RECOGNISING ACHIEVEMENT

ADVANCED GCE UNIT **4757/01**

MATHEMATICS (MEI)

Further Applications of Advanced Mathematics (FP3)

THURSDAY 14 JUNE 2007

Afternoon
Time: 1 hour 30 minutes

Additional materials:
Answer booklet (8 pages)
Graph paper
MEI Examination Formulae and Tables (MF2)

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet.
- Answer any **three** questions.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- The total number of marks for this paper is 72.

ADVICE TO CANDIDATES

- Read each question carefully and make sure you know what you have to do before starting your answer.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.

This document consists of **6** printed pages and **2** blank pages.

Option 1: Vectors

1 Three planes P, Q and R have the following equations.

$$\begin{aligned} \text{Plane } P\text{:} \quad & 8x - \ \ y - 14z = 20 \\ \text{Plane } Q\text{:} \quad & 6x + 2y - \ \ 5z = 26 \\ \text{Plane } R\text{:} \quad & 2x + \ \ y - \ \ \ \ z = 40 \end{aligned}$$

The line of intersection of the planes P and Q is K.

The line of intersection of the planes P and R is L.

(i) Show that K and L are parallel lines, and find the shortest distance between them. [9]

(ii) Show that the shortest distance between the line K and the plane R is $5\sqrt{6}$. [3]

The line M has equation $\mathbf{r} = (\mathbf{i} - 4\mathbf{j}) + \lambda(5\mathbf{i} - 4\mathbf{j} + 3\mathbf{k})$.

(iii) Show that the lines K and M intersect, and find the coordinates of the point of intersection. [7]

(iv) Find the shortest distance between the lines L and M. [5]

Option 2: Multi-variable calculus

2 A surface has equation $z = xy^2 - 4x^2y - 2x^3 + 27x^2 - 36x + 20$.

(i) Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$. [3]

(ii) Find the coordinates of the four stationary points on the surface, showing that one of them is $(2, 4, 8)$. [8]

(iii) Sketch, on separate diagrams, the sections of the surface defined by $x = 2$ and by $y = 4$. Indicate the point $(2, 4, 8)$ on these sections, and deduce that it is neither a maximum nor a minimum. [6]

(iv) Show that there are just two points on the surface where the normal line is parallel to the vector $36\mathbf{i} + \mathbf{k}$, and find the coordinates of these points. [7]

Option 3: Differential geometry

3 The curve C has equation $y = \frac{1}{2}x^2 - \frac{1}{4}\ln x$, and a is a constant with $a \geqslant 1$.

(i) Show that the length of the arc of C for which $1 \leqslant x \leqslant a$ is $\frac{1}{2}a^2 + \frac{1}{4}\ln a - \frac{1}{2}$. [5]

(ii) Find the area of the surface generated when the arc of C for which $1 \leqslant x \leqslant 4$ is rotated through 2π radians **about the *y*-axis**. [5]

(iii) Show that the radius of curvature of C at the point where $x = a$ is $a\left(a + \dfrac{1}{4a}\right)^2$. [5]

(iv) Find the centre of curvature corresponding to the point $(1, \frac{1}{2})$ on C. [5]

C is one member of the family of curves defined by $y = px^2 - p^2 \ln x$, where p is a parameter.

(v) Find the envelope of this family of curves. [4]

[Questions 4 and 5 are printed overleaf.]

Option 4: Groups

4 **(i)** Prove that, for a group of order 10, every proper subgroup must be cyclic. [4]

The set $M = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ is a group under the binary operation of multiplication modulo 11.

(ii) Show that M is cyclic. [4]

(iii) List all the proper subgroups of M. [3]

The group P of symmetries of a regular pentagon consists of 10 transformations

$$\{\mathrm{A, B, C, D, E, F, G, H, I, J}\}$$

and the binary operation is composition of transformations. The composition table for P is given below.

