OCR
RECOGNISING ACHIEVEMENT

ADVANCED GCE

MATHEMATICS (MEI) 4753/01

Methods for Advanced Mathematics (C3)

Candidates answer on the Answer Booklet

OCR Supplied Materials:
- 8 page Answer Booklet
- Graph paper
- MEI Examination Formulae and Tables (MF2)

Other Materials Required:
None

Thursday 15 January 2009
Morning

Duration: 1 hour 30 minutes

475301

INSTRUCTIONS TO CANDIDATES

- Write your name clearly in capital letters, your Centre Number and Candidate Number in the spaces provided on the Answer Booklet.
- Use black ink. Pencil may be used for graphs and diagrams only.
- Read each question carefully and make sure that you know what you have to do before starting your answer.
- Answer **all** the questions.
- Do **not** write in the bar codes.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets **[ ]** at the end of each question or part question.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- The total number of marks for this paper is **72**.
- This document consists of **4** pages. Any blank pages are indicated.

3R–8H16

Section A (36 marks)

1 Solve the inequality $|x - 1| < 3$. **[3]**

2 **(i)** Differentiate $x\cos 2x$ with respect to x. **[3]**

(ii) Integrate $x\cos 2x$ with respect to x. **[4]**

3 Given that $\mathrm{f}(x) = \frac{1}{2}\ln(x - 1)$ and $\mathrm{g}(x) = 1 + \mathrm{e}^{2x}$, show that $\mathrm{g}(x)$ is the inverse of $\mathrm{f}(x)$. **[3]**

4 Find the exact value of $\int_0^2 \sqrt{1 + 4x}\,\mathrm{d}x$, showing your working. **[5]**

5 **(i)** State the period of the function $\mathrm{f}(x) = 1 + \cos 2x$, where x is in degrees. **[1]**

(ii) State a sequence of two geometrical transformations which maps the curve $y = \cos x$ onto the curve $y = \mathrm{f}(x)$. **[4]**

(iii) Sketch the graph of $y = \mathrm{f}(x)$ for $-180° < x < 180°$. **[3]**

6 **(i)** Disprove the following statement.

'If $p > q$, then $\frac{1}{p} < \frac{1}{q}$.' **[2]**

(ii) State a condition on p and q so that the statement is true. **[1]**

7 The variables x and y satisfy the equation $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 5$.

(i) Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = -\left(\frac{y}{x}\right)^{\frac{1}{3}}$. **[4]**

Both x and y are functions of t.

(ii) Find the value of $\frac{\mathrm{d}y}{\mathrm{d}t}$ when $x = 1$, $y = 8$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 6$. **[3]**

Section B (36 marks)

8 Fig. 8 shows the curve $y = x^2 - \frac{1}{8}\ln x$. P is the point on this curve with x-coordinate 1, and R is the point $\left(0, -\frac{7}{8}\right)$.

