Differential equations

First order differential equations

Specifications

Differential Equations

The concept of a differential equation and its order.

The relationship of order to the number of arbitrary constants in the general solution will be expected.

Boundary values and initial conditions, general solutions and particular solutions.

Differential Equations – First Order

Analytical solution of first order linear differential equations of the form

$$\frac{dy}{dx} + Py = Q$$

where P and Q are functions of x.

To include use of an integrating factor and solution by complementary function and particular integral.

Vocabulary and definitioins

- **A differential equation is an equation involving the derivatives of a function.**

  The function is usually called y(x) or y

Some examples:

$$\frac{dy}{dx} = 2x + 1 \qquad \frac{dy}{dx} = 2xy$$

$$\frac{dy}{dx} - \frac{y}{x} = x^2 \qquad y\frac{dy}{dx} = x^2 + y^2$$

$$\frac{d^2y}{dx^2} + y = 0 \qquad \frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = \sin x.$$

- The ORDER of a differential equation is the same as the **highest order of derivation** occuring in the equation

When only the first order derivative, $\frac{dy}{dx}$, is involved (as in the first four examples above), the differential equation is said to be of **first order**. When the second order derivative, $\frac{d^2y}{dx^2}$, is involved (as in the last two examples), the differential equation is said to be of **second order**. Differential equations of order 3, 4, … are defined similarly.

- A differential equation is LINEAR if it is linear in y and the derivative of y.

Any equation containing
powers of y and/or its derivative
or products of y and/or its derivatives are non-linear

Exercises:

1. Write down the order of each of these differential equations.

(a) $x\frac{dy}{dx} + y = x^2.$

(b) $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0.$

(c) $\frac{dy}{dx} = x^3 + y^3.$

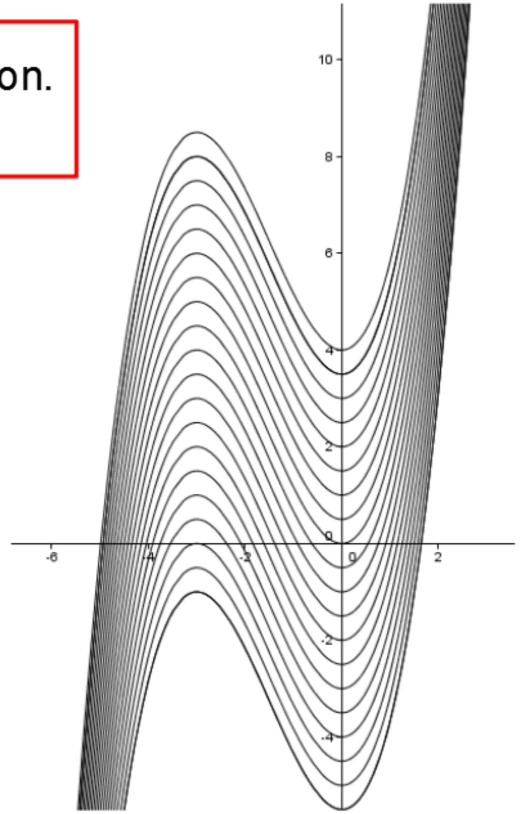
(d) $x\left(\frac{dy}{dx}\right)^2 + y = 1.$

2. State which of the differential equations in Question 1 are linear.

- To solve a differential equation is to find **all the functions** satisfying the equation. All these solutions constitute a **family of solutions**.

