OCR
RECOGNISING ACHIEVEMENT

ADVANCED GCE UNIT **4758/01**

MATHEMATICS (MEI)

Differential Equations

MONDAY 18 JUNE 2007

Morning
Time: 1 hour 30 minutes

Additional materials:
Answer booklet (8 pages)
Graph paper
MEI Examination Formulae and Tables (MF2)

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet.
- Answer any **three** questions.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The acceleration due to gravity is denoted by $g\,\mathrm{m\,s^{-2}}$. Unless otherwise instructed, when a numerical value is needed, use $g = 9.8$.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- The total number of marks for this paper is 72.
- There is an **insert** for use in Question **3**.

ADVICE TO CANDIDATES

- Read each question carefully and make sure that you know what you have to do before starting your answer.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.

This document consists of **4** printed pages and an insert.

1 An object is suspended from one end of a vertical spring in a resistive medium. The other end of the spring is made to oscillate and the differential equation describing the motion of the object is

$$\ddot{y} + 4\dot{y} + 29y = 3\cos t,$$

where y is the displacement at time t of the object from its equilibrium position.

(i) Find the general solution. [11]

(ii) Find the particular solution subject to the conditions $\dot{y} = y = 0$ when $t = 0$. What is the amplitude of the motion for large values of t? [8]

(iii) Find the displacement and velocity of the object when $t = 10\pi$. [2]

At $t = 10\pi$, the upper end of the spring stops oscillating and the differential equation describing the motion of the object is now

$$\ddot{y} + 4\dot{y} + 29y = 0.$$

(iv) Write down the general solution. Describe briefly the motion for $t > 10\pi$. [3]

2 The differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} - 2y = 1 + x^n,$$

where n is a positive constant, is to be solved for $x > 0$.

First suppose that $n \neq 2$.

(i) Find the general solution for y in terms of x. [8]

(ii) Use your general solution to find the limit of y as $x \to 0$. Show how the value of this limit can be deduced from the differential equation, provided that $\frac{\mathrm{d}y}{\mathrm{d}x}$ tends to a finite limit as $x \to 0$. [3]

(iii) Find the particular solution given that $y = -\frac{1}{2}$ when $x = 1$. Sketch a graph of the solution in the case $n = 1$. [4]

Now consider the case $n = 2$.

(iv) Find y in terms of x, given that y has the same value at $x = 1$ as at $x = 2$. [9]

3 There is an insert for use with part (iii) of this question.

Water is draining from a tank. The depth of water in the tank is initially 1 m, and after t minutes the depth is y m.

The depth is first modelled by the differential equation

$$\frac{dy}{dt} = -k\sqrt{y}(1 + 0.1\cos 25t),$$

where k is a constant.

(i) Find y in terms of t and k. [8]

(ii) If the depth of water is 0.5 m after 1 minute, show that $k = 0.586$ correct to three significant figures. Hence calculate the depth after 2 minutes. [4]

An alternative model is proposed, giving the differential equation

$$\frac{dy}{dt} = -0.586\left(\sqrt{y} + 0.1\cos 25t\right). \qquad (*)$$

The insert shows a tangent field for this differential equation.

(iii) Sketch the solution curve starting at $(0, 1)$ and hence estimate the time for the tank to empty. [4]

Euler's method is now used. The algorithm is given by $t_{r+1} = t_r + h$, $y_{r+1} = y_r + h\dot{y}_r$, where $\dot{y}$ is given by (*).

(iv) Using a step length of 0.1, verify that this gives an estimate of $y = 0.935\,54$ when $t = 0.1$ for the solution through $(0, 1)$ and calculate an estimate for y when $t = 0.2$. [6]

(v) Use (*) to show that when the depth of water is less than 1 cm the model predicts that $\frac{dy}{dt}$ is positive for some values of t. [2]

[Question 4 is printed overleaf.]

