Asymptotes and rational functions $f(x) = \dfrac{ax+b}{cx+d}$

Specifications:

Algebra and Graphs

Graphs of rational functions of the form

$$\frac{ax+b}{cx+d}$$

Sketching the graphs.

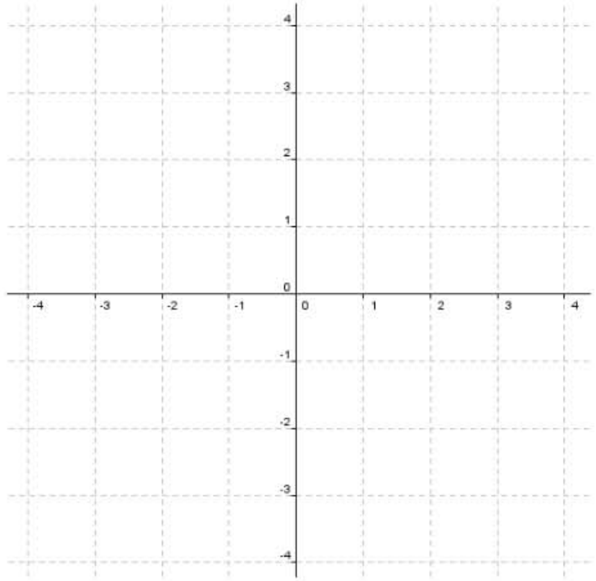
Finding the equations of the asymptotes which will always be parallel to the coordinate axes.

Finding points of intersection with the coordinate axes or other straight lines.

Solving associated inequalities.

Introduction:

Sketch the graph with equation $y = \dfrac{1}{x}$



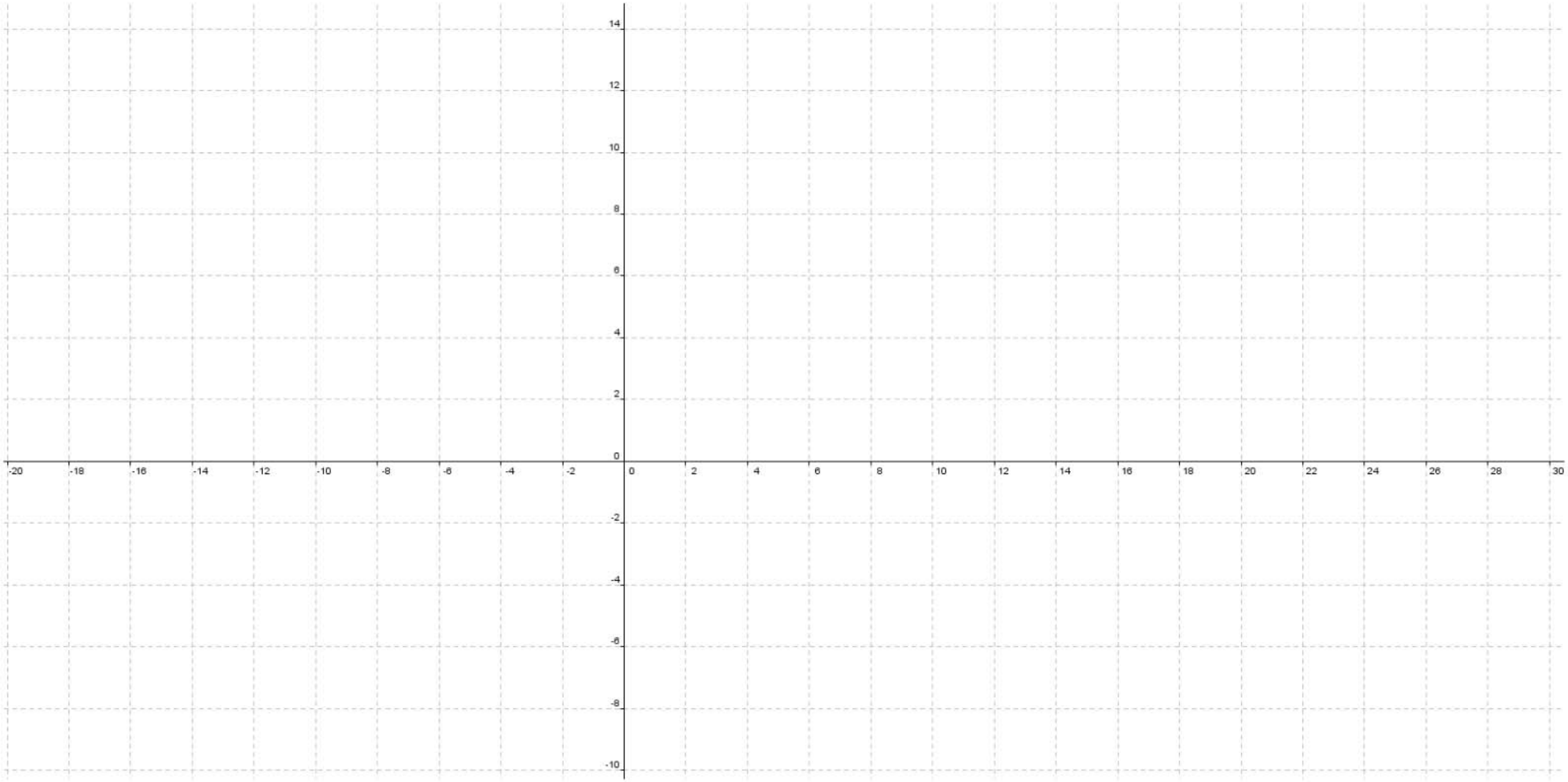
The graph of a rational function of the form $f(x) = \dfrac{ax+b}{cx+d}$ is a transformation (translations and stretches) of the graph $y = \dfrac{1}{x}$

