RECOGNISING ACHIEVEMENT

ADVANCED GCE **4758/01**

MATHEMATICS (MEI)

Differential Equations

THURSDAY 12 JUNE 2008 Morning

Time: 1 hour 30 minutes

Additional materials (enclosed): None

Additional materials (required):
Answer Booklet (8 pages)
Graph paper
MEI Examination Formulae and Tables (MF2)

INSTRUCTIONS TO CANDIDATES

- Write your name in capital letters, your Centre Number and Candidate Number in the spaces provided on the Answer Booklet.
- Read each question carefully and make sure you know what you have to do before starting your answer.
- Answer any **three** questions.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The acceleration due to gravity is denoted by $g\,\mathrm{m\,s^{-2}}$. Unless otherwise instructed, when a numerical value is needed, use $g = 9.8$.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- The total number of marks for this paper is **72**.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.

This document consists of **4** printed pages.

1 Fig. 1 shows a particle of mass 2 kg suspended from a light vertical spring. At time t seconds its displacement is x m below its equilibrium level and its velocity is $v\,\mathrm{m\,s^{-1}}$ vertically downwards. The forces on the particle are

- its weight, $2g$ N
- the tension in the spring, $8(x + 0.25g)$ N
- the resistance to motion, $2kv$ N where k is a positive constant.

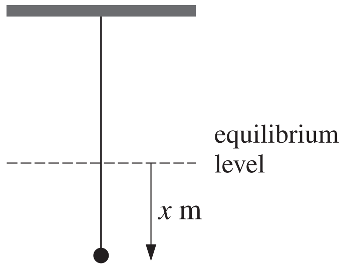


Fig. 1

(i) Use Newton's second law to write down the equation of motion for the particle, justifying the signs of the terms. Hence show that the displacement is described by the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + k\frac{\mathrm{d}x}{\mathrm{d}t} + 4x = 0.$$ [4]

The particle is initially at rest with $x = 0.1$.

(ii) In the case $k = 0$, state the general solution of the differential equation. Find the solution, subject to the given initial conditions. [4]

(iii) In the case $k = 2$, find the solution of the differential equation, subject to the given initial conditions. Sketch a graph of the solution for $t \geqslant 0$. [11]

(iv) Find the range of values of k for which the system is over-damped. Sketch a possible graph of the solution in such a case. [5]

2 The radioactive substance X decays into the substance Y, which in turn decays into Z. At time t hours the masses, in grams, of X, Y and Z are denoted by x, y and z respectively.

Initially there is 8 g of X and there is no Y or Z present.

The differential equation modelling the decay of X is $\frac{\mathrm{d}x}{\mathrm{d}t} = -2x$.

(i) Find x in terms of t. [3]

The differential equation modelling the amount of Y is $\frac{\mathrm{d}y}{\mathrm{d}t} = 2x - y$.

(ii) Using your expression for x found in part **(i)**, solve this equation to find y in terms of t. [9]

(iii) Show that $y > 0$ for $t > 0$. Sketch a graph of y for $t \geqslant 0$. [5]

The differential equation modelling the amount of Z is $\frac{\mathrm{d}z}{\mathrm{d}t} = y$.

(iv) Without solving this equation, show that $x + y + z = 8$. Hence show that $z = 8\left(1 - \mathrm{e}^{-t}\right)^2$. [5]

(v) Calculate the time required for 99% of the total mass to become substance Z. [2]

3 The differential equation $t\frac{\mathrm{d}y}{\mathrm{d}t} + ky = t$, where k is a constant, is to be solved for $t \geqslant 1$, subject to the condition $y = 0$ when $t = 1$.

(i) When $k \neq -1$, find the solution for y in terms of t and k. [10]

(ii) Sketch a graph of the solution for $k = 2$. [2]

(iii) When $k = -1$, find the solution for y in terms of t. [5]

Now consider the differential equation $t\frac{\mathrm{d}y}{\mathrm{d}t} - \sin y = t$, subject to the condition $y = 0$ when $t = 1$. This is to be solved by Euler's method. The algorithm is given by $t_{r+1} = t_r + h$, $y_{r+1} = y_r + h\dot{y}_r$.