	A	B	C	D	E	F	G	H	I	J
A	C	J	G	H	A	B	I	F	E	D
B	F	E	H	G	B	A	D	C	J	I
C	G	D	I	F	C	J	E	B	A	H
D	J	C	B	E	D	G	F	I	H	A
E	A	B	C	D	E	F	G	H	I	J
F	H	I	D	C	F	E	J	A	B	G
G	I	H	E	B	G	D	A	J	C	F
H	D	G	J	A	H	I	B	E	F	C
I	E	F	A	J	I	H	C	D	G	B
J	B	A	F	I	J	C	H	G	D	E

One of these transformations is the identity transformation, some are rotations and the rest are reflections.

(iv) Identify which transformation is the identity, which are rotations and which are reflections. [4]

(v) State, giving a reason, whether P is isomorphic to M. [2]

(vi) Find the order of each element of P. [3]

(vii) List all the proper subgroups of P. [4]

Option 5: Markov chains

5 A computer is programmed to generate a sequence of letters. The process is represented by a Markov chain with four states, as follows.

The first letter is A, B, C or D, with probabilities 0.4, 0.3, 0.2 and 0.1 respectively.

After A, the next letter is either C or D, with probabilities 0.8 and 0.2 respectively.

After B, the next letter is either C or D, with probabilities 0.1 and 0.9 respectively.

After C, the next letter is either A or B, with probabilities 0.4 and 0.6 respectively.

After D, the next letter is either A or B, with probabilities 0.3 and 0.7 respectively.

(i) Write down the transition matrix $\mathbf{P}$. [2]

(ii) Use your calculator to find $\mathbf{P}^4$ and $\mathbf{P}^7$. (Give elements correct to 4 decimal places.) [4]

(iii) Find the probability that the 8th letter is C. [2]

(iv) Find the probability that the 12th letter is the same as the 8th letter. [4]

(v) By investigating the behaviour of $\mathbf{P}^n$ when n is large, find the probability that the $(n+1)$th letter is A when

(A) n is a large even number,

(B) n is a large odd number. [4]

The program is now changed. The initial probabilities and the transition probabilities are the same as before, except for the following.

After D, the next letter is A, B or D, with probabilities 0.3, 0.6 and 0.1 respectively.

(vi) Write down the new transition matrix $\mathbf{Q}$. [1]

(vii) Verify that $\mathbf{Q}^n$ approaches a limit as n becomes large, and hence write down the equilibrium probabilities for A, B, C and D. [4]

(viii) When n is large, find the probability that the $(n+1)$th, $(n+2)$th and $(n+3)$th letters are DDD. [3]