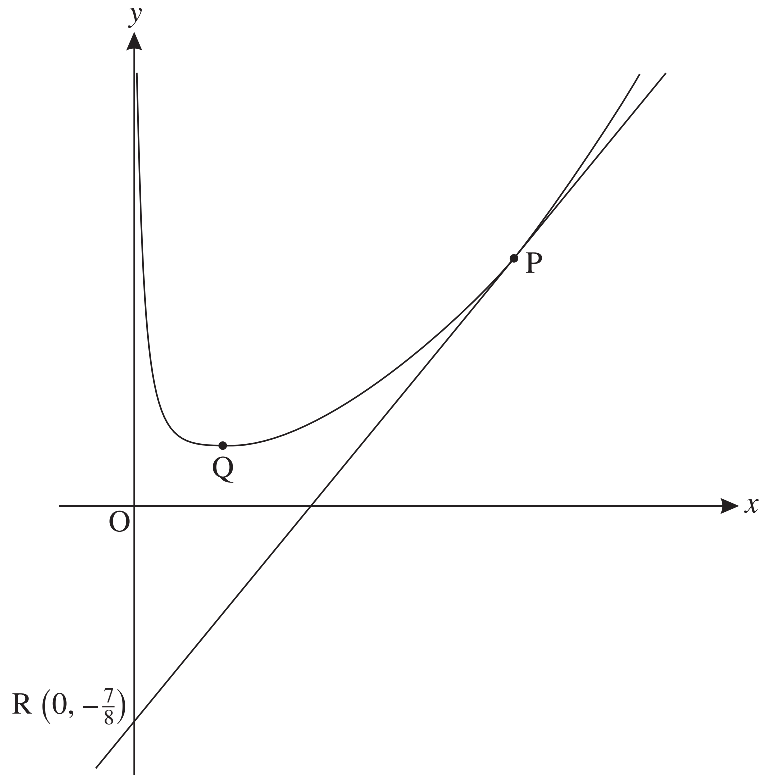


Fig. 8

(i) Find the gradient of PR. **[3]**

(ii) Find $\frac{dy}{dx}$. Hence show that PR is a tangent to the curve. **[3]**

(iii) Find the exact coordinates of the turning point Q. **[5]**

(iv) Differentiate $x \ln x - x$.

Hence, or otherwise, show that the area of the region enclosed by the curve $y = x^2 - \frac{1}{8}\ln x$, the x-axis and the lines $x = 1$ and $x = 2$ is $\frac{59}{24} - \frac{1}{4}\ln 2$. **[7]**

[Question 9 is printed overleaf.]

9 Fig. 9 shows the curve $y = \mathrm{f}(x)$, where $\mathrm{f}(x) = \dfrac{1}{\sqrt{2x - x^2}}$.

The curve has asymptotes $x = 0$ and $x = a$.

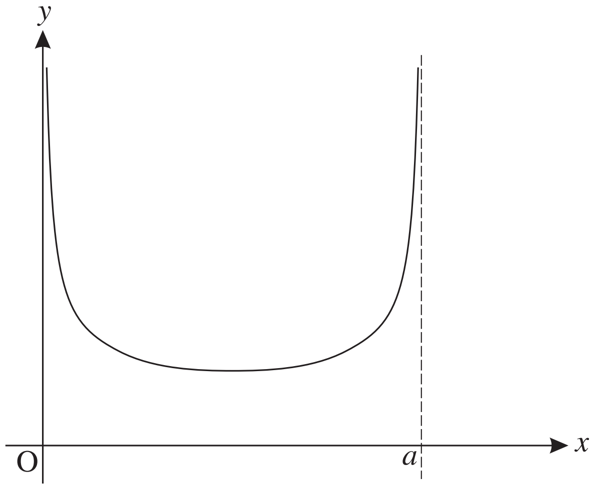


Fig. 9

(i) Find a. Hence write down the domain of the function. **[3]**

(ii) Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{x - 1}{(2x - x^2)^{\frac{3}{2}}}$.

Hence find the coordinates of the turning point of the curve, and write down the range of the function. **[8]**

The function $\mathrm{g}(x)$ is defined by $\mathrm{g}(x) = \dfrac{1}{\sqrt{1 - x^2}}$.

(iii) (*A*) Show algebraically that $\mathrm{g}(x)$ is an even function.

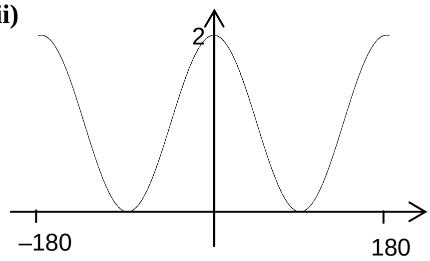
(*B*) Show that $\mathrm{g}(x - 1) = \mathrm{f}(x)$.