This kind of curves is called HYPERBOLA

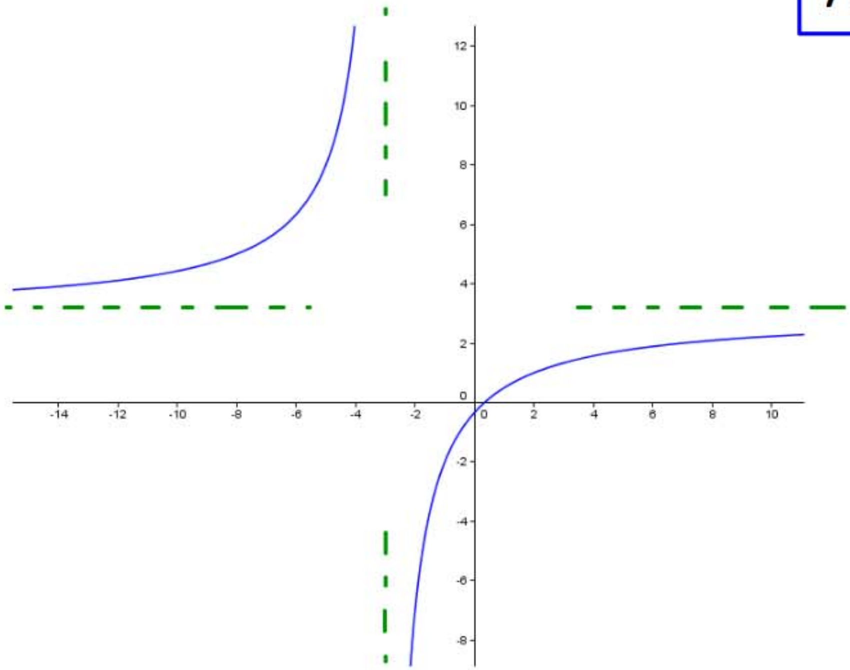
Consider the function $f(x) = \dfrac{3x - 1}{x - 2}$

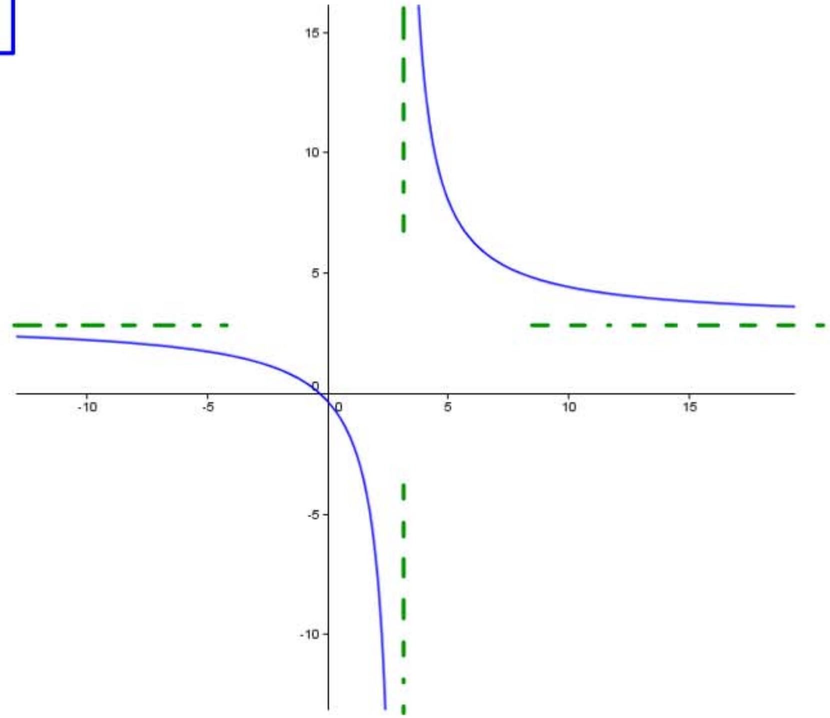
Complete the table of values and sketch the graph of *f*

x	-20	-15	-10	-5	-4	-3	-2	-1	0	1	1.5	1.6	1.7	1.8	1.9	2	2.1	2.2	2.3	2.4	2.5	3	4	5	10	15	20	25	30
y																													



Asymptotes

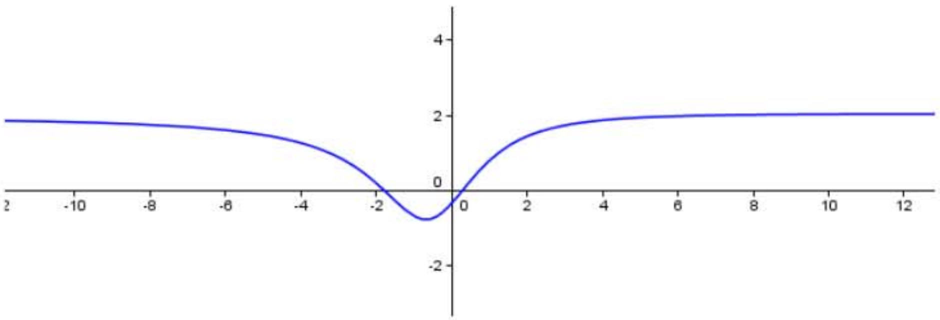


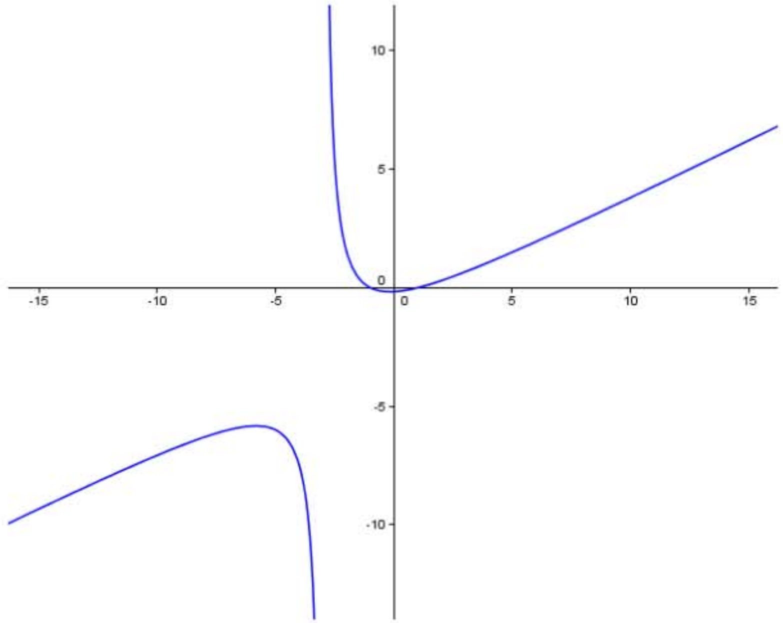


Consider a graph with equation y = f(x)

An asymptote is a line that a curve approaches for large values of |x| and |y|.