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Mark Scheme 4757
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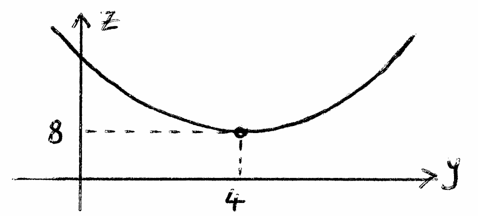
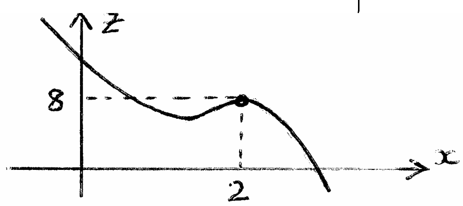
1 (i)	$\mathbf{d}_K = \begin{pmatrix} 8 \\ -1 \\ -14 \end{pmatrix} \times \begin{pmatrix} 6 \\ 2 \\ -5 \end{pmatrix} = \begin{pmatrix} 33 \\ -44 \\ 22 \end{pmatrix} \quad [\ = 11\begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix}\ ]$ $\mathbf{d}_L = \begin{pmatrix} 8 \\ -1 \\ -14 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 15 \\ -20 \\ 10 \end{pmatrix} \quad [\ = 5\begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix}\ ]$ Hence K and L are parallel For a point on K, $z = 0,\ x = 3,\ y = 4$ i.e. $(3,\ 4,\ 0)$ For a point on L, $z = 0,\ x = 6,\ y = 28$ i.e. $(6,\ 28,\ 0)$	M1* A1* A1 M1*A1* A1*	Finding direction of K or L One direction correct ** These marks can be earned anywhere in the question* Correctly shown Finding one point on K or L *or* $(6,\ 0,\ 2)$ *or* $(0,\ 8,\ -2)$ *etc* *Or* $(27,\ 0,\ 14)$ *or* $(0,\ 36,\ -4)$ *etc*
	$\left[\begin{pmatrix} 6 \\ 28 \\ 0 \end{pmatrix} - \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}\right] \times \begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 24 \\ 0 \end{pmatrix} \times \begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix} = \begin{pmatrix} 48 \\ -6 \\ -84 \end{pmatrix}$ Distance is $\dfrac{\sqrt{48^2 + 6^2 + 84^2}}{\sqrt{3^2 + 4^2 + 2^2}} = \dfrac{\sqrt{9396}}{\sqrt{29}} = 18$	M1 M1 A1 **9**	For $(\mathbf{b} - \mathbf{a}) \times \mathbf{d}$ Correct method for finding distance
	OR $\begin{pmatrix} 6 + 3\lambda - 3 \\ 28 - 4\lambda - 4 \\ 2\lambda \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix} = 0$ M1 $-87 + 29\lambda = 0,\quad \lambda = 3$ M1 Distance is $\sqrt{12^2 + 12^2 + 6^2} = 18$ A1		For $(\mathbf{b} + \lambda\mathbf{d} - \mathbf{a}).\mathbf{d} = 0$ Finding λ, and the magnitude
(ii)	Distance from $(3, 4, 0)$ to R is $\left\lvert \dfrac{2 \times 3 + 4 - 0 - 40}{\sqrt{2^2 + 1^2 + 1^2}} \right\rvert$ $= \dfrac{30}{\sqrt{6}} = \dfrac{30\sqrt{6}}{6} = 5\sqrt{6}$	M1A1 ft A1 ag **3**	
(iii)	K, M intersect if $1 + 5\lambda = 3 + 3\mu$ (1) $-4 - 4\lambda = 4 - 4\mu$ (2) $3\lambda = 2\mu$ (3) Solving (2) and (3): $\lambda = 4,\ \mu = 6$ Check in (1): LHS $= 1 + 20 = 21$, RHS $= 3 + 18 = 21$ Hence K, M intersect, at $(21,\ -20,\ 12)$	M1 A1 ft M1M1 M1A1 A1 **7**	At least 2 eqns, different parameters Two equations correct Intersection correctly shown *Can be awarded after M1A1M1M0M0*
	OR M meets P when M1 $8(1 + 5\lambda) - (-4 - 4\lambda) - 14(3\lambda) = 20$ A1 M meets Q when $6(1 + 5\lambda) + 2(-4 - 4\lambda) - 5(3\lambda) = 26$ A1 Both equations have solution $\lambda = 4$ A1 Point is on P, Q and M; hence on K and M M2 Point of intersection is $(21,\ -20,\ 12)$ A1		Intersection of M with **both** P and Q

(iv)	$\left[\begin{pmatrix}6\\28\\0\end{pmatrix}-\begin{pmatrix}1\\-4\\0\end{pmatrix}\right]\cdot\left[\begin{pmatrix}3\\-4\\2\end{pmatrix}\times\begin{pmatrix}5\\-4\\3\end{pmatrix}\right]=\begin{pmatrix}5\\32\\0\end{pmatrix}\cdot\begin{pmatrix}-4\\1\\8\end{pmatrix}=12$ Distance is $\dfrac{12}{\sqrt{4^2+1^2+8^2}}=\dfrac{12}{9}=\dfrac{4}{3}$	M1A1 ft M1 A1 ft A1 **5**	For evaluating $\mathbf{d}_L \times \mathbf{d}_M$ For $(\mathbf{b}-\mathbf{c}).(\mathbf{d}_L \times \mathbf{d}_M)$ Numerical expression for distance