4 The following simultaneous differential equations are to be solved.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -5x + 4y + \mathrm{e}^{-2t},$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = -9x + 7y + 3\mathrm{e}^{-2t}.$$

(i) Show that $\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2} - 2\dfrac{\mathrm{d}x}{\mathrm{d}t} + x = 3\mathrm{e}^{-2t}$. [5]

(ii) Find the general solution for x in terms of t. [8]

(iii) Hence obtain the corresponding general solution for y, simplifying your answer. [4]

(iv) Given that $x = y = 0$ when $t = 0$, find the particular solutions. Find the values of $\dfrac{\mathrm{d}x}{\mathrm{d}t}$ and $\dfrac{\mathrm{d}y}{\mathrm{d}t}$ when $t = 0$. Sketch graphs of the solutions. [7]

RECOGNISING ACHIEVEMENT

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MATHEMATICS (MEI)

Differential Equations

INSERT

MONDAY 18 JUNE 2007

Morning
Time: 1 hour 30 minutes

Candidate Name										
Centre Number						Candidate Number				

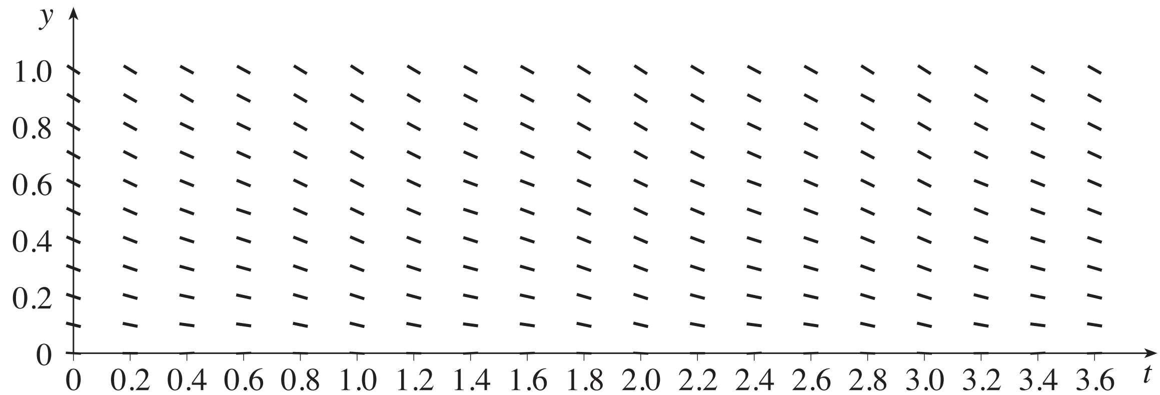
INSTRUCTIONS TO CANDIDATES

- This insert should be used in Question **3**.
- Write your name, centre number and candidate number in the spaces provided and **attach the page to your answer booklet**.

This insert consists of **2** printed pages.

[Turn over

3 (iii)



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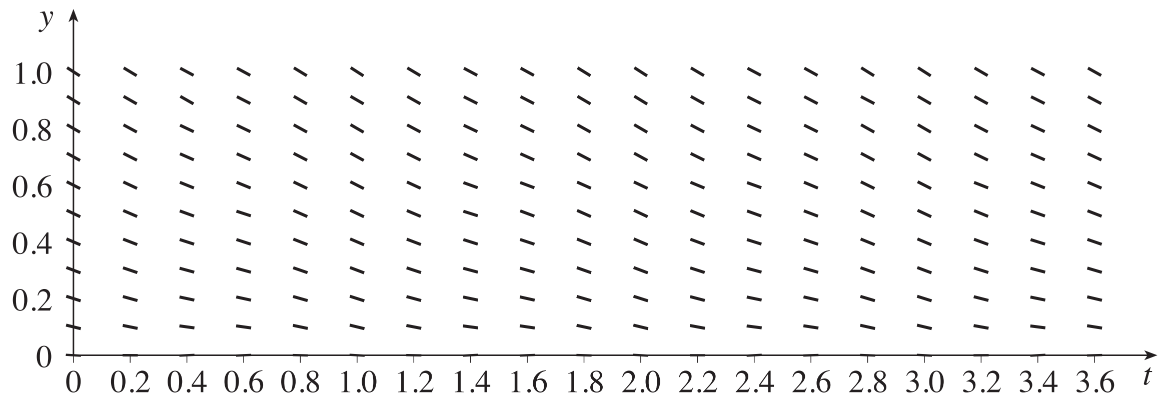
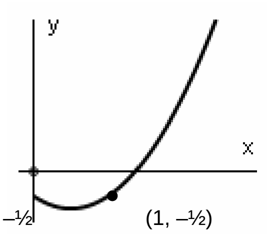
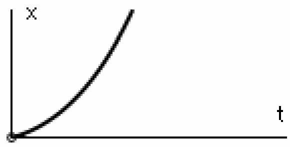
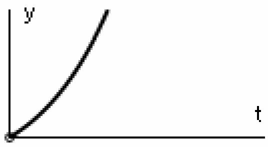