Asymptotes are usually represented by dotted lines





In this chapter, we only study asymptotes parallel to the axes

Vertical asymptotes

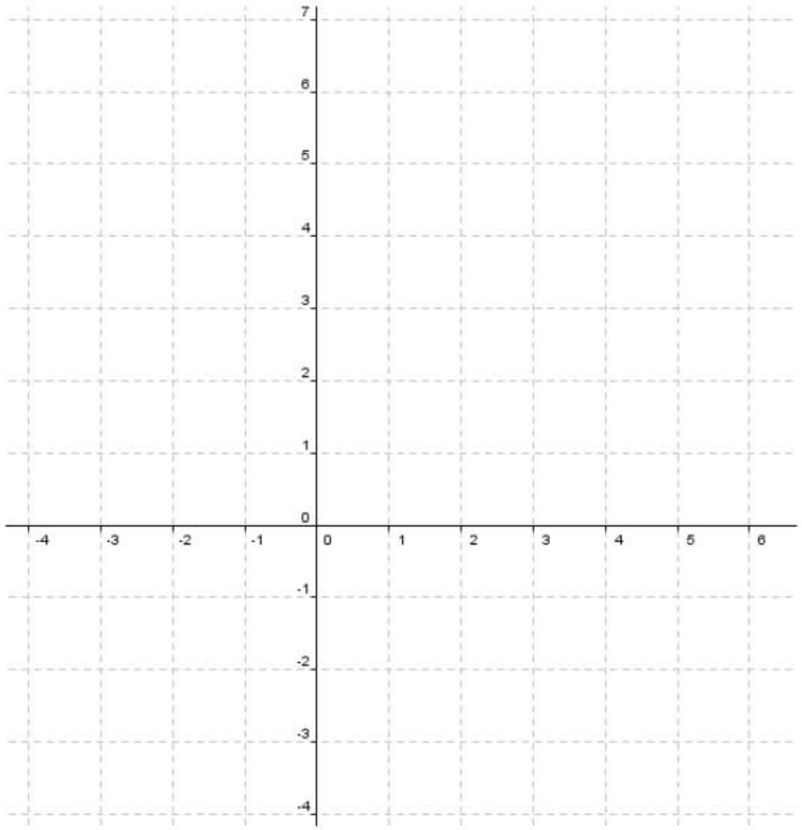
A curve has an equation of the form $y = \dfrac{f(x)}{g(x)}$

If g(a) = 0 then x = a is a "vertical" asymptote of the curve

$$We\ write: when\ x \to a, |y| \to \infty\ or\ |y| \xrightarrow[x \to a]{} \infty$$

Position of the curve relative to the asymptote

Consider the curve with equation $y = \dfrac{4x-3}{x-1}$

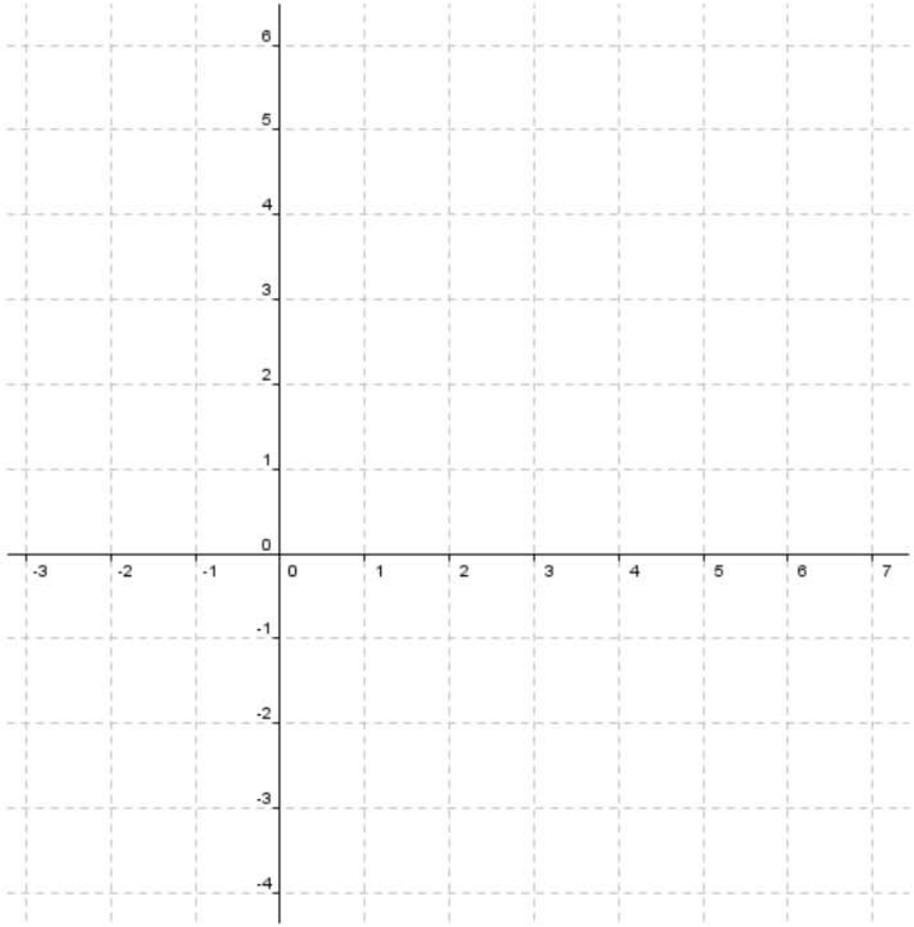


Horizontal asymptote

A curve has equation $y = \dfrac{f(x)}{g(x)}$.

