Complex numbers

specifications

Complex Numbers

Non-real roots of quadratic equations.	Complex conjugates – awareness that non-real roots of quadratic equations with real coefficients occur in conjugate pairs.
Sum, difference and product of complex numbers in the form $x+\mathrm{i}y$.	
Comparing real and imaginary parts.	Including solving equations e.g. $2z+z^{*}=1+\mathrm{i}$ where z^{*} is the conjugate of z.

The sets of numbers

The set of the Natural *numbers*: $\mathbb{N} = \{0,1,2,3,4,5,.....\}$

Note: In some books, "0" is not considered a Natural number

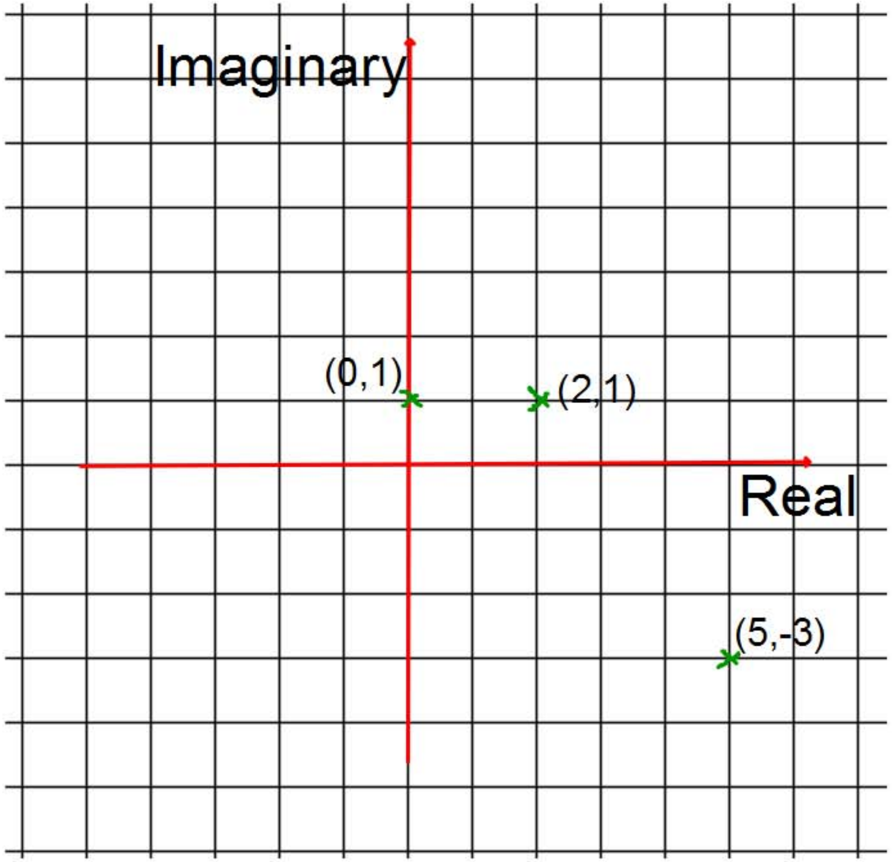
The set of the Integers *numbers*: $\mathbb{Z} = \{...-4,-3,-2,-1,0,1,2,3,4,5,.....\}$

The set of Rational *numbers*: $\mathbb{Q} = \{ \frac{p}{q} \text{ with } p \in \mathbb{Z}, q \in \mathbb{N}, q \neq 0\}$

The set of Real *numbers*: $\mathbb{R} =$ All rational and irrational numbers ($e, \pi, \sqrt{2},...$)

What's next?

To be able to solve equations like $x^2 = -1$
We consider numbers in "two dimensions"



These numbers are called complex numbers:
z=(2,1) , i=(0,1) , w=(5,-3)

Introducing new numbers is improtant, but we also need to introduce operations to manipulate them:

Rule of addition:

$$(a,b)+(c,d)=(a+b,c+d)$$

Rule of multiplication

$$(a,b)\times(c,d)=(ac-bd,ad+bc)$$

$$\lambda\times(a,b)=(\lambda a,\lambda b)$$

Work out

a) z+w

b) z×w

c) i×z

Let's make it simpler...

