# Probability

## Specifications

### Probability

| | |
|---|---|
| Elementary probability; the concept of a random event and its probability. | Assigning probabilities to events using relative frequencies or equally likely outcomes. Candidates will be expected to understand set notation but its use will not be essential. |
| Addition law of probability. Mutually exclusive events. | $P(A \cup B) = P(A) + P(B) - P(A \cap B)$; two events only.<br>$P(A \cup B) = P(A) + P(B)$; two or more events.<br>$P(A') = 1 - P(A)$. |
| Multiplication law of probability and conditional probability. | $P(A \cap B) = P(A) \times P(B \mid A) = P(B) \times P(A \mid B)$; two or more events. |
| Independent events. | $P(A \cap B) = P(A) \times P(B)$; two or more events. |
| Application of probability laws. | Only simple problems will be set that can be solved by direct application of the probability laws, by counting equally likely outcomes and/or the construction and the use of frequency tables or relative frequency (probability) tables. Questions requiring the use of tree diagrams or Venn diagrams will not be set, but their use will be permitted. |

2 The number of MPs in the House of Commons was 645 at the beginning of August 2009. The genders of these MPs and the political parties to which they belonged are shown in the table.

| | | Political Party | | | | |
|---|---|---|---|---|---|---|
| | | **Labour** | **Conservative** | **Liberal Democrat** | **Other** | **Total** |
| **Gender** | **Male** | 255 | 175 | 54 | 35 | 519 |
| | **Female** | 94 | 18 | 9 | 5 | 126 |
| | **Total** | 349 | 193 | 63 | 40 | 645 |

**(a)** One MP was selected at random for an interview. Calculate, to three decimal places, the probability that the MP was:

**(i)** a male Conservative; *(1 mark)*

**(ii)** a male; *(1 mark)*

**(iii)** a Liberal Democrat; *(1 mark)*

**(iv)** Labour, given that the MP was female; *(2 marks)*

**(v)** male, given that the MP was **not** Labour. *(3 marks)*

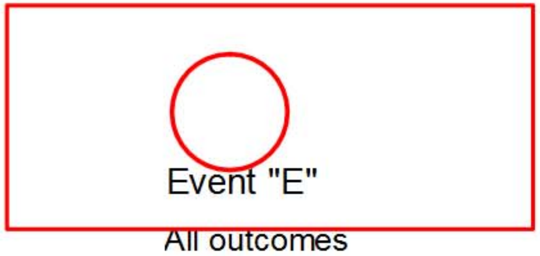
**(b)** Two **female** MPs were selected at random for an enquiry. Calculate, to three decimal places, the probability that both MPs were Labour. *(2 marks)*

**(c)** Three MPs were selected at random for a committee. Calculate, to three decimal places, the probability that exactly one MP was Labour and exactly one MP was Conservative. *(4 marks)*

## Definition and vocabulary

Consider all possible outcomes of an experience.

An EVENT is part of the outcomes.




The probability of the event "E" is $P(E) = \dfrac{\text{Number of element of E}}{\text{Total numbers of outcomes}}$

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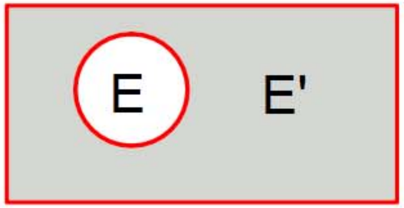
## Complementary event

The complementary event E' , is the event "E does not occur":

$$P(E') = 1 - P(E)$$

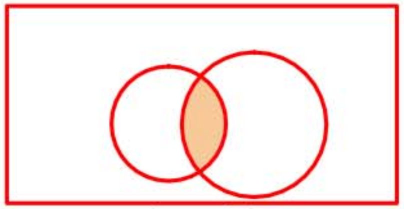



## Union and intersection of events

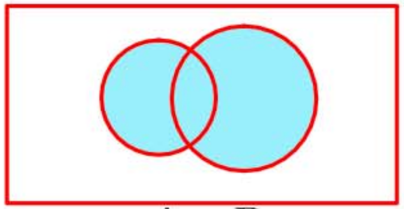
Consider event A and event B

Union: The event $A \cup B$ , A *or* B , is the event "At least one of A or B occurs"

Intersection: The event $A \cap B$ , A *and* B, is the event "Both A and B occurs"