If $y \xrightarrow[x \to \infty]{} k$ *then the line* $y = k$ *is a "horizontal" asymptote of the curve*

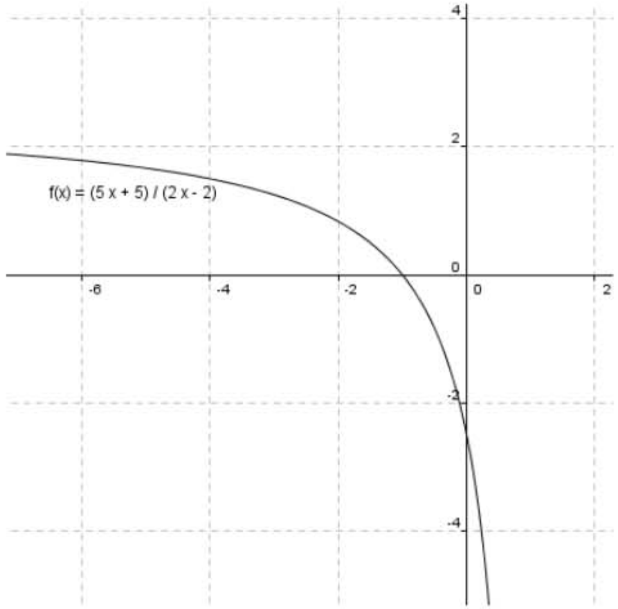
Method: $y = \dfrac{ax+b}{cx+d}$ Divide the numerator an the denomimator by"x"



Numerical example: $y = \dfrac{5x+3}{2x-2}$

Intersection with the axes

A curve has equation $y = \frac{f(x)}{g(x)}$



Intersection with the y-axis: work out y(0)

Intersetion with the x-axis: solve y = 0

Example: $y = \frac{5x+5}{2x-2}$

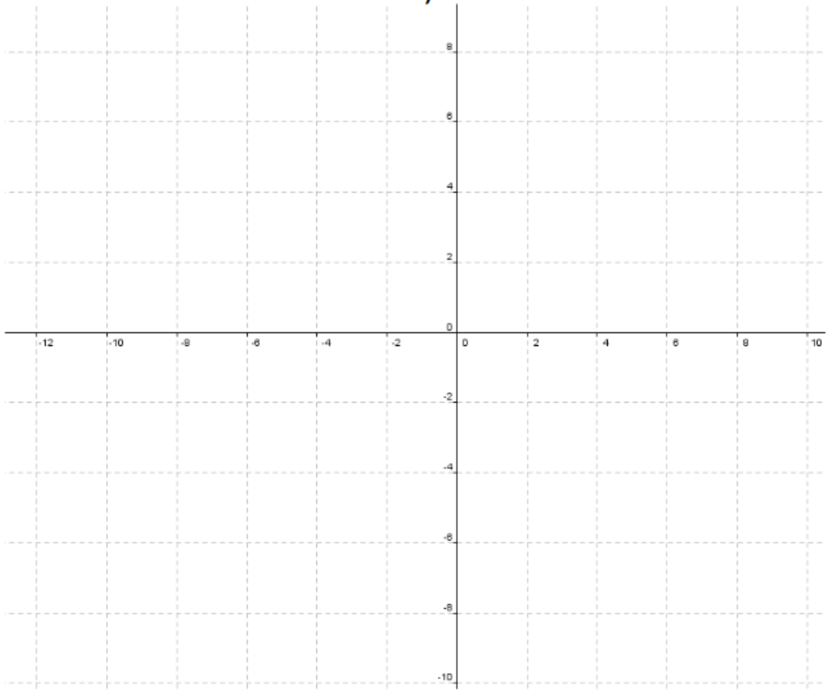
Exercises: For each of the following curves,

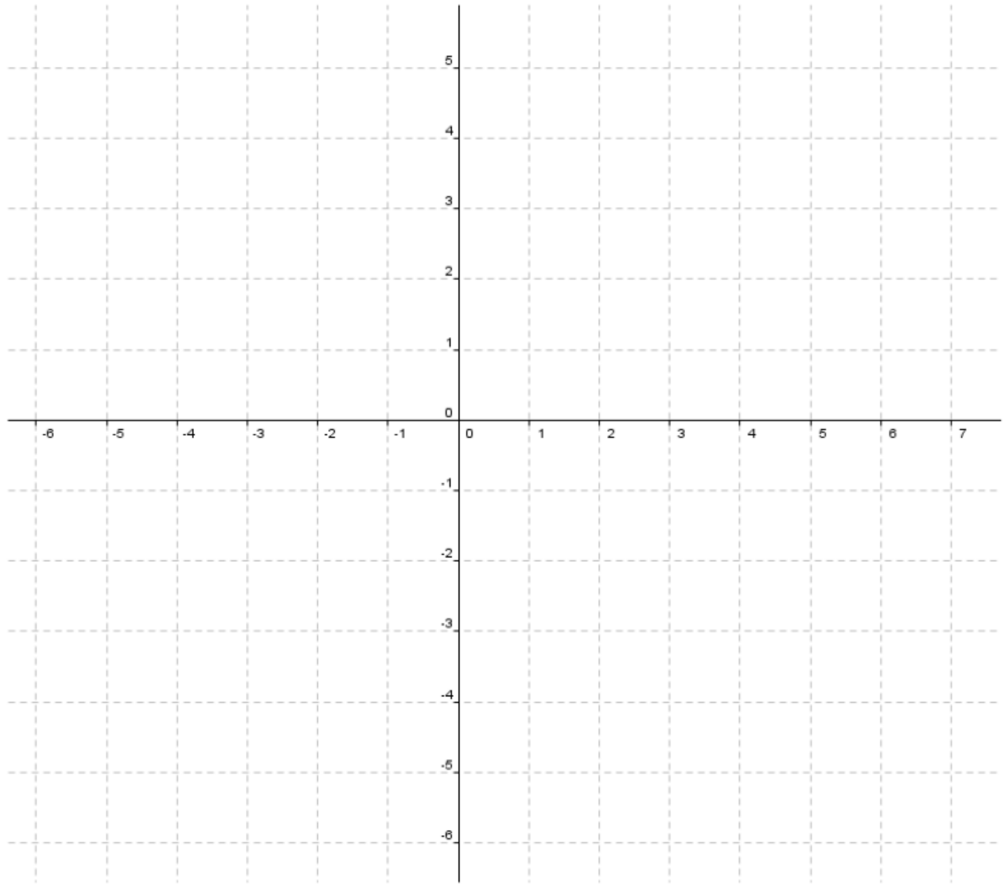
i) State the coordinates of the points where the curve crosses the axes.

