Eigenvalues, Eigenvectors – exam questions

Question 1: Jan 2006 – Q7

The matrix $\mathbf{M} = \begin{bmatrix} 1 & -1 & 1 \\ 3 & -3 & 1 \\ 3 & -5 & 3 \end{bmatrix}$.

(a) Given that $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ are eigenvectors of **M**, find the eigenvalues corresponding to **u** and **v**. *(5 marks)*

(b) Given also that the third eigenvalue of **M** is 1, find a corresponding eigenvector. *(6 marks)*

(c) (i) Express the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ in terms of **u** and **v**. *(1 mark)*

(ii) Deduce that $\mathbf{M}^n \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \lambda^n \mathbf{u} + \mu^n \mathbf{v}$, where λ and μ are scalar constants whose values should be stated. *(4 marks)*

(iii) Hence prove that, for all positive **odd** integers n,

$$\mathbf{M}^n \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2^n \\ 0 \\ 2^n \end{bmatrix}$$ *(3 marks)*

Question 2: June 2006 – Q8

For real numbers a and b, with $b \neq 0$ and $b \neq \pm a$, the matrix

$$\mathbf{M} = \begin{bmatrix} a & b+a \\ b-a & -a \end{bmatrix}$$

(a) (i) Show that the eigenvalues of **M** are b and $-b$. *(3 marks)*

(ii) Show that $\begin{bmatrix} b+a \\ b-a \end{bmatrix}$ is an eigenvector of **M** with eigenvalue b. *(2 marks)*

(iii) Find an eigenvector of **M** corresponding to the eigenvalue $-b$. *(2 marks)*

(b) By writing **M** in the form $\mathbf{UDU}^{-1}$, for some suitably chosen diagonal matrix **D** and corresponding matrix **U**, show that

$$\mathbf{M}^{11} = b^{10}\mathbf{M}$$ *(7 marks)*

Question 3: Jan 2007 – Q6

(a) Find the eigenvalues and corresponding eigenvectors of the matrix

$$\mathbf{X} = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}$$ *(6 marks)*

(b) (i) Write down a diagonal matrix $\mathbf{D}$, and a suitable matrix $\mathbf{U}$, such that

$$\mathbf{X} = \mathbf{U}\mathbf{D}\mathbf{U}^{-1}$$ *(2 marks)*

(ii) Write down also the matrix $\mathbf{U}^{-1}$. *(1 mark)*

(iii) Use your results from parts (b)(i) and (b)(ii) to determine the matrix $\mathbf{X}^5$ in the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where a, b, c and d are integers. *(3 marks)*

Question 4: June 2007 – Q7

(a) The matrix $\mathbf{M} = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}$ represents a shear.

(i) Find $\det \mathbf{M}$ and give a geometrical interpretation of this result. *(2 marks)*

(ii) Show that the characteristic equation of $\mathbf{M}$ is $\lambda^2 - 2\lambda + 1 = 0$, where λ is an eigenvalue of $\mathbf{M}$. *(2 marks)*

(iii) Hence find an eigenvector of $\mathbf{M}$. *(3 marks)*

(iv) Write down the equation of the line of invariant points of the shear. *(1 mark)*

(b) The matrix $\mathbf{S} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ represents a shear.

(i) Write down the characteristic equation of $\mathbf{S}$, giving the coefficients in terms of a, b, c and d. *(2 marks)*

(ii) State the numerical value of $\det \mathbf{S}$ and hence write down an equation relating a, b, c and d. *(2 marks)*

(iii) Given that the only eigenvalue of $\mathbf{S}$ is 1, find the value of $a + d$. *(2 marks)*

Question 5: Jan 2008 – Q4

The matrix $\mathbf{T}$ has eigenvalues 2 and -2, with corresponding eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ respectively.

(a) Given that $\mathbf{T} = \mathbf{U}\mathbf{D}\mathbf{U}^{-1}$, where $\mathbf{D}$ is a diagonal matrix, write down suitable matrices $\mathbf{U}$, $\mathbf{D}$ and $\mathbf{U}^{-1}$. *(3 marks)*

(b) Hence prove that, for all **even** positive integers n,

$$\mathbf{T}^n = \mathrm{f}(n)\,\mathbf{I}$$

where $\mathrm{f}(n)$ is a function of n, and $\mathbf{I}$ is the 2×2 identity matrix. *(5 marks)*

Question 6: Jan 2008 – Q7

The non-singular matrix $\mathbf{M} = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix}$.