(*C*) Hence prove that the curve $y = \mathrm{f}(x)$ is symmetrical, and state its line of symmetry. **[7]**

4753 (C3) Methods for Advanced Mathematics

Section A

1 $\lvert x-1\rvert < 3 \Rightarrow -3 < x-1 < 3$ $\Rightarrow \quad -2 < x < 4$	M1 A1 B1 [3]	or $x - 1 = \pm 3$, or squaring $\Rightarrow$ correct quadratic $\Rightarrow$ $(x + 2)(x - 4)$ (condone factorising errors) or correct sketch showing $y = 3$ to scale $-2 <$ < 4 (penalise $\le$ once only)
2(i) $y = x\cos 2x$ $\Rightarrow \quad \frac{dy}{dx} = -2x\sin 2x + \cos 2x$	M1 B1 A1 [3]	product rule $d/dx\,(\cos 2x) = -2\sin 2x$ oe cao
(ii) $\int x\cos 2x\,dx = \int x\frac{d}{dx}(\frac{1}{2}\sin 2x)dx$ $= \frac{1}{2}x\sin 2x - \int \frac{1}{2}\sin 2x\,dx$ $= \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + c$	M1 A1 A1ft A1 [4]	parts with $u = x$, $v = ½ \sin 2x$ $+\frac{1}{4}\cos 2x$ cao – must have $+ c$
3 *Either* $y = \frac{1}{2}\ln(x-1)$ $\quad x \leftrightarrow y$ $\Rightarrow \quad x = \frac{1}{2}\ln(y-1)$ $\Rightarrow \quad 2x = \ln(y-1)$ $\Rightarrow \quad e^{2x} = y - 1$ $\Rightarrow \quad 1 + e^{2x} = y$ $\Rightarrow \quad g(x) = 1 + e^{2x}$	 M1 M1 E1	or $y = e^{(x-1)/2}$ attempt to invert and interchanging x with y o.e. (at any stage) $e^{\ln y-1} = y - 1$ or $\ln(e^y) = y$ used www
or $gf(x) = g(½ \ln(x-1))$ $= 1 + e^{\ln(x-1)}$ $= 1 + x - 1$ $= x$	M1 M1 E1 [3]	or $fg(x) = \ldots$ (correct way round) $e^{\ln(x-1)} = x - 1$ or $\ln(e^{2x}) = 2x$ www
4 $\int_0^2 \sqrt{1+4x}\,dx$ $\quad$ let $u = 1+4x,\ du = 4dx$ $= \int_1^9 u^{1/2}.\frac{1}{4}du$ $= \left[\frac{1}{6}u^{3/2}\right]_1^9$ $= \frac{27}{6} - \frac{1}{6} = \frac{26}{6} = \frac{13}{3}$ or $4\frac{1}{3}$	M1 A1 B1 M1 A1cao	$u = 1 + 4x$ and $du/dx = 4$ or $du = 4dx$ $\int u^{1/2}.\frac{1}{4}du$ $\int u^{1/2}du = \frac{u^{3/2}}{3/2}$ soi substituting correct limits (u or x) dep attempt to integrate
or $\frac{d}{dx}(1+4x)^{3/2} = 4.\frac{3}{2}(1+4x)^{1/2} = 6(1+4x)^{1/2}$ $\Rightarrow \quad \int_0^2 (1+4x)^{1/2}dx = \left[\frac{1}{6}(1+4x)^{3/2}\right]_0^2$ $= \frac{27}{6} - \frac{1}{6} = \frac{26}{6} = \frac{13}{3}$ or $4\frac{1}{3}$	M1 A1 A1 M1 A1cao [5]	$k(1 + 4x)^{3/2}$ $\int (1+4x)^{1/2}dx = \frac{2}{3}(1+4x)^{3/2}\ldots$ $\times ¼$ substituting limits (dep attempt to integrate)