Mark Scheme 4758
June 2007

1(i)	$\lambda^2+4\lambda+29=0$	M1	Auxiliary equation	
		M1	Solve for complex roots	
	$\lambda=-2\pm 5j$	A1		
	CF $y=\mathrm{e}^{-2t}\left(A\cos 5t+B\sin 5t\right)$	F1	CF for their roots (if complex, must be exp/trig form)	
	PI $y=a\cos t+b\sin t$	B1	Correct form for PI	
	$\dot{y}=-a\sin t+b\cos t, \ddot{y}=-a\cos t-b\sin t$	M1	Differentiate twice	
	$-a\cos t-b\sin t+4\left(-a\sin t+b\cos t\right)$ $+29\left(a\cos t+b\sin t\right)=3\cos t$	M1	Substitute	
	$4b+28a=3$ $-4a+28b=0$	M1	Compare coefficients (both sin and cos)	
	$a=0.105$	M1	Solve for two coefficients	
	$b=0.015$	A1	Both	
	$y=\mathrm{e}^{-2t}\left(A\cos 5t+B\sin 5t\right)+0.105\cos t+0.015\sin t$	F1	GS = PI + CF (with two arbitrary constants)	
				11
(ii)	$t=0, y=0\Rightarrow 0=A+0.105$	M1	Use condition on y	
	$\Rightarrow A=-0.105$	F1		
	$\dot{y}=-2\mathrm{e}^{-2t}\left(A\cos 5t+B\sin 5t\right)$ $+\mathrm{e}^{-2t}\left(-5A\sin 5t+5B\cos 5t\right)-0.105\sin t+0.015\cos t$	M1	Differentiate (product rule)	
	$t=0, \dot{y}=0\Rightarrow 0=-2A+5B+0.015$	M1	Use condition on $\dot{y}$	
	$\Rightarrow B=-0.045$			
	$y=-\mathrm{e}^{-2t}\left(0.105\cos 5t+0.045\sin 5t\right)+0.105\cos t+0.015\sin t$	A1	cao	
	For large t, $y\approx 0.105\cos t+0.015\sin t$	M1	Ignore decaying terms	
	amplitude $\approx\sqrt{0.105^2+0.015^2}\approx 0.106$	M1	Calculate amplitude from solution of this form	
		A1	cao	
				8
(iii)	$y(10\pi)\approx 0.105$	B1	Their a from PI, provided GS of correct form	
	$\dot{y}(10\pi)\approx 0.015$	B1	Their b from PI, provided GS of correct form	
				2
(iv)	$y=\mathrm{e}^{-2t}\left(C\cos 5t+D\sin 5t\right)$	F1	Correct or follows previous CF Must not use same arbitrary constants as before	
	oscillations	B1		
	with decaying amplitude (or tends to zero)	B1	Must indicate that y approaches zero, not that $y\approx 0$ for $t>10\pi$	
				3

2(i)	$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{2}{x}y = \frac{1}{x} + x^{n-1}$	M1	Rearrange
	$I = \exp\left(\int -\frac{2}{x}\mathrm{d}x\right)$	M1	Attempt IF
	$= \exp(-2\ln x)$	M1	Integrate to get $k\ln x$
	$= x^{-2}$	A1	Simplified form of IF
	$\frac{\mathrm{d}}{\mathrm{d}x}\left(yx^{-2}\right) = x^{-3} + x^{n-3}$	M1	Multiply both sides by IF and recognise derivative
	$yx^{-2} = -\frac{1}{2}x^{-2} + \frac{1}{n-2}x^{n-2} + A$	M1 A1	Integrate RHS including constant
	$y = -\frac{1}{2} + \frac{1}{n-2}x^n + Ax^2$	F1	Their integral (with constant) divided by IF
			8
(ii)	From solution, $x \to 0 \Rightarrow y \to -\frac{1}{2}$	B1	Limit consistent with their solution
	From DE, $x = 0 \Rightarrow 0 - 2y = 1$	M1	Use DE with $x = 0$
	$\Rightarrow y = -\frac{1}{2}$	E1	Correctly deduced
			3
(iii)	$y = -\frac{1}{2}, x = 1 \Rightarrow -\frac{1}{2} = -\frac{1}{2} + \frac{1}{n-2} + A$	M1	Use condition
	$\Rightarrow A = -\frac{1}{n-2}$		
	$y = -\frac{1}{2} + \frac{1}{n-2}\left(x^n - x^2\right)$	F1	Consistent with their GS and given condition
	$n = 1, y = -\frac{1}{2} - x + x^2$		
		B1 B1	Shape for $x > 0$ consistent with their solution (provided not y = constant) Through $\left(1, -\frac{1}{2}\right)$ or (0, their value from (ii))
			4
(iv)	$\frac{\mathrm{d}}{\mathrm{d}x}\left(yx^{-2}\right) = x^{-3} + x^{-1}$	M1 F1	Use result from (i) or attempt to solve from scratch Follow work in (i)
	$yx^{-2} = -\frac{1}{2}x^{-2} + \ln x + B$	M1	Integrate
		A1	RHS (accept repeated error in first term from (i))
	$y = -\frac{1}{2} + x^2\ln x + Bx^2$	M1	Divide by IF, including constant (here or later)
	$y(1) = -\frac{1}{2} + B$	M1	Use condition at $x = 1$
	$y(2) = -\frac{1}{2} + 4\ln 2 + 4B$	M1	Use condition at $x = 2$
	$y(1) = y(2) \Rightarrow 3B = -4\ln 2 \Rightarrow B = -\frac{4}{3}\ln 2$	M1	Equate and solve
	$y = -\frac{1}{2} + x^2\left(\ln x - \frac{4}{3}\ln 2\right)$	A1	cao
			9