Work out

1) $(0,1)\times(0,1) =$

2) $(5,0)+(0,1)(6,0) =$

3) $(3,0)-(0,1)(2,0) =$

Rule of addition:

$$(a,b)+(c,d)=(a+b,c+d)$$

Rule of multiplication

$$(a,b)\times(c,d)=(ac-bd,ad+bc)$$

$$\lambda\times(a,b)=(\lambda a,\lambda b)$$

Note : The complex numbers of the type $(a,0)$ is simply written a

(it is effectively a Real number)

From this we can establish the following notation:

The complex number (0,1) is called "i"
and we have $i^2 = (-1,0) = -1$

A complex number $z=(a,b)=(a,0)+(0,1)(b,0)$
can be written $z = a+ib$

where "a" and "b" are two real numbers

"a" is called the REAL part of z

"b" is called the IMAGINARY part of z

Complex numbers and operations

Adding and subtracting complex numbers

Consider two complex numbers

$$z_1 = a_1 + ib_1 \quad and \quad z_2 = a_2 + ib_2$$

$$Then \quad z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$$

$$and \quad z_1 - z_2 = (a_1 - a_2) + i(b_1 - b_2)$$

Examples:

$$z_1 = 3 + 2i \ \ and \ \ z_2 = -2 + 4i$$

$$z_1 + z_2 = 1 + 6i$$

$$z_1 - z_2 = 5 - 2i$$

Multiplying complex numbers

Consider two complex numbers

$$z_1 = a_1 + ib_1 \quad and \quad z_2 = a_2 + ib_2$$

$$Then \quad z_1 \times z_2 = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$$

(just expand the brackets like you would do in algebra)

Examples:

$$z_1 = 3 + 2i \ \ and \ \ z_2 = -2 + 4i$$

$$z_1 \times z_2 = (3 + 2i)(-2 + 4i) = -6 + 12i - 4i + 8i^2$$

$$z_1 \times z_2 = -14 + 8i$$

Complex conjugates

Consider the complex number $z = a + ib$

The complex **conjugate** of z is noted z*

with $z^* = a - ib$

Examples:

z= 3+2i

z*=3-2i

EXERCISE

1 Simplify each of the following:

(a) i^5, **(b)** i^6, **(c)** i^9, **(d)** i^{27},

(e) $(-i)^3$, **(f)** $(-i)^7$, **(g)** $(-i)^{10}$.

2 Simplify:

(a) $(2i)^3$, **(b)** $(3i)^4$, **(c)** $(7i)^2$,

(d) $(-2i)^2$, **(e)** $(-3i)^3$, **(f)** $(-2i)^5$.

3 Solve each of the following equations, giving your answers in terms of i:

(a) $x^2 = -9$, **(b)** $x^2 = -100$, **(c)** $x^2 = -49$,

(d) $x^2 + 1 = 0$, **(e)** $x^2 + 121 = 0$, **(f)** $x^2 + 64 = 0$,

(g) $x^2 + n^2 = 0$, where n is a positive integer.

4 Find the exact solutions of each of the following equations, giving your answers in terms of i:

(a) $x^2 = -5$, **(b)** $x^2 = -3$, **(c)** $x^2 = -8$,

(d) $x^2 + 20 = 0$, **(e)** $x^2 + 18 = 0$, **(f)** $x^2 + 48 = 0$.

1 **(a)** i; **(b)** −1; **(c)** i; **(d)** −i;
(e) i; **(f)** i; **(g)** −1.

2 **(a)** −8i; **(b)** 81; **(c)** −49;
(d) −4; **(e)** 27i; **(f)** −32i.

3 **(a)** −3i, 3i; **(b)** −10i, 10i; **(c)** −7i, 7i; **(d)** −i, i;
(e) −11i, 11i; **(f)** −8i, 8i; **(g)** $-ni$, ni.

4 **(a)** $-\sqrt{5}i, \sqrt{5}i$; **(b)** $-\sqrt{3}i, \sqrt{3}i$; **(c)** $-2\sqrt{2}i, 2\sqrt{2}i$
(d) $-2\sqrt{5}i, 2\sqrt{5}i$; **(e)** $-3\sqrt{2}i, 3\sqrt{2}i$; **(f)** $-4\sqrt{3}i, 4\sqrt{3}i$.

EXERCISE

1 Find the complex conjugate of each of the following:

(a) $3 - \text{i}$, **(b)** $2 + 6\text{i}$, **(c)** $-3 - 8\text{i}$, **(d)** $-7 + 5\text{i}$,
(e) $3 + \sqrt{2}\text{i}$, **(f)** $4 - \sqrt{3}\text{i}$, **(g)** $-1 - \frac{1}{3}\text{i}$

2 Simplify each of the following:

(a) $(3 + \text{i}) + (5 - 2\text{i})$, **(b)** $(3 + \text{i}) - (5 - 2\text{i})$,
(c) $3(3 - 5\text{i}) + 4(1 + 6\text{i})$, **(d)** $3(3 - 5\text{i}) - 4(1 + 6\text{i})$,
(e) $4(8 - 5\text{i}) - 5(1 - 4\text{i})$, **(f)** $6(3 - 4\text{i}) - 2(9 - 6\text{i})$.