2 (i)	$\frac{\partial z}{\partial x} = y^2 - 8xy - 6x^2 + 54x - 36$ $\frac{\partial z}{\partial y} = 2xy - 4x^2$	B2 B1 **3**	Give B1 for 3 terms correct
(ii)	At stationary points, $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$ When $x = 0$, $y^2 - 36 = 0$ $y = \pm 6$; points $(0,\ 6,\ 20)$ and $(0,\ -6,\ 20)$ When $y = 2x$, $4x^2 - 16x^2 - 6x^2 + 54x - 36 = 0$ $-18x^2 + 54x - 36 = 0$ $x = 1,\ 2$ Points $(1,\ 2,\ 5)$ and $(2,\ 4,\ 8)$	M1 M1 A1A1 M1 M1A1 A1 **8**	If A0, give A1 for $y = \pm 6$ or $y = 2,\ 4$ *A0 if any extra points given*
(iii)	When $x = 2$, $z = 2y^2 - 16y + 40$  When $y = 4$, $z = -2x^3 + 11x^2 - 20x + 20$ $\left(\frac{d^2 z}{dx^2} = -12x + 22 = -2 \text{ when } x = 2 \right)$  The point is a minimum on one section and a maximum on the other; so it is neither a maximum nor a minimum	B1 B1 B1 B1 B1 B1 **6**	'Upright' parabola $(2,\ 4,\ 8)$ identified as a minimum (in the first quadrant) 'Negative cubic' curve $(2,\ 4,\ 8)$ identified as a stationary point Fully correct (unambiguous minimum and maximum)
(iv)	Require $\frac{\partial z}{\partial x} = -36$ and $\frac{\partial z}{\partial y} = 0$ When $x = 0$, $y^2 - 36 = -36$ $y = 0$; point $(0,\ 0,\ 20)$ When $y = 2x$, $4x^2 - 16x^2 - 6x^2 + 54x - 36 = -36$ $-18x^2 + 54x = 0$ $x = 0,\ 3$ $x = 0$ gives $(0,\ 0,\ 20)$ same as above $x = 3$ gives $(3,\ 6,\ -7)$	M1 M1 A1 M1 M1 A1 A1 **7**	$\frac{\partial z}{\partial x} = 36$ can earn all M marks Solving to obtain x (or y) *or stating 'no roots' if appropriate (e.g. when +36 has been used)*

3 (i)	$1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=1+\left(x-\frac{1}{4x}\right)^2$ $=1+x^2-\frac{1}{2}+\frac{1}{16x^2}=x^2+\frac{1}{2}+\frac{1}{16x^2}$ $=\left(x+\frac{1}{4x}\right)^2$ Arc length is $\int_1^a\left(x+\frac{1}{4x}\right)\mathrm{d}x$ $=\left[\frac{1}{2}x^2+\frac{1}{4}\ln x\right]_1^a$ $=\frac{1}{2}a^2+\frac{1}{4}\ln a-\frac{1}{2}$	M1 A1 M1 M1 A1 ag **5**	 For $\int\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$
(ii)	Curved surface area is $\int 2\pi x\,\mathrm{d}s$ $=\int_1^4 2\pi x\left(x+\frac{1}{4x}\right)\mathrm{d}x$ $=2\pi\left[\frac{1}{3}x^3+\frac{1}{4}x\right]_1^4$ $=\frac{87\pi}{2}\quad(\approx 137)$	M1 A1 ft M1 A1 A1 **5**	 Any correct integral form (including limits) for $\frac{1}{3}x^3+\frac{1}{4}x$
(iii)	$\rho=\dfrac{\left(1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)^{\frac{3}{2}}}{\frac{\mathrm{d}^2y}{\mathrm{d}x^2}}=\dfrac{\left(a+\frac{1}{4a}\right)^3}{1+\frac{1}{4a^2}}$ $=\dfrac{a\left(a+\frac{1}{4a}\right)^3}{a+\frac{1}{4a}}=a\left(a+\frac{1}{4a}\right)^2$	B1 B1 M1 A1 A1 ag **5**	any form, in terms of x or a any form, in terms of x or a Formula for ρ or κ ρ or κ correct, in any form, in terms of x or a
(iv)	At $(1,\ \frac{1}{2})$, $\rho=(\frac{5}{4})^2=\frac{25}{16}$ $\frac{\mathrm{d}y}{\mathrm{d}x}=1-\frac{1}{4}=\frac{3}{4}$, so $\hat{\mathbf{n}}=\begin{pmatrix}-\frac{3}{5}\\ \frac{4}{5}\end{pmatrix}$ $\mathbf{c}=\begin{pmatrix}1\\ \frac{1}{2}\end{pmatrix}+\frac{25}{16}\begin{pmatrix}-\frac{3}{5}\\ \frac{4}{5}\end{pmatrix}$ Centre of curvature is $\left(\frac{1}{16},\ \frac{7}{4}\right)$	M1 A1 M1 A1A1 **5**	Finding gradient Correct normal vector (not necessarily unit vector); may be in terms of x OR *M2A1 for obtaining equation of normal line at a general point and differentiating partially*