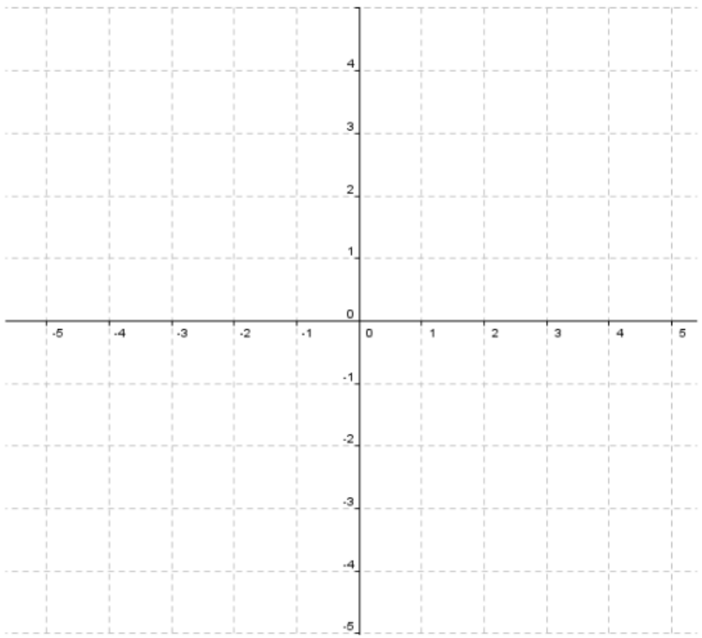
ii) Find the equation of the asymptotes.

iii) Sketch the curve.

$$a)\ y=\frac{5-x}{x+2} \qquad b)\ y=\frac{x+1}{3x-2} \qquad c)\ y=1+\frac{3}{x-2}$$







More practice:

1) For each of the following curves:

(i) find the equations of the asymptotes;

(ii) determine the coordinates of the points where the curve cuts the coordinate axes;

(iii) sketch its graph.

(a) $y = \frac{x-3}{2x+5}$, **(b)** $y = \frac{3x+4}{1-3x}$, **(c)** $y = 3 - \frac{2}{x+2}$.

2) Sketch the graphs of:

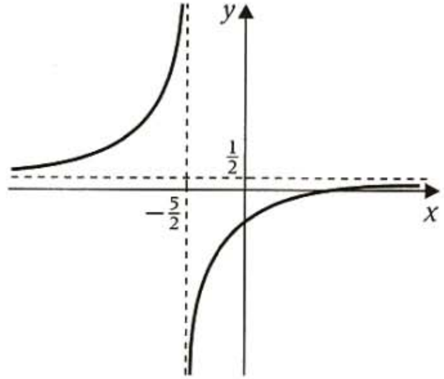
(a) $y = \frac{5x+3}{4x-7}$, **(b)** $y = \frac{4-8x}{3-5x}$, **(c)** $y = 2 - \frac{5x+3}{3x-2}$.

State the equations of any asymptotes and points where the curve crosses the coordinate axes.

Answers

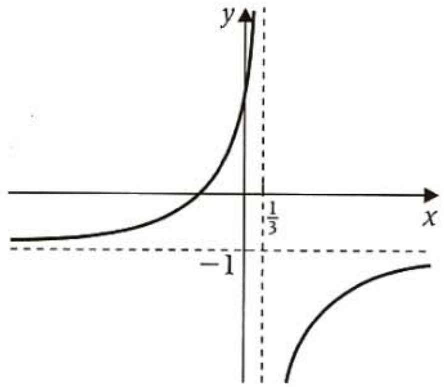
1) **(a)** **(i)** $x = -\frac{5}{2}$ and $y = \frac{1}{2}$, **(ii)** $(0, -\frac{3}{5})$ and $(3, 0)$,

(iii)



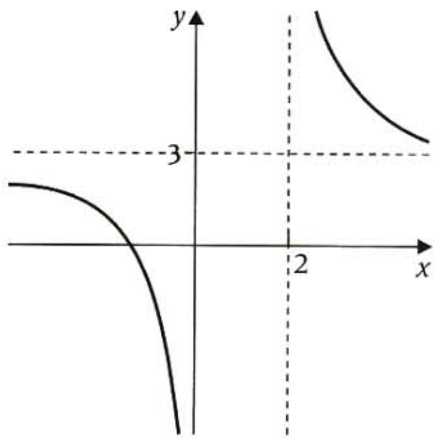
(b) **(i)** $x = \frac{1}{3}$ and $y = -1$, **(ii)** $(0, 4)$ and $(-\frac{4}{3}, 0)$,

(iii)