Consider the differential equations

$$\frac{dy}{dx} = x^2 + 3x \qquad \frac{dy}{dx} = 2xy^2$$

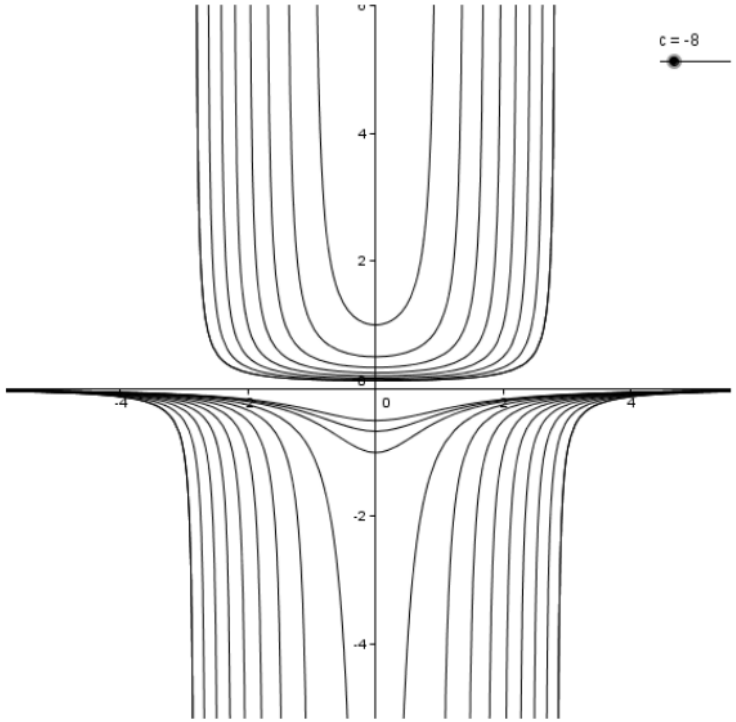


Method 1: Direct integration

$\frac{dy}{dx} = x^2 + 3x$ is of the form $\frac{dy}{dx} = f(x)$

We can integrate this function:

$$y = \frac{1}{3}x^3 + \frac{3}{2}x^2 + c$$



Method 2: Separating variables

$$\frac{dy}{dx} = 2xy^2$$

We are going to re-arrange this equation in the form $g(y)dy = f(x)dx$ *and* integrate both sides

$$\frac{1}{y^2}\frac{dy}{dx} = 2x \qquad so \qquad \int \frac{1}{y^2}dy = \int 2x dx$$

$$-\frac{1}{y} = x^2 + c$$

$$y = -\frac{1}{x^2 + c}$$

- Solutions that involve ARBRITRARY CONSTANTS are called GENERAL SOLUTIONS.

- A solution which contains NO CONSTANT is called a PARTICULAR SOLUTION.

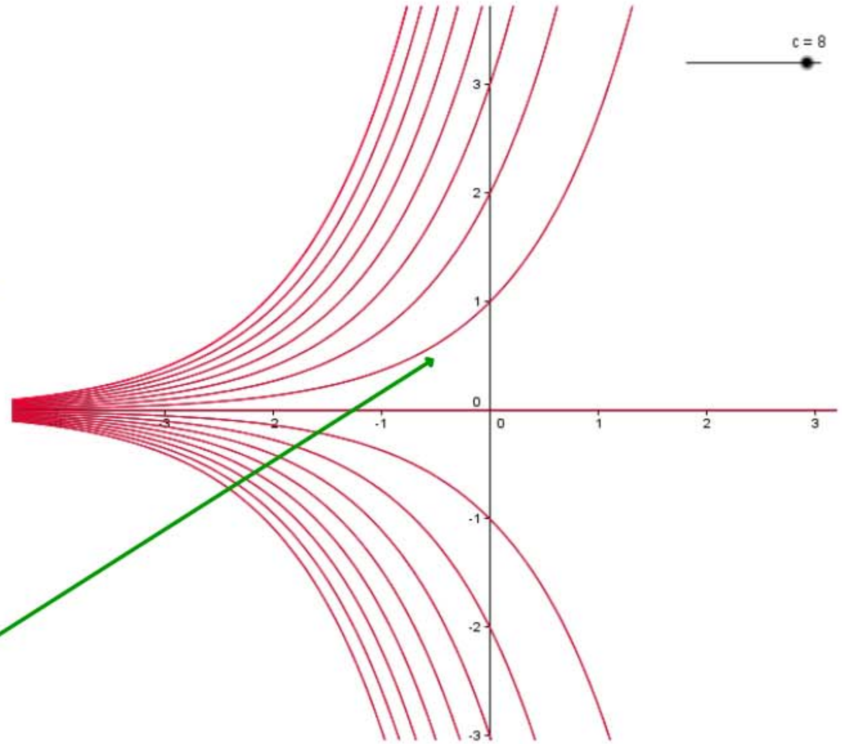
To work out a particular solution, we will be given
- an interval (of validity) for x and/or
- boundary/initial conditions

Worked example:

Consider the differential equation: $\dfrac{dy}{dx} = y$

$a)$ Find the general solutions of this equation.