3(i)	$\int y^{-\frac{1}{2}}\,\mathrm{d}y = \int -k\left(1+0.1\cos 25t\right)\mathrm{d}t$	M1	Separate	
	$2y^{\frac{1}{2}} = -k\left(t+0.004\sin 25t\right)+c$	M1 A1 A1	Integrate LHS RHS (condone no constant)	
	$t=0, y=1 \Rightarrow c=2$	M1 F1	Use condition (must have constant)	
	$y=\left(1-\frac{1}{2}k\left(t+0.004\sin 25t\right)\right)^2$	M1 A1	Rearrange, dealing properly with constant cao	8
(ii)	$t=1, y=0.5 \Rightarrow 2\left(0.5\right)^{\frac{1}{2}} = -k\left(1+0.004\sin 25\right)+2$	M1	Substitute	
	$\Rightarrow k \approx 0.586$	E1	Calculate k (must be from correct solution)	
	$t=2 \Rightarrow y=\left(1-\frac{1}{2}\times 0.586\left(2+0.004\sin 50\right)\right)^2 \approx 0.172$	M1 A1	Substitute cao	4
(iii)	solution curve on insert	M1 A1 A1	Reasonable attempt at curve From (0,1) and decreasing Curve broadly in line with tangent field	
	tank empty after 3.0 minutes	F1	Answer must be consistent with their curve	4
(iv)	$x(0.1)=1+0.1\left(-0.6446\right)$	M1 A1	 –0.6446	
	= 0.93554	E1	Must be clearly shown	
	$x(0.2)=0.93554+0.1\left(-0.51985\right)$	M1 A1	 –0.51985	
	= 0.88356	A1	awrt 0.884	6
(v)	$y<0.01 \Rightarrow \sqrt{y}<0.1 \Rightarrow \sqrt{y}+0.1\cos 25t<0$ for some t	M1	Consider size of $\sqrt{y}$ and sign of $\sqrt{y}+0.1\cos 25t$	
	$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t}>0$ for some values of t	E1	Complete argument	2