(v)	Differentiating partially w.r.t. p $0 = x^2 - 2p\ln x$ $p = \dfrac{x^2}{2\ln x}$ and $y = \dfrac{x^4}{2\ln x} - \dfrac{x^4}{4\ln x}$ $y = \dfrac{x^4}{4\ln x}$	M1 A1 M1 A1 **4**	

4 (i)	By Lagrange's theorem, a proper subgroup has order 2 or 5 A group of prime order is cyclic Hence every proper subgroup is cyclic	M1 A1 M1 A1 **4**	Using Lagrange *(need not be mentioned explicitly)* or equivalent For completion
(ii)	e.g. $2^2=4,\ 2^3=8,\ 2^4=5,\ 2^5=10,$ $2^6=9,\ 2^7=7,\ 2^8=3,\ 2^9=6,\ 2^{10}=1$ 2 has order 10, hence M is cyclic	M1 A1 A1 A1 **4**	Considering order of an element Identifying an element of order 10 (2, 6, 7 or 8) Fully justified For conclusion (can be awarded after M1A1A0)
(iii)	$\{1,\ 10\}$ $\{1,\ 3,\ 4,\ 5,\ 9\}$	B1 B2 **3**	Ignore $\{1\}$ and M Deduct 1 mark (from B1B2) for each (proper) subgroup given in excess of 2
(iv)	E is the identity A, C, G, I are rotations B, D, F, H, J are reflections	B1 M1 A1 A1 **4**	Considering elements of order 2 *(or equivalent) Implied by four of B, D, F, H, J in the same set* Give A1 if one element is in the wrong set; or if two elements are interchanged
(v)	P and M are not isomorphic M is abelian, P is non-abelian	B1 B1 **2**	Valid reason e.g. M has one element of order 2 P has more than one
(vi)	$\begin{array}{c\|c\|c\|c\|c\|c\|c\|c\|c\|c\|c} & \text{A} & \text{B} & \text{C} & \text{D} & \text{E} & \text{F} & \text{G} & \text{H} & \text{I} & \text{J} \\ \hline \text{Order} & 5 & 2 & 5 & 2 & 1 & 2 & 5 & 2 & 5 & 2 \end{array}$	B3 **3**	Give B2 for 7 correct B1 for 4 correct
(vii)	$\{\text{E},\text{B}\},\{\text{E},\text{D}\},\{\text{E},\text{F}\},\{\text{E},\text{H}\},\{\text{E},\text{J}\}$ $\{\text{E},\ \text{A},\ \text{C},\ \text{G},\ \text{I}\}$	M1 A1 ft B2 cao **4**	Ignore $\{\text{E}\}$ and P *Subgroups of order 2* Using elements of order 2 (allow two errors/omissions) Correct or ft. A0 if any others given *Subgroups of order greater than 2* Deduct 1 mark (from B2) for each extra subgroup given