5(i) period 180°	B1 [1]	condone $0 \le x \le 180°$ or π
(ii) one-way stretch in x-direction scale factor ½ translation in y-direction through $\begin{pmatrix}0\\1\end{pmatrix}$	M1 A1 M1 A1 [4]	[either way round…] condone 'squeeze', 'contract' for M1 stretch used and s.f ½ condone 'move', 'shift', etc for M1 'translation' used, +1 unit $\begin{pmatrix}0\\1\end{pmatrix}$ only is M1 A0
(iii) 	M1 B1 A1 [3]	correct shape, touching x-axis at -90°, 90° correct domain (0, 2) marked or indicated (i.e. amplitude is 2)
6(i) e.g $p = 1$ and $q = -2$ $p > q$ but $1/p = 1 > 1/q = -½$	M1 E1 [2]	stating values of p, q with $p \ge 0$ and $q \le 0$ (but not $p = q = 0$) showing that $1/p > 1/q$ - if 0 used, must state that 1/0 is undefined or infinite
(ii) Both p and q positive (or negative)	B1 [1]	or $q > 0$, 'positive integers'
7(i) $\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$ $\Rightarrow \frac{dy}{dx} = -\frac{\frac{2}{3}x^{-1/3}}{\frac{2}{3}y^{-1/3}}$ $= -\frac{y^{1/3}}{x^{1/3}} = -\left(\frac{y}{x}\right)^{\frac{1}{3}}$ *	M1 A1 M1 E1 [4]	Implicit differentiation (must show = 0) solving for dy/dx www. Must show, or explain, one more step.
(ii) $\frac{dy}{dt} = \frac{dy}{dx}.\frac{dx}{dt}$ $= -\left(\frac{8}{1}\right)^{\frac{1}{3}}.6$ $= -12$	M1 A1 A1cao [3]	any correct form of chain rule

8(i) When $x = 1$ $y = 1^2 - (\ln 1)/8 = 1$ Gradient of PR = $(1 + 7/8)/1 = 1\frac{7}{8}$	B1 M1 A1 [3]	 1.9 or better
(ii) $\frac{dy}{dx} = 2x - \frac{1}{8x}$ When $x = 1$, $dy/dx = 2 - 1/8 = 1\frac{7}{8}$ Same as gradient of PR, so PR touches curve	B1 B1dep E1 [3]	cao 1.9 or better dep 1st B1 dep gradients exact
(iii) Turning points when $dy/dx = 0$ $\Rightarrow \quad 2x - \frac{1}{8x} = 0$ $\Rightarrow \quad 2x = \frac{1}{8x}$ $\Rightarrow \quad x^2 = 1/16$ $\Rightarrow \quad x = ¼\ (x > 0)$ When $x = ¼$, $y = \frac{1}{16} - \frac{1}{8}\ln\frac{1}{4} = \frac{1}{16} + \frac{1}{8}\ln 4$ So TP is $(\frac{1}{4}, \frac{1}{16} + \frac{1}{8}\ln 4)$	M1 M1 A1 M1 A1cao [5]	setting their derivative to zero multiplying through by x allow verification substituting for x in y o.e. but must be exact, not $1/4^2$. Mark final answer.
(iv) $\frac{d}{dx}(x\ln x - x) = x.\frac{1}{x} + 1.\ln x - 1 = \ln x$	M1 A1	product rule $\ln x$
Area = $\int_1^2 (x^2 - \frac{1}{8}\ln x)dx$ $= \left[\frac{1}{3}x^3 - \frac{1}{8}(x\ln x - x)\right]_1^2$ $= (\frac{8}{3} - \frac{1}{4}\ln 2 + \frac{1}{4}) - (\frac{1}{3} - \frac{1}{8}\ln 1 + \frac{1}{8})$ $= \frac{7}{3} + \frac{1}{8} - \frac{1}{4}\ln 2$ $= \frac{59}{24} - \frac{1}{4}\ln 2$ *	M1 M1 A1 M1 E1 [7]	correct integral and limits (soi) – condone no dx $\int \ln x dx = x\ln x - x$ used (or derived using integration by parts) $\frac{1}{3}x^3 - \frac{1}{8}(x\ln x - x)$ – bracket required substituting correct limits must show at least one step

