First order linear differential equation

Standard form

A first order linear equation can be re-arrange in the form

$$\frac{dy}{dx}+P(x)y=Q(x) \text{ where } P(x) \text{ and } Q(x) \text{ are two functions.}$$

This form is called the STANDARD form the equation.

Integrating factors

Considering an equation $\frac{dy}{dx}+P(x)y=Q(x)$,

we want to multiple both sides by a function $I(x)$,

so that the left-hand side of the equation becomes the derivative of a product function.

$$\text{i.e } I\times\frac{dy}{dx}+IP\times y=IQ \qquad \text{with } \frac{dI}{dx}=IP$$

Such a function is called an INTEGRATING FACTOR and $I(x)=e^{\int P(x)dx}$

Example:

Find the general solution of the equation $\frac{dy}{dx}-\frac{1}{x}y=x^2$ *where* $x>0$

- The integrating factor $I(x)=e^{\int -\frac{1}{x}dx}=e^{-\ln(x)}=e^{\ln\left(\frac{1}{x}\right)}=\frac{1}{x}$
- *Multiplying* the equation by $I(x)$, it becomes

$$\frac{1}{x}\frac{dy}{dx}-\frac{1}{x^2}y=x \qquad \text{this is} \qquad \frac{d}{dx}\left(\frac{1}{x}\times y\right)=x \text{ and by integrating}$$

$$\frac{1}{x}y=\frac{1}{2}x^2+c \qquad y=\frac{1}{2}x^3+cx \quad c\in\mathbb{R}$$

Substitution

The substitution to use will be given to you in the question.

Use this substitution to transform the given differential equation into one which you can solve using either of the known methods:

(direct integration, separating variables, integrating factors)

Example:

$a)$ Use the substitution $z=\frac{1}{y}$ to transform the diff. eq. $\frac{dy}{dx}+xy=xy^2$ into a diff. eq in z and x.

$b)$ Solve the new differential equation.

$c)$ Find y in terms of x.

$$\text{Solution: } z=\frac{1}{y} \qquad \text{so } y=\frac{1}{z} \text{ and } \frac{dy}{dx}=-\frac{1}{z^2}\frac{dz}{dx}$$

$$\text{after substitution, we have } -\frac{1}{z^2}\frac{dz}{dx}+\frac{x}{z}=\frac{x}{z^2} \qquad \frac{dz}{dx}-xz=-x$$

An integrating factor is $I(x)=e^{\int -xdx}=e^{-\frac{x^2}{2}}$

$$e^{-\frac{x^2}{2}}\frac{dz}{dx}-xe^{-\frac{x^2}{2}}z=-xe^{-\frac{x^2}{2}} \qquad \frac{d}{dx}\left(e^{-\frac{x^2}{2}}z\right)=-xe^{-\frac{x^2}{2}}$$

$$e^{-\frac{x^2}{2}}z=\int -xe^{-\frac{x^2}{2}}dx=e^{-\frac{x^2}{2}}+c \qquad z=1+ce^{\frac{x^2}{2}} \quad \text{this gives } y=\frac{1}{ce^{\frac{x^2}{2}}+1}$$