Pre-multiplication by transition matrix

5 (i)	$\mathbf{P} = \begin{pmatrix} 0 & 0 & 0.4 & 0.3 \\ 0 & 0 & 0.6 & 0.7 \\ 0.8 & 0.1 & 0 & 0 \\ 0.2 & 0.9 & 0 & 0 \end{pmatrix}$	B2 **2**	Give B1 for two columns correct
(ii)	$\mathbf{P}^4 = \begin{pmatrix} 0.3366 & 0.3317 & 0 & 0 \\ 0.6634 & 0.6683 & 0 & 0 \\ 0 & 0 & 0.3366 & 0.3317 \\ 0 & 0 & 0.6634 & 0.6683 \end{pmatrix}$ $\mathbf{P}^7 = \begin{pmatrix} 0 & 0 & 0.3334 & 0.3333 \\ 0 & 0 & 0.6666 & 0.6667 \\ 0.3335 & 0.3333 & 0 & 0 \\ 0.6665 & 0.6667 & 0 & 0 \end{pmatrix}$	B2 B2 **4**	Give B1 for two non-zero elements correct to at least 2dp Give B1 for two non-zero elements correct to at least 2dp
(iii)	$\mathbf{P}^7 \begin{pmatrix} 0.4 \\ 0.3 \\ 0.2 \\ 0.1 \end{pmatrix} = \begin{pmatrix} 0.1000 \\ 0.2000 \\ 0.2334 \\ 0.4666 \end{pmatrix}$ P(8th letter is C) $= 0.233$	M1 A1 **2**	Using $\mathbf{P}^7$ (or $\mathbf{P}^8$) and initial probs
(iv)	$0.1000 \times 0.3366 + 0.2000 \times 0.6683$ $+ 0.2334 \times 0.3366 + 0.4666 \times 0.6683$ $= 0.558$	M1 M1 A1 ft A1 **4**	Using probabilities for 8th letter Using diagonal elements from $\mathbf{P}^4$
(v)(A)	$\mathbf{P}^n \begin{pmatrix} 0.4 \\ 0.3 \\ 0.2 \\ 0.1 \end{pmatrix} \approx \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ \frac{2}{3} & \frac{2}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{2}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 0.4 \\ 0.3 \\ 0.2 \\ 0.1 \end{pmatrix} = \begin{pmatrix} 0.2333 \\ 0.4667 \\ 0.1 \\ 0.2 \end{pmatrix}$ P($(n+1)$ th letter is A) $= 0.233$	M1 A1	Approximating $\mathbf{P}^n$ when n is large and even
(B)	$\mathbf{P}^n \begin{pmatrix} 0.4 \\ 0.3 \\ 0.2 \\ 0.1 \end{pmatrix} \approx \begin{pmatrix} 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{2}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ \frac{2}{3} & \frac{2}{3} & 0 & 0 \end{pmatrix} \begin{pmatrix} 0.4 \\ 0.3 \\ 0.2 \\ 0.1 \end{pmatrix} = \begin{pmatrix} 0.1 \\ 0.2 \\ 0.2333 \\ 0.4667 \end{pmatrix}$ P($(n+1)$ th letter is A) $= 0.1$	M1 A1 **4**	Approximating $\mathbf{P}^n$ when n is large and odd
(vi)	$\mathbf{Q} = \begin{pmatrix} 0 & 0 & 0.4 & 0.3 \\ 0 & 0 & 0.6 & 0.6 \\ 0.8 & 0.1 & 0 & 0 \\ 0.2 & 0.9 & 0 & 0.1 \end{pmatrix}$	B1 **1**	

(vii)	$\mathbf{Q}^n \to \begin{pmatrix} 0.1721 & 0.1721 & 0.1721 & 0.1721 \\ 0.3105 & 0.3105 & 0.3105 & 0.3105 \\ 0.1687 & 0.1687 & 0.1687 & 0.1687 \\ 0.3487 & 0.3487 & 0.3487 & 0.3487 \end{pmatrix}$ Probabilities are 0.172, 0.310, 0.169, 0.349	M1 M1 A2 **4**	Considering $\mathbf{Q}^n$ for large n OR at least two eqns for equilib probs Probabilities from equal columns OR solving to obtain equilib probs Give A1 for two correct
(viii)	$0.3487 \times 0.1 \times 0.1$ $= 0.0035$	M1M1 A1 **3**	Using 0.3487 and 0.1

