

**ADVANCED SUBSIDIARY GCE UNIT
MATHEMATICS (MEI)**

Numerical Methods

WEDNESDAY 20 JUNE 2007

4776/01

Afternoon
Time: 1 hour 30 minutes

Additional materials:

Answer booklet (8 pages)

Graph paper

MEI Examination Formulae and Tables (MF2)

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet.
- Answer **all** the questions.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [] at the end of each question or part question.
- The total number of marks for this paper is 72.

ADVICE TO CANDIDATES

- Read each question carefully and make sure you know what you have to do before starting your answer.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.

Section A (36 marks)

- 1 Show that the equation $x^2 + \sqrt{1+x} = 3$ has a root in the interval $(1, 1.4)$.

Use the bisection method to obtain an estimate of the root with maximum possible error 0.025.

Determine how many additional iterations of the bisection process would be required to reduce the maximum possible error to less than 0.005. [8]

- 2 For the integral $\int_0^{0.5} \frac{1}{1+x^4} dx$, find the values given by the trapezium rule and the mid-point rule, taking $h = 0.5$ in each case.

Hence show that the Simpson's rule estimate with $h = 0.25$ is 0.493 801.

You are now given that the Simpson's rule estimate with $h = 0.125$ is 0.493 952. Use extrapolation to determine the value of the integral as accurately as you can. [8]

- 3 A triangle has sides a , 3 and 4. The angle opposite side a is $(90 + \varepsilon)^\circ$, where ε is small. See Fig. 3.

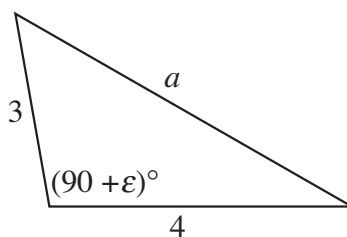


Fig. 3

Use the cosine rule to calculate a when $\varepsilon = 5$.

The approximation

$$\cos (90 + \varepsilon)^\circ \approx -\frac{\pi\varepsilon}{180}$$

with $\varepsilon = 5$ is now used in the cosine rule to find an approximate value for a .

Find the absolute and relative errors in this approximate value of a . [5]

- 4 The number x is represented in a computer program by the approximation X . You are given that $X = x(1 + r)$ where r is small.

(i) State what r represents. [1]

(ii) Use the first two terms in a binomial expansion to show that the relative error in X^n as an approximation to x^n is approximately nr . [2]

(iii) A lazy programmer has approximated π by $\frac{22}{7}$. Find the relative error in this approximation.

Use the result in part (ii) to write down the approximate relative errors in the values of π^2 and $\sqrt{\pi}$ when π is taken as $\frac{22}{7}$. [5]

- 5 The function $f(x)$ has the values shown in the table.

x	-1	0	4
$f(x)$	3	2	9

Use Lagrange's interpolation method to obtain the quadratic function that fits the three data points.

Hence estimate the value of x for which $f(x)$ takes its minimum value. [7]

Section B (36 marks)

- 6 (i) Explain, with the aid of a sketch, the principle underlying the Newton-Raphson method for the solution of the equation $f(x) = 0$. [3]

(ii) Draw a sketch of the function $f(x) = \tan x - 2x$ for $0 \leq x < \frac{1}{2}\pi$ (x in radians). Mark on your sketch the non-zero root, α , of the equation $\tan x - 2x = 0$. Show by means of your sketch that, for some starting values, the Newton-Raphson method will fail to converge to α . Identify two distinct cases that can arise. [6]

(iii) Given that the derivative of $\tan x$ is $1 + \tan^2 x$, show that the Newton-Raphson iteration for the solution of the equation $\tan x - 2x = 0$ is

$$x_{r+1} = x_r - \frac{(\tan x_r - 2x_r)}{(\tan^2 x_r + 1)}$$

Use this iteration with $x_0 = 1.2$ to determine α correct to 4 decimal places.

Show carefully that this iteration is faster than first order. [9]

[Question 7 is printed overleaf.]

7 The function $g(x)$ has the values shown in the table.

x	$g(x)$
1	2.87
2	4.73
3	6.23
4	7.36
5	8.05

- (i) Draw up a difference table for $g(x)$ as far as second differences. State with a reason whether or not $g(x)$ is quadratic. [5]
- (ii) Draw up another difference table, based this time on $x = 1, 3, 5$. Use Newton's forward difference formula to find the quadratic approximation to $g(x)$ based on these three points. Simplify the coefficients of this quadratic. [8]
- (iii) Find the absolute and relative errors when this quadratic is used to estimate $g(2)$ and $g(4)$. [5]

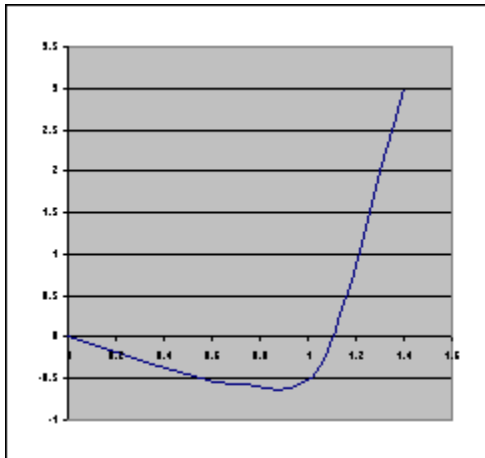
Mark Scheme 4776
June 2007

6(i) Sketch showing curve, root, initial estimate, tangent, intersection of tangent with x-axis as improved estimate

[E1E1E1]

[subtotal 3]

(ii)



Sketch showing root, α

[G2]

E.g. starting values just to the left of the root can produce an x_1 that is the wrong side of the asymptote

[M1]

[E1]

E.g. starting values further left can converge to zero.