$b)$ Work out the particular solution satisfying $y(0) = 1$.



$$\frac{dy}{dx} = y \qquad \frac{1}{y}\frac{dy}{dx} = 1$$

$$\int \frac{1}{y}\,dy = \int dx$$

$$\ln y = x + c$$

$$y = e^{x+c}$$

For $x = 0, y = 1$ *gives* $1 = e^{c}$ *so* $c = 0$

$$y = e^{x}$$

Exercises:

1. Find the general solution of the differential equation.

1 $\dfrac{dy}{dx} = 2x$ **2** $\dfrac{dy}{dx} = y$

3 $\dfrac{dy}{dx} = x^2$ **4** $\dfrac{dy}{dx} = \dfrac{1}{x}, \ x > 0$

5 $\dfrac{dy}{dx} = \dfrac{2y}{x}$ **6** $\dfrac{dy}{dx} = \dfrac{x}{y}$

7 $\dfrac{dy}{dx} = e^y$ **8** $\dfrac{dy}{dx} = \dfrac{y}{x(x+1)}, \quad x > 0$

9 $\dfrac{dy}{dx} = \cos x$ **10** $\dfrac{dy}{dx} = y \cot x, \quad 0 < x < \pi$

2. (a) Obtain the general solution of the differential equation

$$\frac{dy}{dx} + xy^2 = 0.$$

(b) Find the particular solution which satisfies the condition $y(0) = 2$.

3. (a) Obtain an equation representing the family of solutions of the differential equation

$$\frac{dy}{dx} = \frac{1}{x^2}, \qquad x > 0.$$

(b) Find the equation of the member of this family whose graph passes through the point $(1, 0)$.

(c) Sketch this graph.

4. (a) Solve the differential equation

$$\frac{dy}{dx} + y = 0, \qquad 0 \le x \le 2,$$

subject to the boundary condition $y(0) = 3$.

(b) Verify that $y(2) \approx 0.406$.

1) $y = x^2 + c$ 2) $y = Ae^x$ 3) $y = \dfrac{x^3}{3} + c$

4) $y = \ln(Ax)$ 5) $y = Ax^2$ 6) $y^2 - x^2 = 2c$

7) $y = \ln\left(\dfrac{1}{-x-c}\right)$ 8) $y = \dfrac{Ax}{x+1} \ x > 0$ 9) $y = Sin(x) + c$

10) $y = ASin(x)$

2. (a) $y = \dfrac{1}{\frac{1}{2}x^2 + C}$ (b) $y = \dfrac{2}{1+x^2}$

3. (a) $y = -\dfrac{1}{x} + C$ (b) $y = 1 - \dfrac{1}{x}$

4. (a) $y = 3e^{-x}$

Analytical solutions of first order differential equations

To solve a diff. eq. using an analytical method means finding an exact expression of the solution functions.

Method 3: Recognising the derivative of a product function

If a differential equation is of the form

$$u \times \frac{dy}{dx} + \frac{du}{dx} \times y = f(x)$$

$$re-write\ it\ as\quad \frac{d}{dx}(u \times y) = f(x)$$

then integrate: $\quad u \times y = \int f(x)dx$

$$y(x) = \frac{1}{u(x)} \int f(x)dx$$

Find the general solution of the following:

$a)\ x^2 \frac{dy}{dx} + 2xy = 0 \qquad b) \frac{1}{x} \times \frac{dy}{dx} - \frac{y}{x^2} = 2 \qquad c) \cos(x) \frac{dy}{dx} - \sin(x) y = \cos(2x)$

Exercises:

In questions 1–6 find the general solution of the exact differential equation

1 $x\dfrac{dy}{dx} + y = \cos x$

2 $e^{-x}\dfrac{dy}{dx} - e^{-x}y = xe^x$

3 $\sin x\dfrac{dy}{dx} + y\cos x = 3$

4 $\dfrac{1}{x}\dfrac{dy}{dx} - \dfrac{1}{x^2}y = e^x$

5 $x^2e^y\dfrac{dy}{dx} + 2xe^y = x$

6 $4xy\dfrac{dy}{dx} + 2y^2 = x^2$

7 **a** Find the general solution of the differential equation

$$x^2\frac{dy}{dx} + 2xy = 2x + 1.$$

In the following answers, c and A are constants.

1 $y = \frac{1}{x}\sin x + \frac{c}{x}$

2 $y = xe^{2x} - e^{2x} + ce^x$

3 $y = 3x\operatorname{cosec} x + c\operatorname{cosec} x$

4 $y = xe^x + cx$

5 $y = \ln\left[\frac{1}{2} + \frac{c}{x^2}\right]$

6 $y = \pm\sqrt{\left(\frac{1}{6}x^2 + \frac{c}{2x}\right)}$

7 **a** $y = 1 + \frac{1}{x} + \frac{c}{x^2}$

Integrating factors

A first order linear differential equation can be re-arrange in the form

$$\frac{dy}{dx} + P(x)y = Q(x) \qquad where\, P(x)\, and\, Q(x)\, are\ two\ \ functions$$

This is called the STANDARD FORM of the first order linear diff.eq.

