix}$

6 **a** −1, 6.
b An eigenvector corresponding to the eigenvalue −1 is $\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$.
An eigenvector corresponding to the eigenvalue 3 is $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.
An eigenvector corresponding to the eigenvalue 6 is $\begin{pmatrix} 5 \\ 1 \\ 8 \end{pmatrix}$.

Diagonalisation

A **diagonal matrix** is a square matrix which only has non-zero elements on the leading diagonal. For example,

$$\begin{bmatrix} 1 & 0 \\ 0 & -5 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -5 \end{bmatrix}.$$



A transformation represented by a diagonal matrix is easy to picture: stretches in the three direction x, y and z.

Introduction:

The matrix $\mathbf{M}=\begin{pmatrix} 4 & 1 \\ 6 & 3 \end{pmatrix}$

$a)$ Find the eigenvalues λ_1 and λ_2 and the associated eigenvectors $\mathbf{v}_1\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ *and* $\mathbf{v}_2 = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

$b)$ Let's call $\mathbf{V}=[\mathbf{v}_1 \ \ \mathbf{v}_2]=\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ *and* $\mathbf{D}=\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$. *Work out* $\mathbf{VDV}^{-1}$

The process of expressing a matrix in the form $\mathbf{VDV}^{-1}$, where **D** is a diagonal, is called **diagonalisation**.

An $n \times n$ matrix **M** which has n independent eigenvectors can be expressed as $\mathbf{VDV}^{-1}$, where **V** is an invertible matrix and **D** is a diagonal matrix

V consists of the eigenvectors of **M**, and **D** consists of the eigenvalues. The order of the eigenvalues in **D** corresponds to the order of the eigenvectors in **V**

What is the geometrical significance of the diagonalisation?

$$\mathbf{M} = \begin{bmatrix} 4 & 1 \\ 6 & 3 \end{bmatrix} = \mathbf{VDV}^{-1} \ \ with \ \ \mathbf{V} = \begin{bmatrix} 1 & 1 \\ -3 & 2 \end{bmatrix} \ \ and \ \ \mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}$$

It is a change of set of axes:

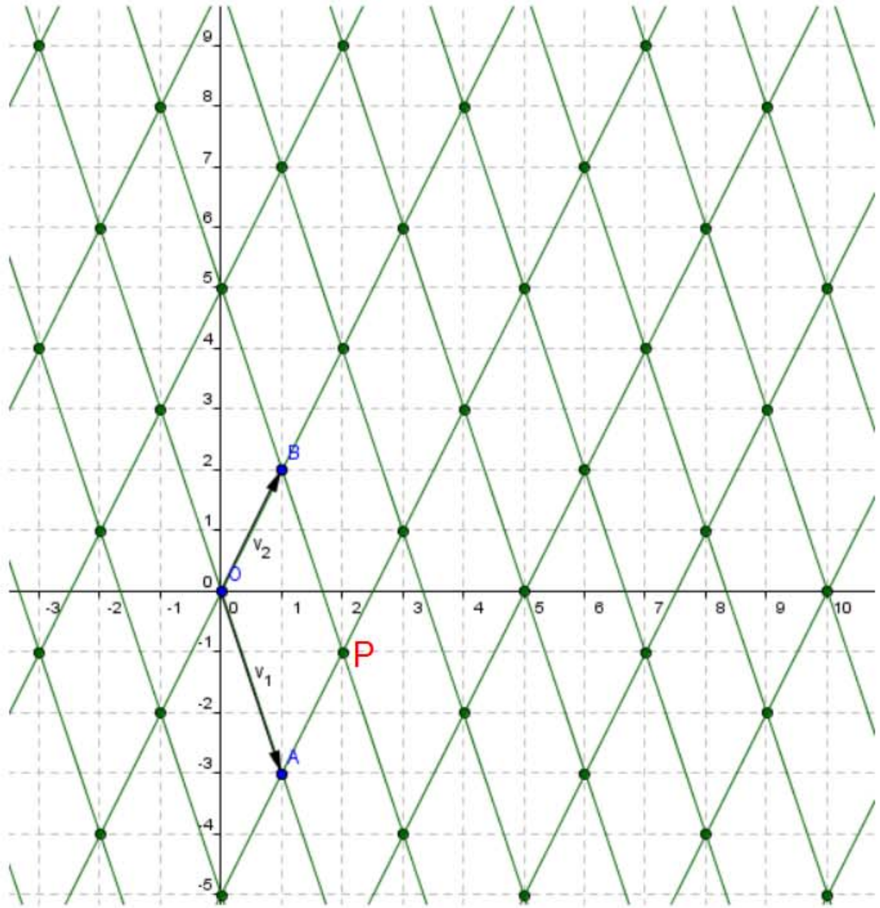
In the set of axes with base vectors (**i**,**j**),

the transformation is **M** (difficult to describe)