3 Simplify each of the following:

(a) $(4 + \text{i})(7 - 2\text{i})$, **(b)** $(3 + 4\text{i})(5 - 3\text{i})$,
(c) $(7 - 5\text{i})(5 + 6\text{i})$, **(d)** $3(3 - 5\text{i})(4 + 3\text{i})$,
(e) $(8 - 5\text{i})(8 - 5\text{i})$, **(f)** $(3 - 4\text{i})^2$.

4 Find the square of each of the following complex numbers:

(a) $3 - \text{i}$, **(b)** $2 + 6\text{i}$,
(c) $-3 - 8\text{i}$, **(d)** $-7 + 5\text{i}$,
(e) $3 + \sqrt{2}\text{i}$, **(f)** $4 - \sqrt{3}\text{i}$.

5 The complex numbers z_1 and z_2 are given by $z_1 = 2 - 3\text{i}$ and $z_2 = -3 + 5\text{i}$.

Find: **(a)** $5z_1 + 3z_2$ **(b)** $3z_1 - 4z_2$ **(c)** $z_1 z_2$

6 Given that z^* is the conjugate of z, find the values of

(i) $z + z^*$, **(ii)** $z - z^*$, **(iii)** zz^*

for each of the following values of z:

(a) $2 + 3\text{i}$, **(b)** $-4 + 2\text{i}$, **(c)** $-5 - 3\text{i}$, **(d)** $6 - 5\text{i}$,
(e) $x + y\text{i}$, where x and y are real.

7 Find the value of the real constant p so that $(3 + 2\text{i})(4 - \text{i}) + p$ is purely imaginary.

8 Find the value of the real constant q so that $(2 + 5\text{i})(4 - 3\text{i}) + q\text{i}$ is real.

9 **(a)** Find $(1 + \text{i})(2 - 3\text{i})$.
(b) Hence, simplify $(1 + \text{i})(2 - 3\text{i})(5 + \text{i})$.

10 **(a)** Find $(2 + \text{i})^2$.
(b) Hence, find: **(i)** $(2 + \text{i})^3$, **(ii)** $(2 + \text{i})^4$.

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1 **(a)** $3 + \text{i}$; **(b)** $2 - 6\text{i}$; **(c)** $-3 + 8\text{i}$; **(d)** $-7 - 5\text{i}$;
(e) $3 - \sqrt{2}\text{i}$; **(f)** $4 + \sqrt{3}\text{i}$; **(g)** $-1 + \frac{1}{3}\text{i}$.

2 **(a)** $8 - \text{i}$; **(b)** $-2 + 3\text{i}$; **(c)** $13 + 9\text{i}$;
(d) $5 - 39\text{i}$; **(e)** 27; **(f)** -12i.

3 **(a)** $30 - \text{i}$; **(b)** $27 + 11\text{i}$; **(c)** $65 + 17\text{i}$;
(d) $81 - 33\text{i}$; **(e)** $39 - 80\text{i}$; **(f)** $-7 - 24\text{i}$.

4 **(a)** $8 - 6\text{i}$; **(b)** $-32 + 24\text{i}$; **(c)** $-55 + 48\text{i}$;
(d) $24 - 70\text{i}$; **(e)** $7 + 6\sqrt{2}\text{i}$; **(f)** $13 - 8\sqrt{3}\text{i}$.

5 **(a)** 1; **(b)** $18 - 29\text{i}$; **(c)** $9 + 19\text{i}$.

6 **(a)** **(i)** 4, **(ii)** 6i, **(iii)** 13;
(b) **(i)** -8, **(ii)** 4i, **(iii)** 20;
(c) **(i)** -10, **(ii)** -6i, **(iii)** 34;
(d) **(i)** 12, **(ii)** -10i, **(iii)** 61;
(e) **(i)** $2x$, **(ii)** $2y\text{i}$, **(iii)** $x^2 + y^2$.

7 $p = -14$. **8** $q = -14$.

9 **(a)** $5 - \text{i}$; **(b)** 26.

10 **(a)** $3 + 4\text{i}$; **(b)** **(i)** $2 + 11\text{i}$, **(ii)** $-7 + 24\text{i}$.

Solving quadratic equations.

1) Solve $x^2 + 9 = 0$
2) work out the discriminant (b^2-4ac) of this quadratic equation.
   what do you notice?