$A \cap B$

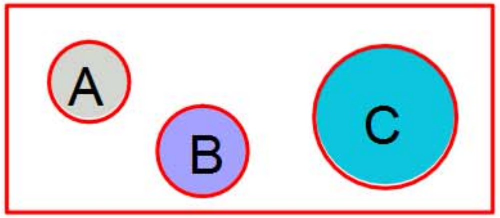


$A \cup B$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

## Mutually exclusive events

Events A, B, C, .., are said mutually exclusive when the occurence of one of them implies that none of the other can have occured.




$$P(A \cap B \cap C \cap ...) = 0$$

In particular, A and A' are mutually exclusive. $P(A \cap A') = 0$

## Addition law for mutually exclusive events

When the events A and B are mutually exclusive, $P(A \cup B) = P(A) + P(B)$

## Exhaustive events

Two events A and B are exhaustive
when it is certain that at least one of them occurs: $P(A \cup B) = 1$

Note: Exhaustive events need not be exclusive

## Exercises

**1** A survey of 1000 people revealed the voting intentions shown. A person is chosen at random from the sample.

Find the probability that the person chosen:

a) intends to vote Conservative

b) is a woman intending to vote Labour

c) is either a woman or intends to vote Conservative

d) is neither a man nor intends to vote LibDem

e) is a man and intends to vote either LibDem or Labour.

| | Women | Men | Total |
|---|---|---|---|
| Con | 153 | 130 | 283 |
| Lab | 220 | 194 | 414 |
| LibDem | 157 | 146 | 303 |
| Total | 530 | 470 | 1000 |

**2** A man tosses two fair dice. One is numbered 1 to 6 in the usual way. The other is numbered 1, 3, 5, 7, 9, 11. The events $A$ to $E$ are defined as follows.

$A$: Both dice show odd numbers.
$B$: The number shown by the normal die is the greater.
$C$: The total of the two numbers shown is greater than 10.
$D$: The total is less than or equal to 4.
$E$: The total is odd.

a) Determine the probability of each event.

b) State:

i) Which pairs of events (if any) are exclusive.

ii) Which pairs of events (if any) are exhaustive.

**3** A fair die is thrown. Events $A, B, C, D$ are defined as follows.

$A$: The score is even.
$B$: The score is divisible by 3.
$C$: The score is not more than 2.
$D$: The score exceeds 3.

a) Verify that $P(A) + P(B) = P(A \cup B) + P(A \cap B)$.

b) Find $P(A'), P(B'), P(C'), P(D')$.

c) Identify two pairs of events that are exclusive, and verify the addition rule in each case.

d) Identify three events that are collectively exhaustive.

e) Find $P(C \cap D)$.

**4** Two fair dice, one red and one green, are thrown and the separate scores are observed. The outcome is denoted by $(r, g)$, where $r$ and $g$ are the scores on the red and green dice respectively.

Represent these outcomes on a $6 \times 6$ grid, with the $r$-axis horizontal and the $g$-axis vertical.

Events $A, B, C$ are defined as follows.

$A$: The score on the red die exceeds the score on the green die.
$B$: The total score is six or more.
$C$: The score on the red die does not exceed 4.

a) Identify on your diagram the sets corresponding to $A, B, C$.

b) Verify that $P(A) + P(B) = P(A \cup B) + P(A \cap B)$.

c) Verify that $P(A) + P(C) = P(A \cup C) + P(A \cap C)$.

d) Identify a pair of events which are exhaustive.

e) Find $P(A'), P(B'), P(C')$.

f) Find $P(A' \cup B), P(A \cap B'), P(B \cup C), P(B' \cap C'), P(B' \cup C')$.

**1** a) 0.283 b) 0.220 c) 0.660 d) 0.373 e) 0.340 **2** a) $P(A) = \frac{1}{2}, P(B) = \frac{1}{4}, P(C) = \frac{5}{12}, P(D) = \frac{1}{9}, P(E) = \frac{1}{2}$
**2** b) i) $\{(A, E), (C, D)\}$ ii) $(A, E)$ **3** a) $P(A) = \frac{1}{2}, P(B) = \frac{1}{3}, P(A \cup B) = \frac{2}{3}, P(A \cap B) = \frac{1}{6}$
**3** b) $P(A') = \frac{1}{2}, P(B') = \frac{2}{3}, P(C') = \frac{2}{3}, P(D') = \frac{1}{2}$, c) $(B, C), (C, D)$ d) $B, C, D$ e) 0
**4** b) $P(A) = \frac{15}{36}, P(B) = \frac{26}{36}, P(A \cup B) = \frac{30}{36}, P(A \cap B) = \frac{11}{36}$ c) $P(A) = \frac{15}{36}, P(C) = \frac{24}{36}, P(A \cup B) = \frac{33}{36}, P(A \cap B) = \frac{6}{36}$ d) $(B, C)$
**4** e) $P(A') = \frac{7}{12}, P(B') = \frac{5}{18}, P(C') = \frac{1}{3}$, f) $P(A' \cup B) = \frac{8}{9}, P(A \cap B') = \frac{1}{9}, P(B \cup C) = 1, P(B' \cap C') = 0, P(B' \cup C') = \frac{11}{18}$