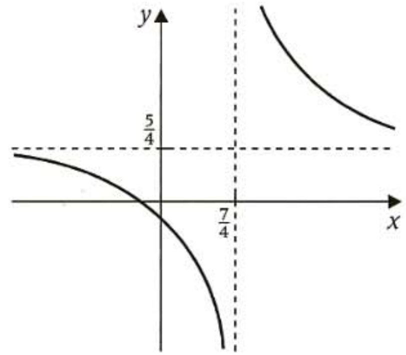


(c) **(i)** $x = 2$ and $y = 3$, **(ii)** $(0, -2)$ and $(-\frac{4}{3}, 0)$,

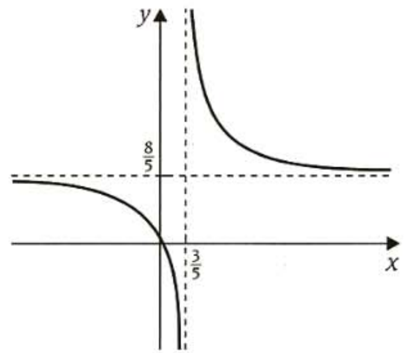
(iii)



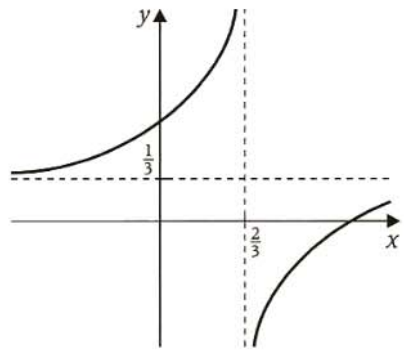
2) **(a)** $x = \frac{7}{4}$ and $y = \frac{5}{4}$,
$(0, -\frac{3}{7})$ and $(-\frac{3}{5}, 0)$



(b) $x = \frac{3}{5}$ and $y = \frac{8}{5}$
$(0, \frac{4}{3})$ and $(\frac{1}{2}, 0)$



(c) $x = \frac{2}{3}$ and $y = \frac{1}{3}$
$(0, \frac{7}{2})$ and $(7, 0)$



6 $a = 4$, $b = 1$.

Points of intersection between a straight line and the graph of a rational functions

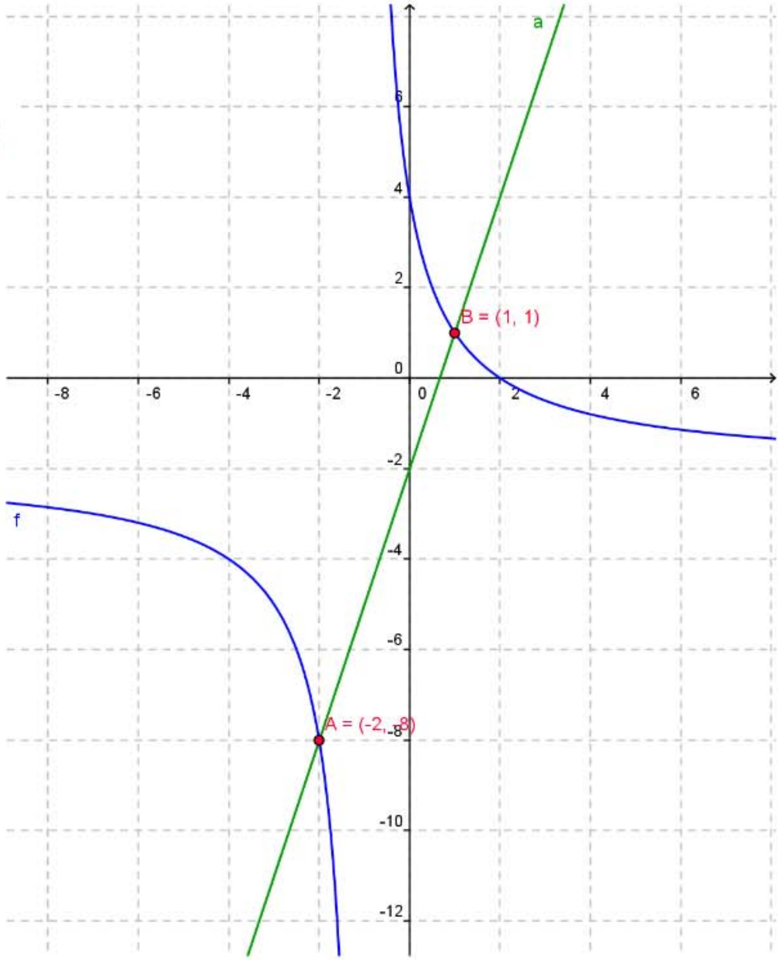
Consider a curve with equation $y = \dfrac{ax+b}{cx+d}$ and a straight line with equation $y = mx + e$

The coordinates of the points of intersection between the two graphs

are solutions to the SIMULTANEOUS equations $\begin{cases} y = \dfrac{ax+b}{cx+d} \\ y = mx + e \end{cases}$.

Example:

Determine the points of intersection of the curve $y = \dfrac{4-2x}{x+1}$ *and* the line $y = 3x - 2$



Have a go!

Find the points of intersection of the following curves and lines:

1 $y = \dfrac{2}{x+1}, \quad y = 2x - 1.$

2 $y = \dfrac{x-1}{2x+3}, \quad y = 3x + 1.$

3 $y = \dfrac{5x}{2x+1}, \quad y = 4 - x.$

4 $y = \dfrac{6x}{2x+5}, \quad y = 3x + 1.$

5 $y = \dfrac{3x-1}{5x-3}, \quad y = 4x - 3.$

Answers

1 $(1, 1)$ and $(-1\frac{1}{2}, -4)$.

2 $(-1, -2)$ and $(-\frac{2}{3}, -1)$.

3 $(2, 2)$ and $(-1, 5)$.

4 $(-1, -2)$, $(-\frac{5}{6}, -\frac{3}{2})$.

5 $(\frac{1}{2}, -1)$ and $(1, 1)$.

Inequalities

Once you have sketched the graph of the rational function, you can use the graph to help you solve inequalitites.