9(i) Asymptotes when $(\sqrt{})(2x - x^2) = 0$ $\Rightarrow$ $x(2 - x) = 0$ $\Rightarrow$ $x = 0$ or 2 so $a = 2$ Domain is $0 < x < 2$	M1 A1 B1ft [3]	 or by verification $x > 0$ and $x < 2$, not $\leq$
(ii) $y = (2x - x^2)^{-1/2}$ let $u = 2x - x^2$, $y = u^{-1/2}$ $\Rightarrow$ $dy/du = -\frac{1}{2} u^{-3/2}$, $du/dx = 2 - 2x$ $\Rightarrow$ $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = -\frac{1}{2}(2x - x^2)^{-3/2}.(2 - 2x)$ $= \frac{x - 1}{(2x - x^2)^{3/2}}$ *	 M1 B1 A1 E1	 chain rule (or within correct quotient rule) $-\frac{1}{2} u^{-3/2}$ or $-\frac{1}{2}(2x - x^2)^{-3/2}$ or $\frac{1}{2}(2x - x^2)^{-1/2}$ in quotient rule $\times (2 - 2x)$ www – penalise missing brackets here
$dy/dx = 0$ when $x - 1 = 0$ $\Rightarrow$ $x = 1$, $y = 1/\sqrt{(2 - 1)} = 1$ Range is $y \geq 1$	M1 A1 B1 B1ft [8]	extraneous solutions M0
(iii) (*A*) $g(-x) = \frac{1}{\sqrt{1-(-x)^2}} = \frac{1}{\sqrt{1-x^2}} = g(x)$	M1 E1	Expression for $g(-x)$ – must have $g(-x) = g(x)$ seen
(*B*) $g(x-1) = \frac{1}{\sqrt{1-(x-1)^2}}$ $= \frac{1}{\sqrt{1-x^2+2x-1}} = \frac{1}{\sqrt{2x-x^2}} = f(x)$	M1 E1	 must expand bracket
(*C*) $f(x)$ is $g(x)$ translated 1 unit to the right. But $g(x)$ is symmetrical about O*y* So $f(x)$ is symmetrical about $x = 1$.	M1 M1 A1	 dep both M1s
or $f(1 - x) = g(-x)$, $f(1 + x) = g(x)$ $\Rightarrow$ $f(1 + x) = f(1 - x)$ $\Rightarrow$ $f(x)$ is symmetrical about $x = 1$.	M1 E1 A1 [7]	or $f(1-x) = \frac{1}{\sqrt{2-2x-(1-x)^2}} = \frac{1}{\sqrt{2-2x-1+2x-x^2}} = \frac{1}{\sqrt{1-x^2}}$ $f(1+x) = \frac{1}{\sqrt{2+2x-(1+x)^2}} = \frac{1}{\sqrt{2+2x-1-2x-x^2}} = \frac{1}{\sqrt{1-x^2}}$

4753 Concepts in Pure Mathematics (C3) (Written Examination)

General Comments

This paper proved to be a straightforward and accessible test of candidates' abilities, and there were many excellent scripts scoring over 60 marks. Even low-scoring candidates managed to obtain around 30 marks. There was no evidence of time problems.

The standard of presentation was, however, variable – some candidates often took pages of working to process relatively simple algebra. As usual there was evidence of sloppy notation, in particular omission of essential brackets from working, which was penalised in 'E' marks.

It is helpful to the marking process if extra sheets are attached, using treasury tags, to the back of answer booklets, rather than inter-leaved.

Comments on Individual Questions

Section A

1) The response of candidates to modulus questions is improving, although there remains evidence of misunderstandings in the notation. For example, some candidates write $|x - 1| < -3$, albeit obtaining the correct limit $x < -2$, and writing $|x| < 4$ scored no marks. Another common error is $x - 1 < -3$. Although some candidates used squaring and a quadratic correctly, the simplest route to the correct answer is to expand the modulus as $-3 < x - 1 < 3 \Rightarrow -2 < x < 4$.

2) (i) Nearly all candidates spotted this as a product rule, and many got it right. The most common errors were in the derivative of cos 2*x*, for example omitting the negative sign or the factor of 2.

(ii) Integration by parts was generally well known, but the usual confusions between integral and derivative results for trigonometric functions led to errors in $v = ½ \sin 2x$. Omitting the arbitrary constant cost quite a few candidates the final 'A' mark.

3) This question was very well answered, with only the weakest candidates failing to get 3 marks, either using $f^{-1}(x) = 1/f(x)$, or expanding $\ln(x - 1)$ as $\ln x - \ln 1$.

4) Although this is a fairly standard integration by substitution, quite a lot of candidates failed to obtain the correct answer. Many are omitting 'dx' and 'du' altogether, or getting du = 4 dx but failing to introduce a factor of ¼ in the integrand. Another quite common error is to (needlessly) convert back to 'x' and use 'u' limits. It would be nice to see an improvement in the use of accurate notation here!