[M1]

[E1]

[subtotal 6]

(iii) Convincing algebra to obtain the N-R formula

[M1A1]

r	0	1	2	3	4
xr	1.2	1.169346	1.165609	1.165561	1.165561
root is	1.1656 to 4 dp				

[M1A1A1]

[A1]

differences from root -0.03065 -0.00374 -4.8E-05 Accept diffs of successive terms

ratio of differences 0.1219 0.012877

[M1A1]

ratio of differences is decreasing (by a large factor), so faster than first order

[E1]

[subtotal 9]

[TOTAL 18]

7

(i)

x	g(x)	Δg	$\Delta^2 g$
1	2.87		
2	4.73	1.86	
3	6.23	1.50	-0.36
4	7.36	1.13	-0.37
5	8.05	0.69	-0.44

[M1A1A1]

Not quadratic

[E1]

Because second differences not constant

[E1]

[subtotal 5]

(ii)

x	g(x)	Δg	$\Delta^2 g$
1	2.87		
3	6.23	3.36	
5	8.05	1.82	-1.54

[B1]

$$Q(x) = 2.87 + 3.36(x - 1)/2 - 1.54(x - 1)(x - 3)/8$$

[M1A1A1A1]

$$= 0.6125 + 2.45x - 0.1925x^2$$

[A1A1A1]

[subtotal 8]

(iii)

x	Q(x)	g(x)	error	rel error
2	4.7425	4.73	0.0125	0.002643
4	7.3325	7.36	-0.0275	-0.00374

Q: [A1A1]

errors: [A1]

rel errors: [M1A1]

[subtotal 5]

[TOTAL 18]

4776: Numerical Methods

General Comments

Though there were some very good scripts, a substantial number of candidates seemed ill prepared for this exam. Some very fundamental ideas – maximum possible error, relative error – seemed unfamiliar; some standard techniques – Newton-Raphson, Lagrange interpolation – were not accurately understood. There was a lot of poor algebra, and a significant minority of candidates worked their solutions with too small a number of significant figures, losing accuracy and losing marks. Over all, there was a sense that some candidates were insufficiently experienced on Numerical Methods papers.

Comments on Individual Questions

1 Bisection

The bisection method was well understood, but the layout of solutions was often poor. Here, as elsewhere in numerical work, a tabular layout is best. Many candidates did not understand maximum possible error. Two inappropriate approaches were seen: some took the value of the function as the maximum possible error; others simply iterated many times more than required in the hope that they would have gone far enough.

2 Numerical integration

The first half of the question was frequently done well, though some candidates still do not appreciate the relationship between the mid-point rule, the trapezium rule and Simpson's rule. The point here is that the weighted average of M and T with $h = 0.5$ gives S with $h = 0.25$. Extrapolation was often not done well, though some candidates showed confidence with one of the several approaches possible. Full marks were available for full extrapolation or for just finding the next term in the sequence of Simpson's rule estimates as the convergence is so rapid.

3 Cosine rule, errors

The basic elements of this question often let candidates down. Some got the cosine rule wrong even though it is GCSE work and in the formula book. Some had their calculators in radian mode even though the question is very clearly set in degrees. Some forgot to take the square root to find a . Some, not reading the question carefully, found the errors in the approximation for the cosine rather than the approximation for a .

4 Relative errors, π

Most candidates were unable to identify r in the formula $X = x(1 + r)$ as the relative error. The binomial expansion was beyond many; though with the result given most could then go on to gain the marks for part (iii). There were no marks for calculating the relative errors from first principles in this part.

5 **Lagrange interpolation**

As usual, there were candidates who confused the x and $f(x)$ values in Lagrange's formula, and those who could not handle the algebra. Some whose working was otherwise correct multiplied the expression for $f(x)$ by 20 in order to eliminate decimal fractions. This question was, nevertheless, often done well.

6 **Newton-Raphson method**

- (i) In part (i), a good, detailed, clearly labelled sketch of the Newton-Raphson method was sufficient to get full marks, though candidates often helped their case if they offered a short written explanation. The major error here was to give another method altogether, such as fixed-point iteration.
- (ii) In part (ii), some candidates sketched $\tan x$ and $2x$ as separate graphs, making it impossible to gain any of the marks. Good solutions identified starting points that would give convergence to the root at zero, or difficulties starting at the turning point or beyond the asymptote.
- (iii) In part (iii), there was some fudging of the derivative of $f(x)$, some work in degrees, and some inaccurate use of calculators. However, there were also many correct solutions. Showing that the iteration is faster than first order defeated quite a few. Many showed that the ratios of differences are not constant. This demonstrates that the process is not first order. Observing that the ratios decrease shows that the process is *faster* than first order.

7 **Difference table, Newton's method**

- (i) The difference table in part (i) was generally done well, though there were some sign errors. The function is not quadratic because the second differences are not equal. Candidates who answered that the function was almost quadratic because the second differences were almost equal were given the credit as they were judged to have understood the point. Those who said that the function was quadratic because the differences were approximately equal did not receive full credit. (Some candidates seemed to believe that the function would be quadratic if the second differences were within 10% of one another.)
- (ii) In part (ii), the method was generally well understood but the algebra defeated some.
- (iii) The final part was, for most, a routine application of the approximating quadratic just found. Those who had made a substantial error in part (ii) frequently had quadratics that gave absurdly large errors. This should have been a clear sign of an earlier mistake.