From textbook:

In order to solve inequalities such as

$$\frac{ax+b}{cx+d} < k \quad \text{or} \quad \frac{ax+b}{cx+d} > k:$$

1 Sketch the graph of $y = \frac{ax+b}{cx+d}$ and the line $y = k$.

2 Solve the equation $\frac{ax+b}{cx+d} = k$.

3 Use the graph to find the possible values of x for which the graph lies below or above the line.

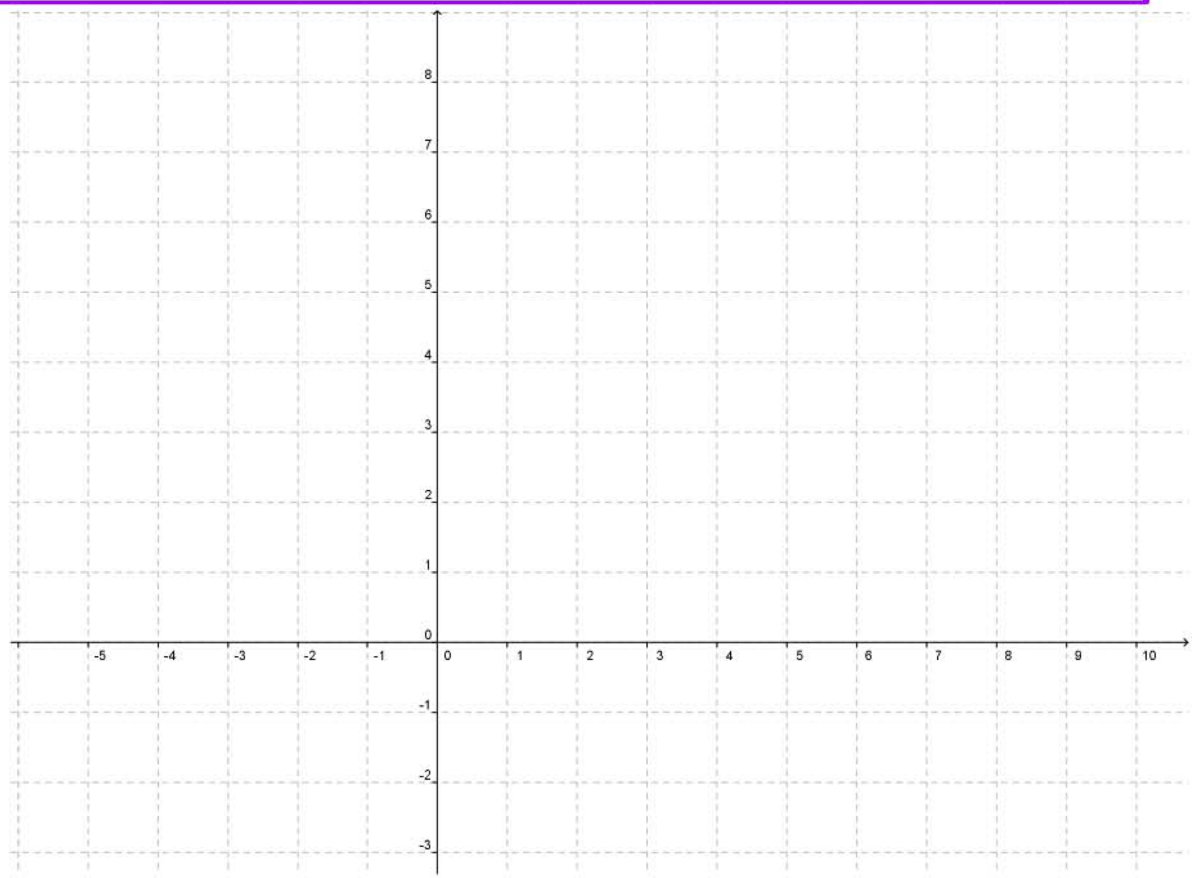
Worked examples:

$$y = \frac{3x-1}{x-2}$$

i) Find the asymptotes and the points of intersection with the axes

ii) Sketch the graph.

iii) Hence, or otherwise, solve $\frac{3x-1}{x-2} > -1$



Practice questions

9 **(a)** Sketch the graph of $y = \dfrac{3 - 4x}{2x - 5}$, stating the coordinates of the points where the curve crosses the coordinate axes and writing down the equations of its asymptotes.

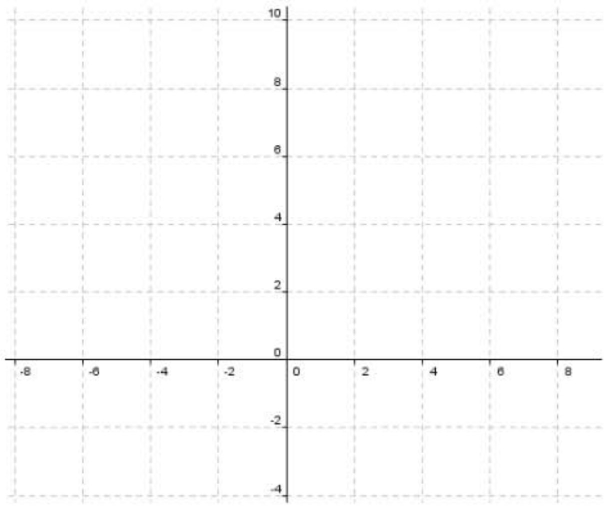
(b) Hence, or otherwise, solve the inequality $\dfrac{3 - 4x}{2x - 5} < 0$. [A]