(a) (i) Show that

$$\mathbf{M}^2 + 2\mathbf{I} = k\mathbf{M}$$

for some integer k to be determined. *(3 marks)*

(ii) By multiplying the equation in part (a)(i) by $\mathbf{M}^{-1}$, show that

$$\mathbf{M}^{-1} = a\mathbf{M} + b\mathbf{I}$$

for constants a and b to be found. *(3 marks)*

(b) (i) Determine the characteristic equation of $\mathbf{M}$ and show that $\mathbf{M}$ has a repeated eigenvalue, 1, and another eigenvalue, 2. *(6 marks)*

(ii) Give a full set of eigenvectors for each of these eigenvalues. *(5 marks)*

(iii) State the geometrical significance of each set of eigenvectors in relation to the transformation with matrix $\mathbf{M}$. *(3 marks)*

Question 7: June 2008 – Q1

Find the eigenvalues and corresponding eigenvectors of the matrix $\begin{bmatrix} 7 & 12 \\ 12 & 0 \end{bmatrix}$. *(6 marks)*

Question 8: June 2008 – Q5

A plane transformation is represented by the 2×2 matrix $\mathbf{M}$. The eigenvalues of $\mathbf{M}$ are 1 and 2, with corresponding eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ respectively.

(a) State the equations of the invariant lines of the transformation and explain which of these is also a line of invariant points. *(3 marks)*

(b) The diagonalised form of $\mathbf{M}$ is $\mathbf{M} = \mathbf{U}\,\mathbf{D}\,\mathbf{U}^{-1}$, where $\mathbf{D}$ is a diagonal matrix.

(i) Write down a suitable matrix $\mathbf{D}$ and the corresponding matrix $\mathbf{U}$. *(2 marks)*

(ii) Hence determine $\mathbf{M}$. *(4 marks)*

(iii) Show that $\mathbf{M}^n = \begin{bmatrix} 1 & \mathrm{f}(n) - 1 \\ 0 & \mathrm{f}(n) \end{bmatrix}$ for all positive integers n, where $\mathrm{f}(n)$ is a function of n to be determined. *(3 marks)*

Question 9: Jan 2009 – Q4

(a) Given that -1 is an eigenvalue of the matrix $\mathbf{M} = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix}$, find a corresponding eigenvector. *(3 marks)*

(b) Determine the other two eigenvalues of $\mathbf{M}$, expressing each answer in its simplest surd form. *(8 marks)*

Eigenvalues, Eigenvectors – exam questions MS

Question 1: Jan 2006 – Q7

7(a)	$\mathbf{M}\begin{bmatrix}1\\1\\2\end{bmatrix}=\begin{bmatrix}2\\2\\4\end{bmatrix}, \mathbf{M}\begin{bmatrix}0\\1\\1\end{bmatrix}=\begin{bmatrix}0\\-2\\-2\end{bmatrix}$	M1 A1 A1	
	$\Rightarrow \lambda_U = 2 \Rightarrow \lambda_V = -2$	B1 B1√	5
(b)	$\mathbf{M}\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}a-b+c\\3a-3b+c\\3a-5b+3c\end{bmatrix}=\begin{bmatrix}a\\b\\c\end{bmatrix}$	M1 A1 dM1	
	$\Rightarrow b = c$ and $3a + c = 4b$	A1 A1	
	Evec. is any non-zero multiple of $\begin{bmatrix}1\\1\\1\end{bmatrix}$	A1	6
c)(i)	$\begin{bmatrix}1\\2\\3\end{bmatrix}=\mathbf{u}+\mathbf{v}$ or $\begin{bmatrix}1\\1\\2\end{bmatrix}+\begin{bmatrix}0\\1\\1\end{bmatrix}$	B1	1
(ii)	$\mathbf{M}^n\begin{bmatrix}1\\2\\3\end{bmatrix}=\mathbf{M}^n(\mathbf{u}+\mathbf{v})=\mathbf{M}^n\mathbf{u}+\mathbf{M}^n\mathbf{v}$	M1 A1	
	$=2^n\,\mathbf{u}+(-2)^n\,\mathbf{v}$	M1 A1	4
(iii)	$\mathbf{M}^n\begin{bmatrix}1\\2\\3\end{bmatrix}=\begin{bmatrix}2^n\\2^n\\2\times2^n\end{bmatrix}-\begin{bmatrix}0\\2^n\\2^n\end{bmatrix}$	M1 B1	
	$=\begin{bmatrix}2^n\\0\\2^n\end{bmatrix}$	A1	3