## Independent events

When the probability of event A occuring is unaffected by whether or not event B occurs the two events are said to be independent

## The multiplication law of independent events

When A and B are independent events, $P(A \cap B) = P(A) \times P(B)$

## Example

The probability of Michelle passing a mathematics exam is 0.3 and the probability of her passing a biology exam is independently 0.45.

Find the probability that she:

**(a)** passes mathematics and fails biology,

**(b)** passes exactly one of the two examinations,

**(c)** passes at least one of the two examinations.

Alternative: probability tree

# Exercises

**1** Alan picks a number at random from the set 1, 2, 3, 4.
Beth picks a number at random from the set 1, 2, 3.
Colin picks a number at random from the set 1, 2.

Find the probability that the three numbers picked are

**(i)** all the same **(ii)** all different

**2** In a college, 50% of students study maths and 35% study science.
55% of students study either maths or science.
A student is picked at random. Find the probability that the student studies

**(a)** both maths and science **(b)** only one of the two subjects

**3** In a raffle, 1000 tickets are sold, numbered from 1 to 1000. One winning ticket is picked at random. Find the probability that the winning number is

**(a)** a multiple of 5

**(b)** a multiple of 5 but not a multiple of 20

**(c)** a multiple of 4 or a multiple of 5

**4** A box contains cubes of different colours and materials. If a cube is taken out at random, the probability that it is red is 0.25, the probability that it is red or wooden is 0.8, and the probability that it is both red and wooden is 0.2.

Find the probability that the ball is

**(a)** neither red nor wooden **(b)** wooden **(c)** wooden but not red

***5** A school offers three languages for GCSE: French, German and Spanish.
All students are required to study at least two of these languages.
90% of students study French and 70% study German. 5% study all three languages.
A student is picked at random. Find the probability that the student studies

**(a)** French and German **(b)** Spanish

**1 (a)** 1, 1, 1  1, 1, 2  1, 2, 1  1, 2, 2  1, 3, 1  1, 3, 2
2, 1, 1  2, 1, 2  2, 2, 1  2, 2, 2  2, 3, 1  2, 3, 2
3, 1, 1  3, 1, 2  3, 2, 1  3, 2, 2  3, 3, 1  3, 3, 2
4, 1, 1  4, 1, 2  4, 2, 1  4, 2, 2  4, 3, 1  4, 3, 2
**(b) (i)** $\frac{1}{12}$ **(ii)** $\frac{1}{3}$
**2 (a)** 0.3 **(b)** 0.25
**3 (a)** $\frac{1}{5}$ **(b)** $\frac{3}{20}$ **(c)** $\frac{2}{5}$
**4 (a)** 0.2 **(b)** 0.75 **(c)** 0.55
**5 (a)** 0.6 **(b)** 0.45

## Condition probability

The probability that the event B occurs given that the event A has occured is denoted P(B|A).
This is described as a CONDITIONAL probability.

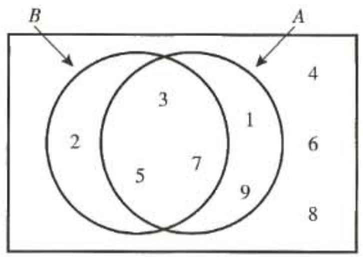
$$P(B \mid A) = \frac{P(B \cap A)}{P(A)}$$

## Examples:

An electronic display is equally likely to show any of the numbers 1, ..., 8, 9. Determine the probability that it shows a prime number (2, 3, 5 or 7):

a) given no knowledge about the number,

b) given the information that the number is odd.

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**Example 2**

Dawn has a collection of photos. This table shows the number of photos in each of four categories.

| | People | Places |
|---|---|---|
| Glossy | 44 | 36 |
| Matt | 26 | 14 |

Dawn picks a photo at random.