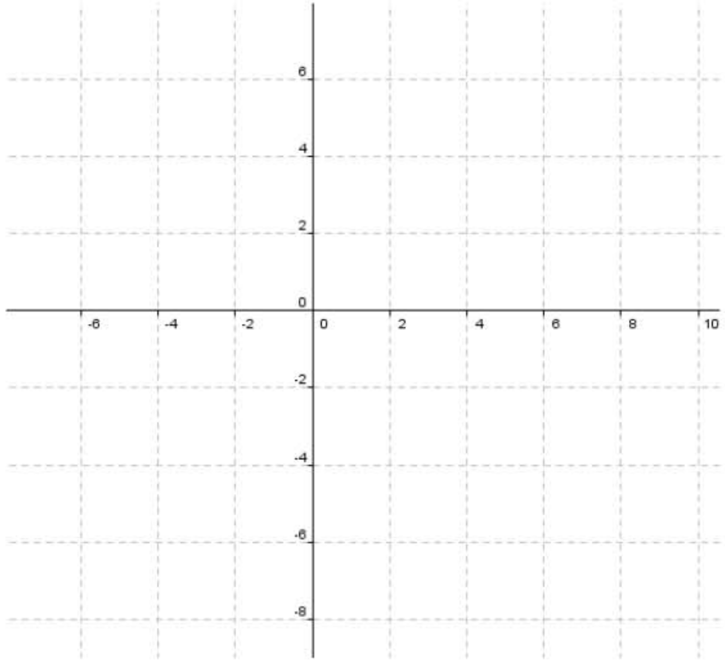
10 **(a)** Sketch the graph of $y = \dfrac{3x + 4}{x - 2}$, stating the coordinates of the points where the curve crosses the coordinate axes and writing down the equations of its asymptotes.

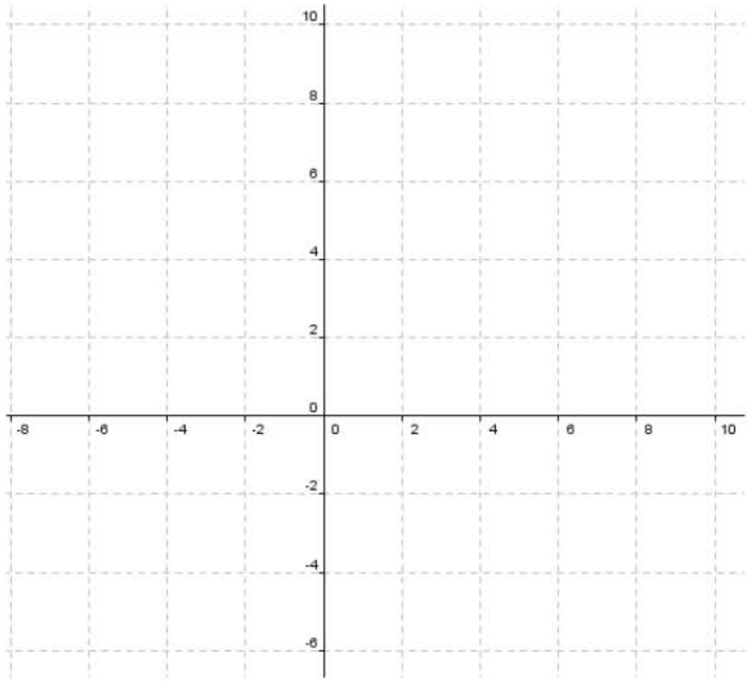
(b) Hence, or otherwise, solve the inequality $\dfrac{3x + 4}{x - 2} > 1$. [A]

11 **(a)** Sketch the curve with equation $y = \dfrac{4x - 3}{x - 1}$, stating the coordinates of the points where the curve crosses the coordinate axes and writing down the equations of its asymptotes.

(b) Solve the inequality $\dfrac{4x - 3}{x - 1} < x + 3$. [A]



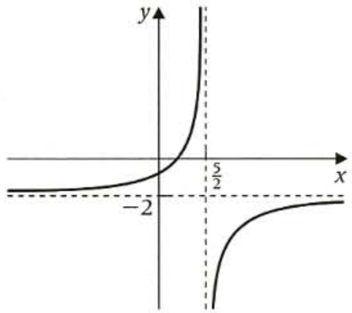




Answers:

9 **(a)** $x = \frac{5}{2}$ and $y = -2$

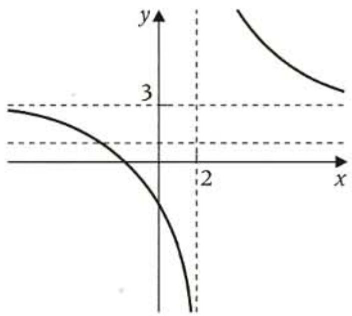
$(0, -\frac{3}{5})$ and $(\frac{3}{4}, 0)$



(b) $x < \frac{3}{4}$ and $x > \frac{5}{2}$.

10 **(a)** $x = 2$ and $y = 3$

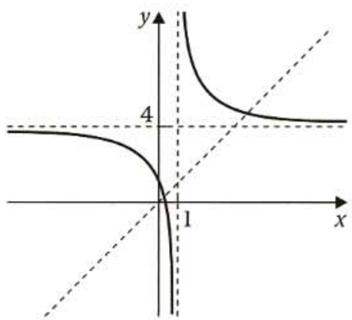
$(0, -2)$ and $(-\frac{4}{3}, 0)$



(b) $x < -3$ and $x > 2$.

11 **(a)** $x = 1$ and $y = 4$

$(0, 3)$ and $(\frac{3}{4}, 0)$



(b) $0 < x < 1$ and $x > 2$.

Key point summary

1 An asymptote is a line that a curve approaches for large values of $|x|$ or $|y|$. It is usually represented by a broken or dotted line.

2 The line $x = a$ is a vertical asymptote of the curve $y = \dfrac{f(x)}{g(x)}$ if $g(a) = 0$.

3 To find a horizontal asymptote of $y = \dfrac{ax + b}{cx + d}$, re-write

the equation as $y = \dfrac{a + \frac{b}{x}}{c + \frac{d}{x}}$.

As $|x| \to \infty$, $\dfrac{1}{x} \to 0$, therefore $y \to \dfrac{a}{c}$.

The horizontal asymptote has equation $y = \dfrac{a}{c}$.

4 In order to solve inequalities such as

$$\frac{ax + b}{cx + d} < k \quad \text{or} \quad \frac{ax + b}{cx + d} > k:$$

1 Sketch the graph of $y = \dfrac{ax + b}{cx + d}$ and the line $y = k$.

2 Solve the equation $\dfrac{ax + b}{cx + d} = k$.

3 Use the graph to find the possible values of x for which the graph lies below or above the line.