BUT

in the set of axes with base vectors ($\mathbf{v}_1, \mathbf{v}_2$),

the tranformation is **D** (a stretch in the $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *direction*)

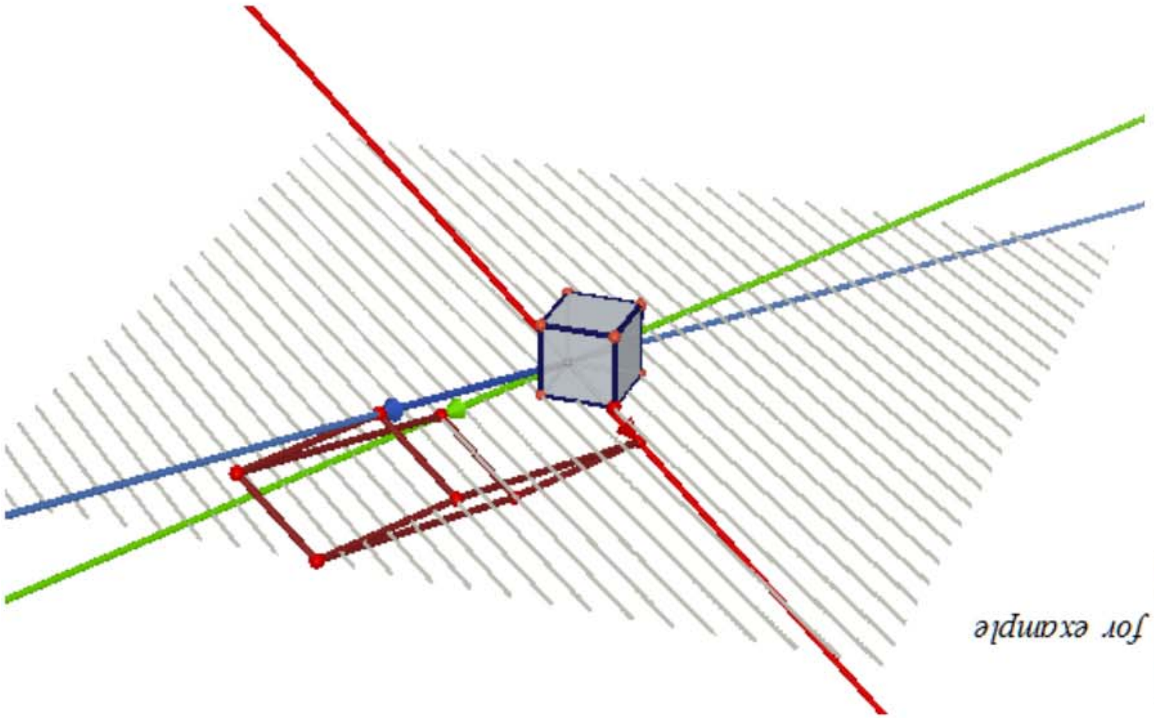


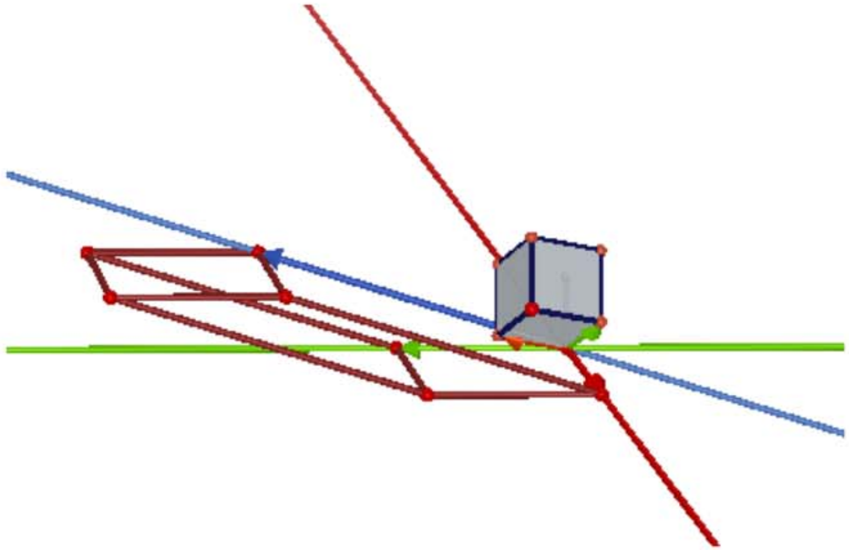
Example:

Consider the point P with coordinates (2,1) in (O,**i**,**j**)

An exercise in 3D:

Find an invertible matrix **V** and a diagonal matrix **D** such that $\begin{bmatrix} 3 & 2 & 2 \\ -1 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}$





$\mathbf{V} = \begin{bmatrix} 0 & 2 & 4 \\ 1 & -1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$ $\mathbf{D} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ *and* $\mathbf{V}^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 2 & -2 \\ 0 & -4 & 4 \\ 1 & 2 & 2 \end{bmatrix}$ *for example*

Diagonalisation and properties

- When a matrix $\mathbf{M}$ can be diagonalised so that $\mathbf{M}=\mathbf{VDV}^{-1}$

  then $\mathbf{V}^{-1}\mathbf{MV} = \mathbf{D}$

- ## Determinants

  When a matrix $\mathbf{M}$ can be diagonalised so that $\mathbf{M}=\mathbf{VDV}^{-1}$

  then $\det(\mathbf{M}) = \det(\mathbf{D}) = \lambda_1 \times \lambda_2 \times \ldots$

- ## Powers of **M**

  If $\mathbf{M} = \mathbf{VDV}^{-1}$, then $\boxed{\mathbf{M}^n = \mathbf{VD}^n\mathbf{V}^{-1}}$

Prove it by induction

Exercises:

1. Write down the eigenvalues and eigenvectors of the matrix $\mathbf{M} = \mathbf{VDV}^{-1}$, where

$$\mathbf{D} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \text{ and } \mathbf{V} = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 0 & 1 \\ 2 & -1 & 1 \end{bmatrix}.$$

2. Express $\begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix}$ in the form $\mathbf{VDV}^{-1}$, where $\mathbf{D}$ is a diagonal matrix.

3. The matrix $\mathbf{M}$ is of the form $\mathbf{V}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\mathbf{V}^{-1}$. Find the determinant of $\mathbf{M}$.

4. Diagonalise each of the following matrices.

(a) $\begin{bmatrix} 5 & 5 & 7 \\ -10 & 3 & 5 \\ 18 & 8 & 10 \end{bmatrix}$, (b) $\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 5 \\ 1 & 1 & -1 \end{bmatrix}$, (c) $\begin{bmatrix} 2 & 0 & -2 \\ 0 & 4 & 0 \\ -2 & 0 & 5 \end{bmatrix}$.

5. The matrix $\mathbf{M} = \begin{bmatrix} 2 & 3 & 0 \\ 2 & 4 & 1 \\ 0 & -3 & 2 \end{bmatrix}$.

(a) Show that $\mathbf{M}$ has an eigenvector $\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$. Find the corresponding eigenvalue.

(b) Hence find two further eigenvalues and corresponding eigenvectors.

(c) Express $\mathbf{M}$ in the form $\mathbf{VDV}^{-1}$.

6. (a) Express $\mathbf{M} = \frac{1}{6}\begin{bmatrix} -1 & 9 & -5 \\ 1 & 3 & -1 \\ -4 & 0 & -2 \end{bmatrix}$ in the form $\mathbf{VDV}^{-1}$, where $\mathbf{D}$ is a diagonal matrix.

(b) Hence find $\mathbf{M}^n$ for any integral n.

7. The matrix $\mathbf{A}$ is given by $\mathbf{A} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$.

(a) (i) Find the eigenvalues of $\mathbf{A}$.

(ii) For each eigenvalue find a corresponding eigenvector.