5) (i) The ‘period’ has not been asked for in recent papers, so tripped up quite a few candidates, mistaking it for ‘domain’. ‘2’ was a common error here, but we generously condoned $0 < x < 180$.

(ii) There was the usual plethora of alternatives to ‘translate’ and ‘stretch’ (i.e. ‘move’, ‘squeeze’, ‘contract’, ‘shift’ etc.). We insisted on ‘translate’ (notwithstanding the correct vector being specified) and ‘stretch’.

(iii) We prefer candidates **not** to use graph paper for a ‘sketch’. Most candidates gave us a sketch with the correct domain, but there were many errors in its shape.

6) (i) The success of candidates in this question did not seem to correlate that well with marks in the other questions. Many seemed to ‘discover’ incorrect counter-examples. We allowed $p = 0$ or $q = 0$ for M1 and E1 if accompanied by explanations such as ‘infinite’ or ‘undefined’.

(ii) We wanted conditions of some generality here, although logically speaking stating it for a single value of p and q can lead to a true statement!

7) (i) Implicit differentiation was quite well done, although some differentiated the power incorrectly. The most common error was to start $dy/dx = \ldots$, and the crucial zero on the right hand side was sometimes missing. Candidates who start d/dx (LHS) = d/dx(RHS) usually got it right. The algebra required to navigate to the given answer was often incorrect.

(ii) Many candidates who failed to score in part (i) got 3 easy marks here. Errors were usually associated with evaluating $-(8/1)^{1/3}$ incorrectly, often omitting the negative sign.

Section B

8) This question had plenty of accessible marks, thought the final part proved testing for even the best candidates.

(i) Most candidates got this correct. The most common errors were to getting the gradient fraction the wrong way round, or use the derivative intended for part (ii).

(ii) The derivative was sometimes written as $2x - 1/8\,x$, leading to a fortuitously correct gradient.

(iii) Equating their derivative to zero gave a very accessible ‘M’ mark However, in the case where the derivative was correct, a surprising number of candidates showed algebraic immaturity by failing to solve the subsequent equation by multiplying through by x (or equivalent). Also, many candidates missed out on the final ‘A’ mark by approximating their y-coordinate.

(iv) Most spotted the product rule to differentiate $x \ln x$, though some used $u = x \ln x$ and $v = x$, and others forgot to take away the '1' from the derivative of x. This might be caused by a lack of organisation of side working.

Having achieved the $\ln x$ from the first part, some candidates then failed to spot the connection with the $1/8 \ln x$ in the area integral, using integration by parts. Many also failed to put the $x \ln x - x$ in a bracket. Only the 'A' grade candidates negotiated the fractional arithmetic convincingly to derive the given result.

9) Again, this question offered plenty of accessible marks.

(i) Most candidates set $2x - x^2$ to zero and solved to get $a = 2$ (some requiring the quadratic formula!). However, the domain was less well done.

(ii) About a half of the candidates used a quotient rule rather than a chain rule, which gave more opportunities for error. We withheld the final 'E' mark if there were missing brackets, at any stage, round the '$2 - 2x$'.

The second part, finding the turning point, offered a very straightforward 3 marks, but the range was quite often omitted or incorrect (e.g. $y > 1$ or $x \geq 1$).

(iii) Most candidates offered an algebraic proof of the even-ness of g, though some wrote $1 - (-x^2)$ or $1 - -x^2$. Sometimes the way the argument is written makes the implication unclear. For example, the 'proof' below is incomplete:

g is even if $g(x) = g(-x)$

So $\dfrac{1}{\sqrt{1-x^2}} = \dfrac{1}{\sqrt{1-(-x)^2}}$

On the other hand, the argument convinces if it is written as follows:

$$g(-x) = \frac{1}{\sqrt{1-(-x)^2}} = \frac{1}{\sqrt{1-x^2}} = g(x)$$

The proof that $g(x - 1) = f(x)$ was generally sound, though some omitted brackets round the $(x - 1)^2$ and lost the 'E1'.

Part (C) required candidates to state the symmetry of g in the y-axis, the translation of g to f one unit to the right, and the new axis of symmetry $x = 1$.