(iv) Using a step length of 0.1, perform two iterations of the algorithm to estimate the value of y when $t = 1.2$. [4]

If the algorithm is carried out with a step length of 0.05, the estimate for y when $t = 1.2$ is $y \approx 0.2138$.

(v) Explain with a reason which of these two estimates for y when $t = 1.2$ is likely to be more accurate. Hence, or otherwise, explain whether these estimates are likely to be overestimates or underestimates. [3]

4 The simultaneous differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 4x - 6y - 9\sin t,$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = 3x - 5y - 7\sin t,$$

are to be solved.

(i) Show that $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \frac{\mathrm{d}x}{\mathrm{d}t} - 2x = -9\cos t - 3\sin t$. [6]

(ii) Find the general solution for x. [9]

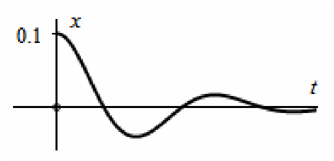
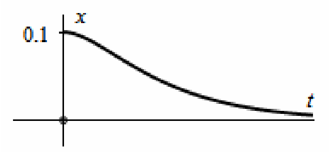
(iii) Hence find the corresponding general solution for y. [3]

It is given that x is bounded as $t \to \infty$.

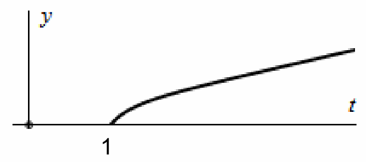
(iv) Show that y is also bounded as $t \to \infty$. [2]

(v) Given also that $y = 0$ when $t = 0$, find the particular solutions for x and y. Write down the expressions for x and y as $t \to \infty$. [4]

4758 Differential Equations

1 (i)	$2\ddot{x} = 2g - 8(x + 0.25g) - 2kv$	M1	N2L equation with all forces using given expressions for tension and resistance	
	Weight positive as down, tension negative as up.	B1		
	Resistance negative as opposes motion.	B1		
	$\Rightarrow \ddot{x} + k\dot{x} + 4x = 0$	E1	Must follow correct N2L equation	4
(ii)	$x = A\cos 2t + B\sin 2t$	B1		
	$t = 0, x = 0.1 \Rightarrow A = 0.1$	M1	Find the coefficient of cos	
	$\dot{x} = -2A\sin 2t + 2B\cos 2t$ so $t = 0, \dot{x} = 0 \Rightarrow B = 0$	M1	Find the coefficient of sin	
	$x = 0.1\cos 2t$	A1	cao	4
(iii)	$\alpha^2 + 2\alpha + 4 = 0$	M1	Auxiliary equation	
	$\alpha = -1 \pm \sqrt{3}\,j$	A1		
		M1	CF for complex roots	
	$x = e^{-t}\left(C\cos\sqrt{3}\,t + D\sin\sqrt{3}\,t\right)$	F1	CF for their roots	
	$t = 0, x = 0.1 \Rightarrow C = 0.1$	M1	Condition on x	
	$\dot{x} = -e^{-t}\left(C\cos\sqrt{3}\,t + D\sin\sqrt{3}\,t\right) + e^{-t}\left(-\sqrt{3}\,C\sin\sqrt{3}\,t + \sqrt{3}\,D\cos\sqrt{3}\,t\right)$	M1	Differentiate (product rule)	
	$0 = -C + \sqrt{3}\,D$	M1	Condition on $\dot{x}$	
	$D = \frac{0.1}{\sqrt{3}}$			
	$x = 0.1e^{-t}\left(\cos\sqrt{3}\,t + \frac{1}{\sqrt{3}}\sin\sqrt{3}\,t\right)$	A1	cao	
		B1 B1 B1	Curve through (0,0.1) with zero gradient Oscillating Asymptote x = 0	11
(iv)	$k^2 - 4 \cdot 1 \cdot 4 > 0$	M1	Use of discriminant	
		A1	Correct inequality	
	(As *k* is positive) $k > 4$	A1	Accept $k < -4$ in addition (but not *k* > –4)	
		B1 B1	Curve through (0,0.1) Decays without oscillating (at most one intercept with positive *t* axis)	5