A quadratic equations can be written

$$ax^2 + bx + c = 0$$

The DISCRIMINANT is the value $b^2 - 4ac$

- **If** $b^2 - 4ac > 0$**, there are two roots** $x_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}$ *and* $x_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$
- **If** $b^2 - 4ac = 0$**, the re is a repeated root** $x_0 = -\frac{b}{2a}$
- **If** $b^2 - 4ac < 0$**, there are two complex conjugate roots** $x_1 = \frac{-b+i\sqrt{-b^2+4ac}}{2a}$ *and* $x_2 = \frac{-b-i\sqrt{-b^2+4ac}}{2a}$

Solving $x^2 - 4x + 13 = 0$

$$x^2 - 4x + 13 = 0$$

Discriminant: $(-4)^2 - 4 \times 1 \times 13 = 16 - 42 = -26$

The complex roots are $\dfrac{4 \pm i\sqrt{26}}{2}$, *meaning*

$$x_1 = \frac{4 + i\sqrt{26}}{2} \quad or \quad x_2 = \frac{4 - i\sqrt{26}}{2}$$

EXERCISE

Find the complex roots of each of the following equations:

1 $x^2 - 2x + 5 = 0$

2 $x^2 + 4x + 13 = 0$

3 $x^2 - 2x + 10 = 0$

4 $x^2 - 6x + 25 = 0$

5 $x^2 - 8x + 20 = 0$

6 $x^2 + 4x + 5 = 0$

7 $x^2 - 12x + 40 = 0$

8 $x^2 + 2x + 50 = 0$

9 $x^2 + 8x + 17 = 0$

10 $x^2 - 10x + 34 = 0$

11 $2x^2 - 2x + 5 = 0$

12 $9x^2 + 6x + 10 = 0$

13 $4x^2 - 8x + 5 = 0$

14 $5x^2 - 6x + 5 = 0$

15 $13x^2 + 10x + 13 = 0$

16 $x^2 - 2x + 4 = 0$

17 $x^2 + 4x + 9 = 0$

18 $x^2 - 6x + 16 = 0$

19 $x^2 + 8x + 19 = 0$

20 $x^2 - x + 1 = 0$

EXERCISE

1 $1 \pm 2i$. **2** $-2 \pm 3i$. **3** $1 \pm 3i$. **4** $3 \pm 4i$.

5 $4 \pm 2i$. **6** $-2 \pm i$. **7** $6 \pm 2i$. **8** $-1 \pm 7i$.

9 $-4 \pm i$. **10** $5 \pm 3i$. **11** $\frac{1}{2} \pm \frac{3}{2}i$. **12** $-\frac{1}{3} \pm i$.

13 $1 \pm \frac{1}{2}i$. **14** $\frac{3}{5} \pm \frac{4}{5}i$. **15** $-\frac{5}{13} \pm \frac{12}{13}i$. **16** $1 \pm \sqrt{3}i$.

17 $-2 \pm \sqrt{5}i$. **18** $3 \pm \sqrt{7}i$. **19** $-4 \pm \sqrt{3}i$. **20** $\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$.

Equality of two complex numbers

Consider two complex numbers

$$z_1 = a_1 + ib_1 \qquad and \qquad z_2 = a_2 + ib_2$$

$$z_1 = z_2 \ \ when \ \ a_1 = a_2 \qquad and \qquad b_1 = b_2$$

$$or \qquad \text{Re}(z_1) = \text{Re}(z_2) \ \ and \ \ \text{Im}(z_1) = \text{Im}(z_2)$$

Example: $z_1 = 3+ib$ and $z_2 = x+5+9i$

Knowing that $z_1 = z_2$, work out x and b

Find z so that

$$z + 2z^* = 6+3i$$

Method:

EXERCISE

1 Find the value of each of the real constants a and b such that $(a + 3\text{i})^2 = 8b + 30\text{i}$.

2 Find the value of each of the real constants p and q such that $(3 + 4\text{i})(p + 2\text{i}) = q + 26\text{i}$.

3 Find the value of each of the real constants t and u such that $(2 + 2\text{i})^2(t + 3\text{i}) = u + 32\text{i}$.

4 Find the complex number z so that $3z + 4z^* = 28 + \text{i}$.

5 Find the complex number z so that $z + 3z^* = 12 - 8\text{i}$.

6 Find the complex number z so that $z - 4\text{i}z^* + 2 + 7\text{i} = 0$.

1 $a = 5$, $b = 2$. 2 $p = 5$, $q = 7$. 3 $t = 4$, $u = -24$.
4 $z = 4 - \text{i}$. 5 $z = 3 + 4\text{i}$. 6 $z = 2 + \text{i}$.