Question 2: June 2006 – Q8

(a)(i)	Char. Equation is	M1	
	$\lambda^2-0\lambda+\{-a^2-(b^2-a^2)\}=0$	A1	
	i.e. $\lambda^2-b^2=0$		
	and $\lambda=\pm b$	A1	3
(ii)	$\lambda=b \Rightarrow (a-b)x+(a+b)y=0$	M1	
	$\Rightarrow y=\dfrac{b-a}{b+a}x \Rightarrow$ evecs. $\alpha\begin{bmatrix}b+a\\b-a\end{bmatrix}$	A1	2
(iii)	$\lambda=-b \Rightarrow (a+b)x+(a+b)y=0$ (etc.)	M1	
	$\Rightarrow y=-x \Rightarrow$ evecs. $\beta\begin{bmatrix}1\\-1\end{bmatrix}$	A1	2
(b)	$\mathbf{D}=\begin{bmatrix}b&0\\0&-b\end{bmatrix}$, $\mathbf{U}=\begin{bmatrix}b+a&1\\b-a&-1\end{bmatrix}$	B1 B1	
	and $\mathbf{U}^{-1}=\dfrac{-1}{2b}\begin{bmatrix}-1&-1\\a-b&a+b\end{bmatrix}$	B1√	
	$\mathbf{D}^{11}=\begin{bmatrix}b^{11}&0\\0&-b^{11}\end{bmatrix}$	B1	
	$\mathbf{M}^n=\mathbf{U}\,\mathbf{D}^n\,\mathbf{U}^{-1}$ used	M1	
	$\mathbf{M}^{11}=\dfrac{1}{2}b^{10}\begin{bmatrix}b+a&1\\b-a&-1\end{bmatrix}\begin{bmatrix}1&1\\a-b&a+b\end{bmatrix}$		
	or $\dfrac{1}{2}b^{10}\begin{bmatrix}b+a&-1\\b-a&1\end{bmatrix}\begin{bmatrix}1&1\\b-a&-a-b\end{bmatrix}$	A1	
	$=\dfrac{1}{2}b^{10}\begin{bmatrix}2a&2(a+b)\\2(b-a)&-2a\end{bmatrix}=b^{10}\,\mathbf{M}$	A1	7
	Alternative to (a)(ii)		
	$\mathbf{M}\begin{bmatrix}b+a\\b-a\end{bmatrix}=\begin{bmatrix}ab+a^2+b^2-a^2\\b^2-a^2-ab+a^2\end{bmatrix}=b\begin{bmatrix}b+a\\b-a\end{bmatrix}$	M1 A1	
	Alternative to (b)		
	NB $\mathbf{D}^{11}=b^{10}\,\mathbf{D}$	B1	
	Then $\mathbf{M}^{11}=\mathbf{U}\,\mathbf{D}^{11}\,\mathbf{U}^{-1}$	M2 A1	
	$=b^{10}\,\mathbf{U}\,\mathbf{D}\,\mathbf{U}^{-1}=b^{10}\,\mathbf{M}$	M2 A1	
	Total		14