Post-multiplication by transition matrix

5 (i)	$\mathbf{P}=\begin{pmatrix}0&0&0.8&0.2\\0&0&0.1&0.9\\0.4&0.6&0&0\\0.3&0.7&0&0\end{pmatrix}$	B2 **2**	Give B1 for two rows correct
(ii)	$\mathbf{P}^4=\begin{pmatrix}0.3366&0.6634&0&0\\0.3317&0.6683&0&0\\0&0&0.3366&0.6634\\0&0&0.3317&0.6683\end{pmatrix}$ $\mathbf{P}^7=\begin{pmatrix}0&0&0.3335&0.6665\\0&0&0.3333&0.6667\\0.3334&0.6666&0&0\\0.3333&0.6667&0&0\end{pmatrix}$	B2 B2 **4**	Give B1 for two non-zero elements correct to at least 2dp Give B1 for two non-zero elements correct to at least 2dp
(iii)	$(0.4\quad 0.3\quad 0.2\quad 0.1)\mathbf{P}^7$ $=(0.1000\quad 0.2000\quad 0.2334\quad 0.4666)$ P(8th letter is C) $=0.233$	M1 A1 **2**	Using $\mathbf{P}^7$ (or $\mathbf{P}^8$) and initial probs
(iv)	$0.1000\times0.3366\ +\ 0.2000\times0.6683$ $+\ 0.2334\times0.3366\ +\ 0.4666\times0.6683$ $=0.558$	M1 M1A1 ft A1 **4**	Using probabilities for 8th letter Using diagonal elements from $\mathbf{P}^4$
(v)(A) (B)	$\mathbf{u}\,\mathbf{P}^n\approx(0.4\quad 0.3\quad 0.2\quad 0.1)\begin{pmatrix}⅓&⅔&0&0\\⅓&⅔&0&0\\0&0&⅓&⅔\\0&0&⅓&⅔\end{pmatrix}$ $=(0.2333\quad 0.4667\quad 0.1\quad 0.2)$ P($(n+1)$ th letter is A) $=0.233$ $\mathbf{u}\,\mathbf{P}^n\approx(0.4\quad 0.3\quad 0.2\quad 0.1)\begin{pmatrix}0&0&⅓&⅔\\0&0&⅓&⅔\\⅓&⅔&0&0\\⅓&⅔&0&0\end{pmatrix}$ $=(0.1\quad 0.2\quad 0.2333\quad 0.4667)$ P($(n+1)$ th letter is A) $=0.1$	M1 A1 M1 A1 **4**	Approximating $\mathbf{P}^n$ when n is large and even Approximating $\mathbf{P}^n$ when n is large and odd
(vi)	$\mathbf{Q}=\begin{pmatrix}0&0&0.8&0.2\\0&0&0.1&0.9\\0.4&0.6&0&0\\0.3&0.6&0&0.1\end{pmatrix}$	B1 **1**	

(vii)	$\mathbf{Q}^n \to \begin{pmatrix} 0.1721 & 0.3105 & 0.1687 & 0.3487 \\ 0.1721 & 0.3105 & 0.1687 & 0.3487 \\ 0.1721 & 0.3105 & 0.1687 & 0.3487 \\ 0.1721 & 0.3105 & 0.1687 & 0.3487 \end{pmatrix}$ Probabilities are 0.172, 0.310, 0.169, 0.349	M1 M1 A2 **4**	Considering $\mathbf{Q}^n$ for large n OR at least two eqns for equilib probs Probabilities from equal rows OR solving to obtain equilib probs Give A1 for two correct
(viii)	$0.3487 \times 0.1 \times 0.1$ $= 0.0035$	M1M1 A1 **3**	Using 0.3487 and 0.1

4757: Further Applications of Advanced Mathematics (FP3)

General Comments

There were some excellent scripts, with about 15% of candidates scoring more than 60 marks (out of 72). However, a lot of able candidates clearly found answering three long themed questions to be a difficult task, and overall the marks were somewhat disappointing. When things go astray part-way through a question, it is important to carry on with the later parts, but not all candidates have the confidence to do this.

Some candidates indicated that they were short of time; and indeed only a very few answered more than the three questions required.
The five questions seemed to offer roughly comparable challenges to the candidates; the average marks (out of 24) ranged from about 13 for Q3 to about 17 for Q2 and Q4. The most popular question was Q1 (attempted by about 85% of the candidates) and the least popular was Q3 (attempted by about 40% of the candidates). The most common combinations of questions seemed to be Q1 Q2 Q4 or Q1 Q4 Q5 or Q1 Q2 Q3.