The principle: to multiple both side of the equation, in its standard form, by a function I(x), called the integrating factor, so that the left-hand side of equation becomes the derivative of a product.

Let's work out an expression of the integrating factor:

Let's call the integrating factor $I(x)$

We now multiply the equation by $I(x)$

$$I(x)\frac{dy}{dx} + I(x)P(x)y = Q(x)$$

We would like that $\frac{d}{dx}(I(x)) = I(x)P(x)$

$or\ \ for\ \ short\ \ \frac{dI}{dx} = IP$

This is a differential equation we can solve by separating the variables:

$$\frac{1}{I}\frac{dI}{dx} = P$$

$$\int \frac{1}{I} dI = \int\ Pdx$$

$$\ln(I) = \int\ Pdx$$

$$I = e^{\int\ p(x)dx}$$

Knowing the function P, we can now work out the functionm $I(x)$, an integrating factor

Method 4: Integrating factor

The standard form of the equation : $\frac{dy}{dx}+Py=Q$

is equivalent to $I\frac{dy}{dx}+IPy=IQ$ *where* the integrating factor $I(x)=e^{\int P(x)dx}$

is equivalent to $I\frac{dy}{dx}+\frac{dI}{dx}y=IQ$

is equivalent to $\frac{d}{dx}(Iy)=IQ$

To solve, integrate the equation (both sides)

Important: The equation must be written in its standard form before applying the integrating factor.

Worked exercise:

(a) Find the integrating factor of the differential equation $\frac{dy}{dx}-\frac{y}{x}=x^2$, where $x>0$.

(b) Hence find

(i) the general solution,

(ii) the particular solution satisfying the condition $y(1)=0$.

An integrating factor is $I=e^{\int -\frac{1}{x}dx}=e^{-\ln x}=\frac{1}{x}$

By multiplying the equation by $\frac{1}{x}$, *we have*

$\frac{1}{x}\frac{dy}{dx}-\frac{1}{x^2}y=x$

This is $\frac{d}{dx}\left(\frac{1}{x}\times y\right)=x$ and by integrating both sides:

$$\frac{y}{x}=\frac{1}{2}x^2+c$$

$$y=\frac{1}{2}x^3+cx$$

ii) When $x=1, y=0$ *so* $0=\frac{1}{2}+c \qquad c=-\frac{1}{2}$

The particular solution is $y=\frac{1}{2}(x^2-x)$

Exercises

1.Work out the general solution of the following differential equations

1 $\frac{dy}{dx} + 2y = e^x$

2 $\frac{dy}{dx} + y \cot x = 1$

3 $\frac{dy}{dx} + y \sin x = e^{\cos x}$

4 $\frac{dy}{dx} - y = e^{2x}$

5 $\frac{dy}{dx} + y \tan x = x \cos x$

6 $\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}$

7 $x^2 \frac{dy}{dx} - xy = \frac{x^3}{x + 2}$ $x > -2$

8 $3x \frac{dy}{dx} + y = x$

9 $(x + 2) \frac{dy}{dx} - y = (x + 2)$

10 $x \frac{dy}{dx} + 4y = \frac{e^x}{x^2}$

2. **11** Find y in terms of x given that

$x \frac{dy}{dx} + 2y = e^x$ and that $y = 1$ when $x = 1$.

12 Solve the differential equation, giving y in terms of x, where

$x^3 \frac{dy}{dx} - x^2y = 1$ and $y = 1$ at $x = 1$.

13 **a** Find the general solution of the differential equation

$\left(x + \frac{1}{x}\right) \frac{dy}{dx} + 2y = 2(x^2 + 1)^2,$

giving y in terms of x.

b Find the particular solution which satisfies the condition that $y = 1$ at $x = 1$.

3. (a) Find the integrating factor of the differential equation $\frac{dy}{dx} + \frac{2y}{x} = 4x, \quad x > 0.$

(b) Hence find the general solution.

4. (a) Find the general solution of the differential equation $\frac{dy}{dx} + 2y = e^{2x}$.

(b) Hence find the particular solution satisfying the condition $y(0) = 0$.

5. Solve the differential equation

$$x^2 \frac{dy}{dx} + xy = 1,$$

where $x > 0$, subject to the condition that $y(1) = 0$.

In the following answers, c is a constant.