Question 3: Jan 2007 – Q6

6(a)	Char. Eqn. is $\lambda^2-5\lambda-6=0$	B1	
	Solving $\Rightarrow \lambda=-1$ or 6	M1 A1	
	Subst[g]. either λ back	M1	
	$\lambda=-1 \Rightarrow x+y=0 \Rightarrow$ evecs. $\alpha\begin{bmatrix}1\\-1\end{bmatrix}$	A1	
	$\lambda=6 \Rightarrow 5x-2y=0 \Rightarrow$ evecs. $\beta\begin{bmatrix}2\\5\end{bmatrix}$	A1	6
b)(i)	$\mathbf{D}=\begin{bmatrix}-1&0\\0&6\end{bmatrix}$ $\quad\mathbf{U}=\begin{bmatrix}1&2\\-1&5\end{bmatrix}$	B1F B1F	2
(ii)	$\mathbf{U}^{-1}=\frac{1}{7}\begin{bmatrix}5&-2\\1&1\end{bmatrix}$	B1F	1
(iii)	$\mathbf{X}^5=\mathbf{U}\,\mathbf{D}^5\,\mathbf{U}^{-1}$	M1	
	$=\frac{1}{7}\begin{bmatrix}1&2\\-1&5\end{bmatrix}\begin{bmatrix}-1&0\\0&6^5\end{bmatrix}\begin{bmatrix}5&-2\\1&1\end{bmatrix}$	B1F	
	$=\begin{bmatrix}2221&2222\\5555&5554\end{bmatrix}$	A1	3
			12

Question 4: June 2007 – Q7

(a)(i)	det $\mathbf{M}=1 \Rightarrow$ **area** invariant	B1B1	2
(ii)	$\lambda^2-(\text{trace }\mathbf{M})\lambda+(\det\mathbf{M})=0$	M1 A1	2
(iii)	$\lambda=1$ subst[d]. back $\Rightarrow -2x+2y=0$	M1 A1	
	and evec. is $\alpha\begin{pmatrix}1\\1\end{pmatrix}$	A1	3
(iv)	$y=x$ (since $\lambda=1$) or vector eqn.	B1	1
(b)(i)	$\lambda^2-(a+d)\lambda+(ad-bc)=0$	B1 B1	2
(ii)	det $\mathbf{S}=1$	B1	
	$\Rightarrow ad-bc=1$	B1√	2
(iii)	λ=1 twice gives Char. Eqn. $\lambda^2-2\lambda+1=0$	M1	
	$\Rightarrow a+d=2$	A1	2
	Or Subst[g]. $\lambda=1$ in Char. Eqn.		
	$\Rightarrow 1-(a+d)+(ad-bc)=0$	(M1)	
	and $ad-bc=1 \Rightarrow a+d=2$	(A1)	(2)
	Total		**14**

Question 5: Jan 2008 – Q4

(a)	$\mathbf{D}=\begin{bmatrix}2&0\\0&-2\end{bmatrix}$, $\mathbf{U}=\begin{bmatrix}1&2\\1&3\end{bmatrix}$,	B1B1	
	$\mathbf{U}^{-1}=\begin{bmatrix}3&-2\\-1&1\end{bmatrix}$	B1	3
(b)	$\mathbf{T}^n=\begin{bmatrix}1&2\\1&3\end{bmatrix}\begin{bmatrix}2^n&0\\0&2^n\end{bmatrix}\begin{bmatrix}3&-2\\-1&1\end{bmatrix}$	B1 M1	
	$=\begin{bmatrix}2^n&2\times2^n\\2^n&3\times2^n\end{bmatrix}\begin{bmatrix}3&-2\\-1&1\end{bmatrix}$	m1 A1	
	or $\begin{bmatrix}1&2\\1&3\end{bmatrix}\begin{bmatrix}3\times2^n&-2\times2^n\\-2^n&2^n\end{bmatrix}$		
	$=2^n\begin{bmatrix}1&0\\0&1\end{bmatrix}$	A1	5
	Alternative for (b):		
	$\mathbf{D}^n=\begin{bmatrix}2^n&0\\0&2^n\end{bmatrix}$	(B1)	
	$\mathbf{T}^n=\mathbf{U}\,(2^n\,\mathbf{I})\,\mathbf{U}^{-1}$	(M1)	
	$=2^n\,(\mathbf{U}\,\mathbf{I}\,\mathbf{U}^{-1})$	(m2)	
	$=2^n\,\mathbf{I}$	(A1)	(5)
	Total		**8**