2 (i)	$x = A\mathrm{e}^{-2t}$	M1	Any valid method	
	$t=0, x=8 \Rightarrow A=8$	M1	Condition on x	
	$x = 8\mathrm{e}^{-2t}$	A1		3
(ii)	$\dot{y} + y = 16\mathrm{e}^{-2t}$	M1	Substitute for x	
	$\alpha + 1 = 0 \Rightarrow \alpha = -1$	M1	Auxiliary equation	
	CF $y = B\mathrm{e}^{-t}$	A1		
	PI $y = a\mathrm{e}^{-2t}$	B1		
	$-2a\mathrm{e}^{-2t} + a\mathrm{e}^{-2t} = 16\mathrm{e}^{-2t}$	M1	Differentiate and substitute	
	$a = -16$	A1	cao	
	GS $y = -16\mathrm{e}^{-2t} + B\mathrm{e}^{-t}$	F1	Their PI + CF (with one arbitrary constant)	
	$t=0, y=0 \Rightarrow B=16$	M1	Condition on y	
	$y = 16\left(\mathrm{e}^{-t} - \mathrm{e}^{-2t}\right)$	F1	Follow a non-trivial GS	
	Alternative mark scheme for first 7 marks:			
		M1	*Substitute for x*	
	$I = e^{t}$	*M1*	*Attempt integrating factor*	
		A1	*IF correct*	
	$d(y e^{t})/dt = 16e^{-t}$	*B1*		
		M1	*Integrate*	
	$y e^{t} = -16e^{-t} + B$	*A1*	*cao*	
	$y = -16e^{-2t} + Be^{-t}$	*F1*	*Divide by their I (must divide constant)*	9
(iii)	$y = 16\mathrm{e}^{-t}\left(1 - \mathrm{e}^{-t}\right)$	M1	Or equivalent (NB $\mathrm{e}^{-t} > \mathrm{e}^{-2t}$ needs justifying)	
	$16\mathrm{e}^{-t} > 0$ and $t > 0 \Rightarrow \mathrm{e}^{-t} < 1$ hence $y > 0$	E1	Complete argument	
		B1	Starts at origin	
		B1	General shape consistent with their solution and $y > 0$	
		B1	Tends to zero	5
(iv)	$\frac{\mathrm{d}}{\mathrm{d}t}(x+y+z) = (-2x) + (2x-y) + (y) = 0$	M1	Consider sum of DE's	
	$\Rightarrow x+y+z = c$	E1		
	Hence initial conditions $\Rightarrow x+y+z=8$	E1		
	$z = 8 - x - y$	M1	Substitute for x and y and find z	
	$z = 8\left(1 - 2\mathrm{e}^{-t} + \mathrm{e}^{-2t}\right) = 8\left(1-\mathrm{e}^{-t}\right)^2$	E1	Convincingly shown (x, y must be correct)	5
(v)	$0.99 \times 8 = 8\left(1-\mathrm{e}^{-t}\right)^2$	B1	Correct equation (any form)	
	$t = -0.690638$ or 5.29581			
	99% is Z after 5.30 hours	B1	Accept value in [5.29, 5.3]	2