1 $y = \frac{1}{3}e^x + ce^{-2x}$

2 $y = -\cot x + c \operatorname{cosec} x$

3 $y = xe^{\cos x} + ce^{\cos x}$

4 $y = e^{2x} + ce^x$

5 $y = \left(\frac{x^2}{2} + c\right) \cos x$

6 $y = \frac{1}{x} \ln x + \frac{c}{x}$

7 $y = x \ln(x + 2) + cx$

8 $y = \frac{1}{4}x + cx^{-\frac{1}{3}}$

9 $y = (x + 2) \ln (x + 2) + c(x + 2)$

10 $y = \frac{1}{x^3}e^x - \frac{1}{x^4}e^x + \frac{c}{x^4}$

(a) x^2 (b) $y = x^2 + \frac{C}{x^2}$

(a) $y = \frac{1}{4}e^{2x} + Ce^{-2x}$ (b) $y = \frac{1}{4}\left(e^{2x} - e^{-2x}\right)$

$y = \frac{\ln x}{x}$

Transforming differential equation using a substitution

The substitution to use will be given in the exam questions. Just follow the instructions...

Worked example:

$a)$ Use the substitution $z = \frac{1}{y}$ to transform thedifferential equation $\frac{dy}{dx} + xy = xy^2$,

into a differential equation in z *and* x.

$b)$ solve the new differential equation, using an integrating factor.

$c)$ Find the general solution of the original equation, giving y in terms of x.

Your go...

A) Use the substitution $z = \frac{y}{x}$ to transform the differential equation $\frac{dy}{dx} = \frac{x^2 + 3y^2}{2xy}$, $x > 0$, into a differential equation in z and x. By first solving this new equation, find the general solution of the original equation, giving y^2 in terms of x.

B) Use the substitution $u = y - x$ to transform the differential equation

$$\frac{dy}{dx} = \frac{y - x + 2}{y - x + 3}$$

into a differential equation in u and x. By first solving this new equation, show that the general solution of the original equation may be written in the form

$\therefore\ (y - x)^2 + 6y - 4x - 2c = 0$, where c is an arbitrary constant.

Exercises:

In questions 1–4 ,use the substitution $z = \frac{y}{x}$ to transform the given homogeneous differential equation into a differential equation in z and x. By first solving the transformed equation, find the general solution of the original equation, giving y in terms of x.

1 $\frac{dy}{dx} = \frac{y}{x} + \frac{x}{y}, \quad x > 0, y > 0$

2 $\frac{dy}{dx} = \frac{y}{x} + \frac{x^2}{y^2}, \quad x > 0$

3 $\frac{dy}{dx} = \frac{y}{x} + \frac{y^2}{x^2}, \quad x > 0$

4 $\frac{dy}{dx} = \frac{x^3 + 4y^3}{3xy^2}, \ x > 0$

5 Use the substitution $z = y^{-2}$ to transform the differential equation

$$\frac{dy}{dx} + (\tfrac{1}{2}\tan x)\, y = -(2\sec x)y^3, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

into a differential equation in z and x. By first solving the transformed equation, find the general solution of the original equation, giving y in terms of x.

6 Use the substitution $z = x^{\frac{1}{2}}$ to transform the differential equation

$$\frac{dx}{dt} + t^2 x = t^2 x^{\frac{1}{2}}$$

into a differential equation in z and t. By first solving the transformed equation, find the general solution of the original equation, giving x in terms of t.

7 Use the substitution $z = y^{-1}$ to transform the differential equation

$$\frac{dy}{dx} - \frac{1}{x}y = \frac{(x+1)^3}{x}y^2$$

into a differential equation in z and x. By first solving the transformed equation, find the general solution of the original equation, giving y in terms of x.

8 Use the substitution $z = y^2$ to transform the differential equation

$$2(1 + x^2)\frac{dy}{dx} + 2xy = \frac{1}{y}$$

into a differential equation in z and x. By first solving the transformed equation,

a find the general solution of the original equation, giving y in terms of x.

b Find the particular solution for which $y = 2$ when $x = 0$.

In the following answers, c is a constant and A is a positive constant.

1 $y^2 = 2x^2(\ln x + c)$

2 $y^3 = 3x^3(\ln x + c)$

3 $y = \frac{-x}{\ln x + c}$

4 $y^3 = x^3(Ax - 1)$

5 $y = \sqrt{\frac{\cos x}{4x + c}}$

6 $x = (1 + ce^{-\frac{1}{6}t^3})^2$

7 $y = \frac{4x}{4c - (x+1)^4}$

8 **a** $y = \sqrt{\frac{x + c}{1 + x^2}}$ **b** $y = \sqrt{\frac{x + 4}{1 + x^2}}$