Mechanics 3

A knowledge of the topics and associated formulae from Modules Mechanics 1, Core 1 and Core 2 is required. A knowledge of the trigonometric identity $\sec^2 x = 1 + \tan^2 x$ is also required. Candidates may use relevant formulae included in the formulae booklet without proof.

Dimensional analysis

Specification

Dimensional Analysis

Finding dimensions of quantities. Prediction of formulae. Checks on working, using dimensional consistency.	Use of the notation $[x]$ and finding the dimensions of quantities in terms of M, L and T. Using this method to predict the indices in proposed formulae, for example, for the period of a simple pendulum. Use to find units, and as a check on working.

Remember your GCSE ?

Dimensional analysis

- Each of these expressions represents a length, an area or a volume. Indicate which it is by writing L, A or V. Each letter represents a length.

a x^2 **b** $2y$ **c** πa **d** πab

e xyz **f** $3x^3$ **g** x^2y **h** $2xy$

u $\frac{(ab + bc)}{d}$ **v** $\frac{ab}{2}$ **w** $(a + b)^2$ **x** $4a^2 + 2ab$

y $3abc + 2abd + 4bcd + 2acd$ **z** $4\pi r^3 + \pi r^2 h$

- Indicate whether each of these expressions is consistent (C) or inconsistent (I). Each letter represents a length.

a $a + b$ **b** $a^2 + b$ **c** $a^2 + b^2$ **d** $ab + c$

e $ab + c^2$ **f** $a^3 + bc$ **g** $a^3 + abc$ **h** $a^2 + abc$

- What power * would make each expression consistent?

a $\pi abc + a^*b$ **b** $\frac{\pi r^* h}{2} + \pi h^* + \frac{rh^2}{2}$

c $\pi a(b^* + ac)$ **d** $a^*b + ab^* + c^3$

Let's take this principal a bit further...

The essence of dimensional analysis is very simple: if you are asked how hot it is outside, the answer is never "2 o'clock". You've got to make sure that the units agree. Quantities which come with units are said to have *dimensions*. In contrast, pure numbers such as 2 or π are said to be *dimensionless*.

the units are hiding within the variables. Nonetheless, it's worth our effort to dig them out. In most situations, it is useful to identify three fundamental dimensions: length L, mass M and time T. The dimensions of all other quantities should be expressible in terms of these. We will denote the dimension of a quantity Y as $[Y]$. Some basic examples include,

Notation

Examples:

$$[\text{Area}] = L^2$$
$$[\text{Speed}] = LT^{-1}$$
$$[\text{Acceleration}] = LT^{-2}$$
$$[\text{Force}] = MLT^{-2}$$
$$[\text{Energy}] = ML^2T^{-2}$$

About Force and Energy

- The unit used for the intensity of a force is N (Newton)

  But the dimension of a force is MLT^{-2}.

  We determine the dimension using the formula $F = ma$:

  $[F] = M \times LT^{-2}$ or $[F] = kg.m.s^{-2}$ (kilograms.metres.second^{-2})

- A similar approach is considered for energy.

  The unit used for the energy is J (Joules)

  But the dimension of the energy is ML^2T^{-2}.

  Using the kinematic energy for example: $E = \frac{1}{2}mv^2$

  $[E] = M \times (LT^{-1})^2 = ML^2T^{-2}$

Example 1 Derived Dimensions of Mechanical Quantities

- The derived dimension of work is

$$[\text{work}] = [\text{force}][\text{length}] = \quad$$

- Power is defined to be the rate of change in time of work so the dimensions are

$$[\text{power}] = \frac{[\text{work}]}{[\text{time}]} = \quad$$

Exercises:

- In the formulae booklet, we can read:

Universal law of gravitation

$$\text{Force} = \frac{Gm_1 m_2}{d^2}$$

Work out the dimension of the gravitational constant G.

- To check your answer.

Here is what you can read on Wikipedia:

Kepler's laws of planetary motion

For circular orbits, Kepler's 3rd Law is also commonly represented as

$$\frac{4\pi^2}{T^2} = \frac{GM}{R^3}$$

Where T is the period,
G is gravitational constant,
M is the mass of the larger body, and
R is the distance between the centers of mass of the two bodies.

Work out the dimension of G using this formula.

Exam question

The magnitude of the resistance force on a moving body is to be modelled as having magnitude kv^n, where v is the speed of the body and k and n are constants.

(a) If $n = 2$, find the dimensions of k. *(3 marks)*

(b) If the dimensions of k are $\mathrm{ML}^{-\frac{1}{2}}\mathrm{T}^q$, find n and q. *(5 marks)*

(a)	$\mathrm{MLT}^{-2} = [k]\mathrm{L}^2\mathrm{T}^{-2}$	M1A1	
	$[k] = \mathrm{ML}^{-1}$	A1	3
(b)	$\mathrm{MLT}^{-2} = \mathrm{ML}^{-\frac{1}{2}}\mathrm{T}^q\mathrm{L}^n\mathrm{T}^{-n}$	M1	
	$1 = -\frac{1}{2} + n$	M1	
	$n = \frac{3}{2}$	A1	
	$-2 = q - n$	M1	
	$q = -\frac{1}{2}$	A1	5
	Total		8

FYI:

International System of Units

Table 1. SI base units

Base quantity	SI base unit Name	Symbol
length	meter	m
mass	kilogram	kg
time	second	s
electric current	ampere	A
thermodynamic temperature	kelvin	K
amount of substance	mole	mol
luminous intensity	candela	cd

Table 2 Dimensions of Some Common Mechanical Quantities

$M \equiv \text{mass}, \; L \equiv \text{length}, \; T \equiv \text{time}$

Quantity	Dimension	MKS unit
Angle	dimensionless[1]	Dimensionless = radian
Steradian	dimensionless	Dimensionless = radian^2
Area	L^2	m^2
Volume	L^3	m^3
Frequency	T^{-1}	$s^{-1} = \text{hertz} = \text{Hz}$
Velocity	$L \cdot T^{-1}$	$m \cdot s^{-1}$
Acceleration	$L \cdot T^{-2}$	$m \cdot s^{-2}$
Angular Velocity	T^{-1}	$\text{rad} \cdot s^{-1}$
Angular Acceleration	T^{-2}	$\text{rad} \cdot s^{-2}$
Density	$M \cdot L^{-3}$	$kg \cdot m^{-3}$
Momentum	$M \cdot L \cdot T^{-1}$	$kg \cdot m \cdot s^{-1}$
Angular Momentum	$M \cdot L^2 \cdot T^{-1}$	$kg \cdot m^2 \cdot s^{-1}$
Force	$M \cdot L \cdot T^{-2}$	$kg \cdot m \cdot s^{-2} = \text{newton} = N$
Work, Energy	$M \cdot L^2 \cdot T^{-2}$	$kg \cdot m^2 \cdot s^{-2} = \text{joule} = J$
Torque	$M \cdot L^2 \cdot T^{-2}$	$kg \cdot m^2 \cdot s^{-2}$
Power	$M \cdot L^2 \cdot T^{-3}$	$kg \cdot m^2 \cdot s^{-3} = \text{watt} = W$
Pressure	$M \cdot L^{-1} \cdot T^{-2}$	$kg \cdot m^{-1} \cdot s^{-2} = \text{pascal} = Pa$