**(a)** Given that the photo is glossy, what is the probability that it shows people?

**(b)** Given that the photo shows places, what is the probability that it is matt?

# June 2011

6 A housing estate consists of 320 houses: 120 detached and 200 semi-detached. The numbers of children living in these houses are shown in the table.

| | Number of children | | | | |
|---|---|---|---|---|---|
| | **None** | **One** | **Two** | **At least three** | **Total** |
| **Detached house** | 24 | 32 | 41 | 23 | 120 |
| **Semi-detached house** | 40 | 37 | 88 | 35 | 200 |
| **Total** | 64 | 69 | 129 | 58 | 320 |

A house on the estate is selected at random.

$D$ denotes the event 'the house is detached'.
$R$ denotes the event 'no children live in the house'.
$S$ denotes the event 'one child lives in the house'.
$T$ denotes the event 'two children live in the house'.
($D'$ denotes the event 'not $D$'.)

(a) Find:

(i) $P(D)$; *(1 mark)*

(ii) $P(D \cap R)$; *(1 mark)*

(iii) $P(D \cup T)$; *(2 marks)*

(iv) $P(D \mid R)$; *(2 marks)*

(v) $P(R \mid D')$. *(3 marks)*

(b) (i) Name two of the events $D$, $R$, $S$ and $T$ that are mutually exclusive. *(1 mark)*

(ii) Determine whether the events $D$ and $R$ are independent. Justify your answer. *(2 marks)*

(c) Define, in the context of this question, the event:

(i) $D' \cup T$; *(2 marks)*

(ii) $D \cap (R \cup S)$. *(2 marks)*

# Exercises:

**1** A card is drawn at random from this pack.

1 2 3 4 5 6 7 8 9 10 11 12

**(a)** Given that the card is black, what is the probability that the number on it is even?

**(b)** Given that the number on the card is even, what is the probability that the card is black?

**2** This table shows the percentages of stamps of different types in Bharat's collection.

| | UK | Foreign |
|---|---|---|
| Used | 36% | 24% |
| Unused | 14% | 26% |

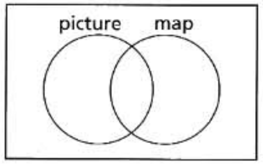
Bharat picks a stamp at random from the collection.

**(a)** What is the probability that the stamp is used?

**(b)** Given that the stamp is used, what is the probability that it is foreign?

**(c)** Given that it is a UK stamp, what is the probability that it is unused?

**3** 30% of the pages in a book have a picture on them. 10% have a map. 3% have both a picture and a map.




**(a)** Copy this Venn diagram and write the percentage of pages in each part.

A page is picked at random.

**(b)** Given that the page has a picture, what is the probability that it also has a map?

**(c)** Given that the page has a map, what is the probability that it also has a picture?

**(d)** What is the probability that the page has neither a map nor a picture?

**(e)** Given that the page does not have a picture, what is the probability that it does not have a map?

**4** Students at a college study **two** subjects from this list: painting, sculpture, ceramics. 45% of the students study painting and sculpture, 30% study painting and ceramics and the rest study sculpture and ceramics.

A student is chosen at random.

**(a)** Given that the student studies painting, what is the probability that they also study ceramics?

**(b)** Given that the student studies ceramics, what is the probability that they also study sculpture?

**5** Each student in a college studies one and only one of the three languages French, German and Spanish. The percentage breakdown of the students by gender and by language studied is given in the table.

| | French | German | Spanish |
|---|---|---|---|
| Male | 30% | 8% | 6% |
| Female | 21% | 18% | 17% |

A student is picked at random.

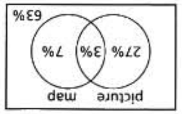
**(a)** Given that the student is female, what is the probability that she studies French?

**(b)** Given that the student studies either German or Spanish, what is the probability that the student is male?

**1** (a) $\frac{4}{7}$ (b) $\frac{2}{3}$

**2** (a) 0.6 (b) 0.4 (c) 0.28

**3** (a)




(b) 0.1 (c) 0.3 (d) 0.63 (e) 0.9

**4** (a) 0.4 (b) 0.45 (to 2 d.p.) or $\frac{5}{11}$

**5** (a) $\frac{3}{8}$ (b) $\frac{2}{7}$

## Key point summary

1. Probability is measured on a scale from 0 to 1.
2. If a trial can result in one of $n$ equally likely outcomes and an event consists of $r$ of these, then the probability of the event happening as a result of the trial is $\frac{r}{n}$.
3. Two events are **mutually exclusive** if they cannot both happen.
4. If $A$ and $B$ are **mutually exclusive** events $P(A \cup B) = P(A) + P(B)$.
5. The event of $A$ not happening as the result of a trial is called the **complement** of $A$ and is usually denoted $A'$.
6. Two events are **independent** if the probability of one happening is unaffected by whether or not the other happens.
7. If $A$ and $B$ are **independent** events $P(A \cap B) = P(A)P(B)$.
8. $P(A | B)$ denotes the probability that event $A$ happens given that event $B$ happens.
9. If events $A$ and $B$ are **independent** $P(A | B) = P(A)$.
10. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
11. $P(A \cap B) = P(A)P(B | A) = P(B)P(A | B)$.