(b) Given that $\mathbf{U} = \begin{bmatrix} a & 5 \\ -3 & b \end{bmatrix}$, write down the values of a and b such that

$\mathbf{U}^{-1}\mathbf{AU} = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}$. [NEAB]

8. The matrix $\mathbf{A}$ is given by $\begin{bmatrix} 7 & 4 \\ -1 & 3 \end{bmatrix}$. The plane transformation $\mathbf{T}$ is such that

$\mathbf{T}: \begin{bmatrix} x \\ y \end{bmatrix} \to \mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix}$.

(a) (i) Show that $\mathbf{A}$ has only one eigenvalue. Find this eigenvalue and a corresponding eigenvector.

(ii) Hence, or otherwise, determine a Cartesian equation of the fixed line of $\mathbf{T}$.

(b) Under $\mathbf{T}$, a square with area 1 cm^2 is transformed into a parallelogram with area $d\text{ cm}^2$. Find the value of d.

[AEB, 1996]

9. Express $\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$ in the form $\mathbf{VDV}^{-1}$, where $\mathbf{D}$ is a diagonal matrix.

Answers

1. $2,\begin{bmatrix}3\\1\\2\end{bmatrix};\quad -1,\begin{bmatrix}2\\0\\-1\end{bmatrix};\quad 5,\begin{bmatrix}1\\1\\1\end{bmatrix}$

2. $\mathbf{D}=\begin{bmatrix}1&0\\0&2\end{bmatrix};\quad \mathbf{V}=\begin{bmatrix}2&1\\-3&-1\end{bmatrix}$

3. 6

4. (a) $\mathbf{D}=\begin{bmatrix}1&0&0\\0&-2&0\\0&0&19\end{bmatrix};\quad \mathbf{V}=\begin{bmatrix}11&2&1\\-90&21&0\\58&-17&2\end{bmatrix}$

(b) $\mathbf{D}=\begin{bmatrix}1&0&0\\0&6&0\\0&0&-2\end{bmatrix};\quad \mathbf{V}=\begin{bmatrix}1&16&0\\-1&19&1\\0&5&-1\end{bmatrix}$

(c) $\mathbf{D}=\begin{bmatrix}4&0&0\\0&1&0\\0&0&6\end{bmatrix};\quad \mathbf{V}=\begin{bmatrix}0&2&1\\1&0&0\\0&1&-2\end{bmatrix}$

5. (a) 2 (b) $1,\begin{bmatrix}-3\\1\\3\end{bmatrix};\quad 5,\begin{bmatrix}1\\1\\-1\end{bmatrix}$ (c) $\mathbf{D}=\begin{bmatrix}1&0&0\\0&2&0\\0&0&5\end{bmatrix};\quad \mathbf{V}=\begin{bmatrix}-3&1&1\\1&0&1\\3&-2&-1\end{bmatrix}$

6. (a) $\mathbf{V}=\begin{bmatrix}2&1&-1\\1&0&1\\-1&1&2\end{bmatrix};\quad \mathbf{D}=\begin{bmatrix}1&0&0\\0&-1&0\\0&0&0\end{bmatrix}$

(b) $\mathbf{M}^n=\mathbf{V}\mathbf{D}^n\mathbf{V}^{-1}$: $\mathbf{D}^n=\begin{bmatrix}1&0&0\\0&-1&0\\0&0&0\end{bmatrix}$ for n odd; $\mathbf{D}^n=\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$ for n even

Hence $\mathbf{M}^n=\mathbf{M}$ for n odd

and $\mathbf{M}^n=\frac{1}{6}\begin{bmatrix}5&3&-1\\1&3&-1\\2&-6&4\end{bmatrix}$ for n even

7. (a)(i) $-1,\ 5$ (ii) $\begin{bmatrix}2\\-1\end{bmatrix},\ \begin{bmatrix}1\\1\end{bmatrix}$ (b) $a=6,\quad b=5$

8. (a)(i) $(\lambda-5)^2=0\ \Rightarrow \lambda=5:\ \begin{bmatrix}2\\-1\end{bmatrix}$ (ii) $y=-\frac{1}{2}x$ (b) 25

9. $\begin{bmatrix}1&1\\1&-1\end{bmatrix}\begin{bmatrix}-1&0\\0&3\end{bmatrix}\begin{bmatrix}1&1\\1&-1\end{bmatrix}^{-1}$