Question	Answer	Mark	Guidance	Total
3 (i)	$\dot{y}+\frac{k}{t}y=1$	M1	Divide by *t* (condone LHS only)	
	$I=\exp\left(\int \frac{k}{t}\,\mathrm{d}t\right)=\exp(k\ln t)=t^k$	M1	Attempt integrating factor	
		A1	Integrating factor	
	$t^k\dot{y}+kt^{k-1}y=t^k$	F1	Multiply DE by their *I*	
	$\frac{\mathrm{d}}{\mathrm{d}t}\left(yt^k\right)=t^k$	M1	LHS	
	$yt^k=\int t^k\,\mathrm{d}t$	M1	Integrate	
	$=\frac{1}{k+1}t^{k+1}+A$	A1	cao (including constant)	
	$y=\frac{1}{k+1}t+At^{-k}$	F1	Divide by their *I* (must divide constant)	
	$t=1, y=0 \Rightarrow 0=\frac{1}{k+1}+A \Rightarrow A=-\frac{1}{k+1}$	M1	Use condition	
	$y=\frac{1}{k+1}\left(t-t^{-k}\right)$	F1	Follow a non-trivial GS	10
(ii)	$y=\frac{1}{3}\left(t-t^{-2}\right)$ 	B1 B1 B1	Shape consistent with their solution for $t \geq 1$ Passes through (1, 0) Behaviour for large *t*	3
(iii)	$yt^{-1}=\int t^{-1}\,\mathrm{d}t$	M1	Follow their (i)	
	$=\ln t+B$	A1	cao	
	$y=t(\ln t+B)$	F1	Divide by their *I* (must divide constant)	
	$t=1, y=0 \Rightarrow B=0 \Rightarrow y=t\ln t$	A1	cao	4
(iv)	$\frac{\mathrm{d}y}{\mathrm{d}t}=1+t^{-1}\sin y$	M1	Rearrange DE (may be implied)	
	t *y* d*y*/d*t* 1 0 1 1.1 0.1 1.0908 1.2 0.2091	M1 A1 A1	Use algorithm *y*(1.1) *y*(1.2)	4
(v)	0.2138 as smaller step size	B1	Must give reason	
	Decreasing step length has increased estimate. Assuming this estimate is more accurate, decreasing step length further will increase estimate further, so true value likely to be greater.	M1	Identify effect of decreasing step length	
	Hence underestimates.	A1	Convincing argument	
	Alternative mark scheme for last 2 marks:			
	dy/dt seems to be increasing, hence Euler's method	*M1*	*Identify derivative increasing*	
	will underestimate true value + sketch (or explanation).	*A1*	*Convincing argument*	3

4 (i)	$\ddot{x} = 4\dot{x} - 6\dot{y} - 9\cos t$	M1	Differentiate first equation	
	$= 4\dot{x} - 6(3x - 5y - 7\sin t) - 9\cos t$	M1	Substitute for $\dot{y}$	
	$y = \frac{1}{6}(4x - \dot{x} - 9\sin t)$	M1	y in terms of $x, \dot{x}$	
	$\ddot{x} = 4\dot{x} - 18x + 5(4x - \dot{x} - 9\sin t) + 42\sin t - 9\cos t$	M1	Substitute for y	
	$\ddot{x} + \dot{x} - 2x = -3\sin t - 9\cos t$	E1	LHS	
		E1	RHS	6
(ii)	$\alpha^2 + \alpha - 2 = 0$	M1	Auxiliary equation	
	$\alpha = 1$ or -2	A1		
	CF $x = A\mathrm{e}^t + B\mathrm{e}^{-2t}$	F1	CF for their roots	
	PI $x = a\cos t + b\sin t$	B1	PI of this form	
	$(-ac - bs) + (-as + bc) - 2(ac + bs) = -3s - 9c$	M1	Differentiate twice and substitute	
	$-a + b - 2a = -9$	M1	Compare coefficients (2 equations)	
	$-b - a - 2b = -3$	M1	Solve (2 equations)	
	$\Rightarrow a = 3, b = 0$	A1		
	$x = 3\cos t + A\mathrm{e}^t + B\mathrm{e}^{-2t}$	F1	Their PI + CF (with two arbitrary constants)	9
(iii)	$y = \frac{1}{6}(4x - \dot{x} - 9\sin t)$	M1	y in terms of $x, \dot{x}$	
	$= \frac{1}{6}\left(12\cos t + 4A\mathrm{e}^t + 4B\mathrm{e}^{-2t} + 3\sin t - A\mathrm{e}^t + 2B\mathrm{e}^{-2t} - 9\sin t\right)$	M1	Differentiate x and substitute	
	$y = 2\cos t - \sin t + \frac{1}{2}A\mathrm{e}^t + B\mathrm{e}^{-2t}$	A1	Constants must correspond with those in x	3
(iv)	x bounded $\Rightarrow A = 0$	M1	Identify coefficient of exponentially growing term must be zero	
	$\Rightarrow y$ bounded	E1	Complete argument	2
(v)	$t = 0, y = 0 \Rightarrow 0 = B + 2 \Rightarrow B = -2$	M1	Condition on y	
	$x = 3\cos t - 2\mathrm{e}^{-2t}$, $y = 2\cos t - \sin t - 2\mathrm{e}^{-2t}$	F1	Follow their (non-trivial) general solutions	
	$x = 3\cos t$	A1	cao	
	$y = 2\cos t - \sin t$	A1	cao	4