# Exam questions

**1** At a station, the probability that the first train in the morning is late is 0.4. Thereafter, the probability of a train being late is 0.6 if the previous train was late and 0.2 if the previous train was on time.

Find the probability that on a particular morning:

a) the first three trains are all late;

b) the first three trains are all on time;

c) exactly one of the first three trains is late.

**2** Shahid, Tracy and Dwight are friends who all have birthdays during January. Assuming that each friend's birthday is equally likely to be on any one of the 31 days of January, find the probability that:

a) Shahid's birthday is on January 3rd;

b) both Shahid's and Tracy's birthdays are on January 3rd;

c) all three friends' birthdays are on the same day;

d) all three friends' birthdays are on different days.

**3** At a university, 60% of students are studying arts subjects and the rest are studying science subjects. Of the arts students, 55% are female, and of the science students, 35% are female.

a) Find the probability that a student, selected at random:
   i) is female and studying an arts subject;
   ii) is male and studying a science subject;
   iii) is male;
   iv) is studying science, given that the student is male.

b) Of all the female students, one third of those studying a science subject and one half of those studying an arts subject live in a hall of residence. Find the probability that a female student, who lives in a hall of residence, is studying a science subject.

**4** Amy, Bruce and Carmen take part in an archery competition. The final test of the competition involves these three archers trying to hit the target centre.

The independent probabilities that Amy, Bruce and Carmen hit the target centre are 0.2, 0.3 and 0.6 respectively.

Find the probability that the target centre will be hit by:

a) Bruce only;

b) Carmen only;

c) exactly one of the three archers;

d) Bruce, given that exactly one of the three archers hits the target centre.

*(AQA, 2002)*

**5** A rugby club has three categories of membership: adult, social and junior. The number of members in each category, classified by gender, is shown in the table.

| | Adult | Social | Junior |
|---|---|---|---|
| Female | 25 | 35 | 40 |
| Male | 95 | 25 | 80 |

One member is chosen, at random, to cut the ribbon at the opening of the new clubhouse.

a) Find the probability that:
   i) a female member is chosen;
   ii) a junior member is chosen;
   iii) a junior member is chosen, given that a female member is chosen.

b) $V$ denotes the event that a female member is chosen.
   $W$ denotes the event that an adult member is chosen.
   $X$ denotes the event that a junior member is chosen.

   For the events $V$, $W$ and $X$:
   i) Write down two which are mutually exclusive.
   ii) Find two which are neither mutually exclusive nor independent. Justify your answer.

*(AQA, 2003)*

**1** a) 0.144 b) 0.384 c) 0.272 **2** a) $\frac{1}{31}$ b) $\frac{1}{961}$ c) $\frac{1}{961}$ d) $\frac{870}{961}$ **3** a) i) 0.33 ii) 0.26 iii) 0.53 iv) $\frac{26}{53} = 0.491$
**3** b) $\frac{28}{127} = 0.220$ **4** a) 0.096 b) 0.336 c) 0.488 d) $\frac{12}{61} = 0.197$ **5** a) i) $\frac{1}{3}$ ii) 0.4 iii) 0.4 b) i) $(W, X)$
**5** ii) $(V, W)$: $P(V \cap W) > 0$ and $P(V \mid W) = \frac{5}{24} \neq P(V) = \frac{1}{3}$

**6** Keith, Yasmin and Suzie are friends who often go to a jazz club on a Friday night. The probability that Keith goes to the club on a particular Friday is 0.7 and is independent of whether or not Yasmin and Suzie go.

The probability that Yasmin goes to the club is 0.8 if Keith goes and 0.6 if Keith does not go.