Question 6: Jan 2008 – Q7

a)(i)	$\mathbf{M}^2 = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix}$	M1	
	$= \begin{bmatrix} 4 & -3 & 3 \\ 3 & -2 & 3 \\ 3 & -3 & 4 \end{bmatrix}$	A1	
	$\mathbf{M}^2 + 2\mathbf{I} = \begin{bmatrix} 4 & -3 & 3 \\ 3 & -2 & 3 \\ 3 & -3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$		
	$= \begin{bmatrix} 6 & -3 & 3 \\ 3 & 0 & 3 \\ 3 & -3 & 6 \end{bmatrix} = 3\mathbf{M}$	A1	3
(ii)	Multiplying by $\mathbf{M}^{-1}$	M1	
	to get $\mathbf{M} + 2\mathbf{M}^{-1} = 3\mathbf{I}$	A1	
	so that $\mathbf{M}^{-1} = \frac{3}{2}\mathbf{I} - \frac{1}{2}\mathbf{M}$	A1	3
b)(i)	Char. eqn. is $\lambda^3 - 4\lambda^2$	M1A1	
	$+ 5\lambda - 2 = 0$	A1A1	
	ie $(\lambda - 2)(\lambda - 1)^2 = 0$	M1	
	giving $\lambda_1 = 1$ (twice) and $\lambda_2 = 2$	A1	6
(ii)	$\lambda = 1 \Rightarrow x - y + z = 0$ (thrice)	B1	
	Any two independent eigenvectors	M1	
	(eg) $\alpha\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \beta\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$	A1	
	$\lambda = 2 \Rightarrow -y + z = 0$; $x - 2y + z = 0 \Rightarrow x = y = z$; $x - y = 0$	M1	
	$\gamma\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	A1	5
(iii)	For $\lambda = 1$, eigenvectors represent a plane	M1	
	of invariant points	A1	
	For $\lambda = 2$, eigenvectors represent an invariant line	B1	3
	Total		**20**

Question 7: June 2008 – Q1

1	Attempt at char eqn $\lambda^2 - 7\lambda - 144 = 0$	M1	
	Solving quadratic to find evals	M1	
	$\lambda = 16$ or -9	A1	
	$\lambda = 16 \Rightarrow -9x + 12y = 0 \Rightarrow y = \frac{3}{4}x$	M1	
	$\Rightarrow$ evecs $\alpha\begin{bmatrix} 4 \\ 3 \end{bmatrix}$	A1	
	$\lambda = -9 \Rightarrow 16x + 12y = 0 \Rightarrow y = -\frac{4}{3}x$		
	$\Rightarrow$ evecs $\beta\begin{bmatrix} 3 \\ -4 \end{bmatrix}$	A1	6
	Total		**6**

Question 8: June 2008 – Q5

5(a)	$y = 0$ (or "x-axis") and $y = x$	B1,B1	
	$y = 0$ is a line of invariant points since $\lambda = 1$	B1	3
b)(i)	$\mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $\mathbf{U} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$,	B1,B1	2
(ii)	$\mathbf{U}^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$	B1	
	$\mathbf{M} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$	M1	
	$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$ or $\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$	A1	
	$= \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$	A1	4
(iii)	$\mathbf{D}^n = \begin{bmatrix} 1 & 0 \\ 0 & 2^n \end{bmatrix}$	B1	
	$\mathbf{M}^n = \mathbf{U}\,\mathbf{D}^n\,\mathbf{U}^{-1}$	M1	
	$= \begin{bmatrix} 1 & 2^n - 1 \\ 0 & 2^n \end{bmatrix}$	A1	3
	Total		**12**

Question 9: Jan 2009 – Q4

4(a)	Subst^g. $\lambda = -1$ into $\det(\mathbf{M} - \lambda\mathbf{I}) = 0$	M1	
	Solving between $x + y + z = 0$ and $x + y + 2z = 0$	dM1	
	Eigenvector(s) $\alpha\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$	A1	3
(b)	Attempt at Char. Eqn.	M1	
	$\lambda^3 - 5\lambda^2 - 5\lambda + 1 = 0$	A1 × 3	
	Use of division/factor theorem etc.	M1	
	$(\lambda + 1)(\lambda^2 - 6\lambda + 1)$	A1	
	Solving remaining quadratic factor	M1	
	$\lambda_{2,3} = 3 \pm 2\sqrt{2}$	A1	8
	Total		**11**