Comments on Individual Questions

1 **Vectors**

(i) Most candidates realised that they should start by finding the directions of, and points lying on, the lines *K* and *L*. Showing that the lines are parallel was usually done correctly, but finding the distance between them caused problems, and was sometimes not even attempted. When it was recognised as the distance from a point to a line it was often found efficiently and accurately.
Quite a number of candidates took *L* to be the line of intersection of *Q* and *R* (instead of *P* and *R*); fortunately this misread did not significantly alter the work to be done throughout the question.

(ii) Surprisingly, this part was quite often omitted, presumably because it was not recognised as the simple problem of finding the distance from a point to a plane.

(iii) The correct point of intersection was very often found, but many candidates did not check properly that the lines do intersect.

(iv) The method for finding the shortest distance between skew lines was well understood, and usually applied correctly.

2 **Multi-variable calculus**

(i) Almost every candidate found the partial derivatives correctly.

(ii) The method for finding stationary points was well known, and was very often carried out completely correctly. The case $x = 0$ was sometimes overlooked; and sometimes, having obtained $x = 0$ *or* $y = 2x$, it was assumed that $y = 0$ when $x = 0$.

(iii) The section $x = 2$ was usually sketched correctly; but on the section $y = 4$ most candidates showed $(2, 4, 8)$ as a point of inflection instead of a maximum.

(iv) This was reasonably well done; although quite a large proportion put $\partial z/\partial x$ equal to +36 instead of −36. As this also leads to exactly two points, candidates were not alerted to their error.

3 **Differential geometry**

(i) The arc length was often found correctly. However, many candidates were unable to simplify $\mathrm{d}s/\mathrm{d}x$ and this prevented success in this part and in part (ii).

(ii) Most candidates began correctly with $\int 2\pi x\,\mathrm{d}s$ for the surface area, but many could not proceed beyond this, even when part (i) had been answered correctly.

(iii) Most candidates obtained a correct expression for the radius of curvature, but a large number failed to earn the final mark for simplifying it to the required form.

(iv) This was quite well answered, with many candidates finding the centre of curvature correctly.

(v) Most of the candidates who attempted this part knew what was required, and often found the envelope correctly.

4 **Groups**

(i) This was quite well answered, although some candidates simply stated that all groups of order 2 or 5 are cyclic, without giving a reason (for example, that 2 and 5 are prime numbers).

(ii) This was also well done, with most candidates selecting a generating element and calculating all its powers.

(iii) The correct subgroups were often found, and it was quite rare for 'extra' ones to be given; the subgroup of order 5 was sometimes omitted. There was sometimes an unnecessarily large amount of working, such as finding the subgroup generated by each of the 10 elements.

(iv) E was almost always stated to be the identity; and the reflections were very often correctly found by considering the elements of order 2.

(v) Most candidates stated that the groups were not isomorphic; when a reason was given it was usually based on the orders of the elements (for example, *P* has more elements of order 2, or *P* has no element of order 10). Strangely, very few candidates referred to the commutativity of *M* and the non-commutativity of *P*.

(vi) The orders of the elements were usually found correctly.

(vii) Most candidates used their answer to part (vi) appropriately to write down the required subgroups.

5 **Markov chains**

(i) The transition matrix was almost always given correctly.

(ii) Calculators were accurately used, and most candidates obeyed the instruction to give the elements to 4 decimal places.

(iii) The probability was usually found correctly. Just a few candidates used $\mathbf{P}^8$ instead of $\mathbf{P}^7$.

(iv) Almost all candidates wrongly assumed that the 8th and 12th letters were independent when finding the probability that they were the same. Only a handful of candidates used the diagonal elements of $\mathbf{P}^4$ as the required conditional probabilities.

(v) This part was very well answered.

(vi) The great majority wrote down the new transition matrix correctly.

(vii) The wording of the question was intended to encourage finding the limiting matrix $\mathbf{Q}^n$ when *n* is large, and hence writing down the equilibrium probabilities from that matrix, and when it was used this method was usually successful. Nevertheless, a large number of candidates preferred to find the equilibrium probabilities by solving simultaneous equations, and this method was much more prone to error.

(viii) Most candidates calculated this probability as $p_D{}^3$ instead of $p_D \times 0.1 \times 0.1$.