The probability that Suzie goes to the club:

is 0.9 if both Keith and Yasmin go;
is 0.65 if Keith goes but Yasmin does not;
is 0.55 if Yasmin goes but Keith does not;
is 0.4 if neither Keith nor Yasmin go. *continued*

By drawing a tree diagram, or otherwise, find the probability that on a particular Friday:

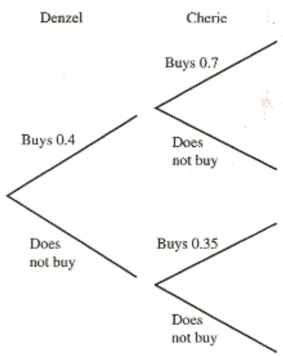
a) all three friends go to the club;

b) Keith goes to the club but Yasmin and Suzie do not;

c) Yasmin and Suzie go to the club but Keith does not;

d) Suzie goes to the club. *(AQA, 2003)*

**7** Denzel and Cherie are friends who often go to the cinema together. On such visits there is a probability of 0.4 that Denzel will buy popcorn. The probability that Cherie will buy popcorn is 0.7 if Denzel buys popcorn and 0.35 if he does not.




a) When Denzel and Cherie visit the cinema together:

i) find the probability that both buy popcorn;

ii) show that the probability that neither buys popcorn is 0.39;

iii) find the probability that exactly one of them buys popcorn.

b) Sohaib sometimes joins Denzel and Cherie on their cinema visits. On these occasions, the probability that Sohaib buys popcorn is 0.55 if both Denzel and Cherie buy popcorn and 0.25 if exactly one of Denzel and Cherie buys popcorn.

Find the probability that when Denzel, Cherie and Sohaib visit the cinema together:

i) all three buy popcorn;

ii) Cherie and Sohaib buy popcorn but Denzel does not. *(AQA, 2003)*

**8** Transport inspectors carry out roadside safety tests on lorries.

a) The probability of a randomly selected lorry failing the test is 0.25. A transport inspector chooses two lorries at random. Find the probability that:

i) both will fail the test;

ii) exactly one will pass the test.

b) Of 12 lorries parked outside a transport café, four would fail if tested.

i) An inspector chooses two of these twelve lorries at random. Find the probability that both will pass the test.

ii) An inspector chooses three of these lorries at random. Find the probability that two will pass the test and one will fail the test.

iii) An inspector chooses four of these lorries at random. Find the probability that at least one will fail the test. *(AQA, 2003)*

**9** The senior driving examiner at a test centre decides to collect some data about the candidates who are taking a driving test for the first time. The candidates are asked how many driving lessons they had before they took the test. The results are given in the table below.

| Number of lessons | Gender | |
|---|---|---|
| | Male | Female |
| 20 or fewer | 35 | 47 |
| between 21 and 50 | 87 | 75 |
| more than 50 | 8 | 21 |

A candidate is selected at random.

$M$ is the event 'The candidate is male'.
$T$ is the event 'The candidate had 20 or fewer lessons'.
$S$ is the event 'The candidate had more than 50 lessons'.
$M'$ is the event 'Not $M$'.
$S'$ is the event 'Not $S$'.

Find: a) $P(M \cap S)$; b) $P(T)$; c) $P(M|T)$; d) $P(M' \cap T)$;
e) $P(M'|S')$. *(AQA, 2003)*

**10** A blood test for determining the type of hepatitis from which a patient is suffering involves testing for the Australian antigen. A 'positive' result to this test indicates that a patient has hepatitis C.

Unfortunately, this test is not completely reliable.

If an individual has hepatitis C, there is a probability of 0.8 that the test will give a 'positive' result. If a patient does **not** have hepatitis C, there is a probability of 0.1 that the test will give a 'positive' result.

It is known that 18% of patients suffering from hepatitis have type C.

Find the probability that a person chosen at random from all hepatitis sufferers:

a) has type C and gives a 'positive' result to the test;

b) gives a 'positive' result to the test;

c) does not have hepatitis C, given that the test has given a 'positive' result. *(AQA, 2002)*

**6** a) 0.504 b) 0.049 c) 0.099 d) 0.742

**7** a) i) 0.28 iii) 0.33 b) i) 0.154 ii) 0.0525 **8** a) i) $\frac{1}{16}$ ii) $\frac{3}{8}$ b) i) $\frac{14}{33} = 0.424$ ii) $\frac{28}{55} = 0.509$ iii) $\frac{85}{99} = 0.859$

**9** a) $\frac{8}{273} = 0.029$ b) $\frac{82}{273} = 0.300$ c) $\frac{35}{82} = 0.427$ d) $\frac{47}{273} = 0.172$ e) $\frac{1}{2}$ **10** a) 0.144 b) 0.226 c) $\frac{41}{113} = 0.363$