4(i)	$\ddot{x} = -5\dot{x} + 4\dot{y} - 2\mathrm{e}^{-2t}$	M1	Differentiate	
	$= -5\dot{x} + 4\left(-9x + 7y + 3\mathrm{e}^{-2t}\right) - 2\mathrm{e}^{-2t}$	M1	Substitute for $\dot{y}$	
	$= -5\dot{x} - 36x + \frac{28}{4}\left(\dot{x} + 5x - \mathrm{e}^{-2t}\right) + 10\mathrm{e}^{-2t}$	M1 M1	y in terms of $x, \dot{x}$ Substitute for y	
	$\ddot{x} - 2\dot{x} + x = 3\mathrm{e}^{-2t}$	E1		
				5
(ii)	$\lambda^2 - 2\lambda + 1 = 0$	M1	Auxiliary equation	
	$\lambda = 1$ (repeated)	A1		
	CF $x = (A + Bt)\mathrm{e}^t$	F1	CF for their roots	
	PI $x = a\mathrm{e}^{-2t}$	B1	Correct form for PI	
	$\dot{x} = -2a\mathrm{e}^{-2t}, \ddot{x} = 4a\mathrm{e}^{-2t}$	M1	Differentiate twice	
	$4a\mathrm{e}^{-2t} - 2\left(-2\mathrm{e}^{-2t}\right) + a\mathrm{e}^{-2t} = 3\mathrm{e}^{-2t}$	M1	Substitute and compare	
	$a = \frac{1}{3}$	A1		
	GS $x = \frac{1}{3}\mathrm{e}^{-2t} + (A + Bt)\mathrm{e}^t$	F1	GS = PI + CF (with two arbitrary constants)	
				8
(iii)	$y = \frac{1}{4}\left(\dot{x} + 5x - \mathrm{e}^{-2t}\right)$	M1	y in terms of $x, \dot{x}$	
	$= \frac{1}{4}\left(-\frac{2}{3}\mathrm{e}^{-2t} + B\mathrm{e}^t + (A + Bt)\mathrm{e}^t + \frac{5}{3}\mathrm{e}^{-2t} + 5(A + Bt)\mathrm{e}^t - \mathrm{e}^{-2t}\right)$	M1 F1	Differentiate x $\dot{x}$ follows their x (but must use product rule)	
	$y = \frac{1}{4}\mathrm{e}^t\left(6A + B + 6Bt\right)$	A1	cao	
				4
(iv)	$\frac{1}{3} + A = 0$	M1	Condition on x	
	$\frac{1}{4}(6A + B) = 0$	M1	Condition on y	
	$A = -\frac{1}{3}, B = 2$			
	$x = \frac{1}{3}\mathrm{e}^{-2t} + \left(2t - \frac{1}{3}\right)\mathrm{e}^t$ $y = 3t\mathrm{e}^t$	A1	Both solutions correct	
	$t = 0 \Rightarrow \dot{x} = 1, \dot{y} = 3$	B1	Both values correct	
	 	B1 B1 B1	x through origin and consistent with their solution for large t (but not linear) y through origin and consistent with their solution for large t (but not linear) Gradient of both curves at origin consistent with their values of $\dot{x}, \dot{y}$	
				7

4758: Differential Equations (DE)

General Comments

The standard of work was generally very good, demonstrating a clear understanding of the techniques required. Candidates commonly answered questions 1 and 4. Question 3 was the least popular choice. Candidates often produced accurate work, but errors in integration were relatively common. Also there were again many candidates omitting or not dealing properly with the constant of integration when solving first order differential equations. It is vital for candidates to realise that the constant must always be included.

Comments on Individual Questions

1 Second order differential equations

- (i) This was often completely correct. The commonest error was with the coefficients of the particular integral.
- (ii) The particular solution was often correct, but errors with one of the constants were common. Most candidates were unable to calculate the amplitude, either omitting this calculation or just using one of the coefficients.
- (iii) The calculations of displacement and velocity were often correct.
- (iv) Most candidates did not seem aware of the connection between the two parts of the motion. Most stated a general solution using the same constants as in part (i). Many then proceeded to say that the motion was negligible because of the factor $e^{-10\pi}$, in contradiction to their answers to part (iii).

2 First order differential equations

- (i) Many candidates completed this correctly, but many made errors in the integrating factor, commonly getting x^2 rather than x^{-2}. Some omitted the constant of integration.
- (ii) Many candidates were able to identify the limit of their solution, but few were able to use the differential equation to deduce the limit.
- (iii) Most candidates used the condition to find the particular solution, but some found it only for the case $n = 1$. Sketches often did not show the known information about the solution, such as the given conditions.
- (iv) This was often done well, but some candidates used their previous solution rather than solving the differential equation. Some used the condition in part (iii) rather than the new condition.

3 Modelling a water tank emptying by separating variables, tangent field and numerically.

- (i) The separation of variables was often started well, but errors in integration and omission of the constant caused problems for many.
- (ii) The calculations were often hampered by errors in the solution for y.
- (iii) The solution curve was almost always done well.
- (iv) The numerical solution was often done well, but some candidates did not show sufficient working for a given answer.
- (v) Many candidates made some progress with this part, but answers often were unclear or incomplete.

4 Simultaneous differential equations

(i) The elimination of *y* was often done well, although a few differentiated the first equation with respect to *x* rather than *t*.

(ii) The solution for *x* was often done very well.

(iii) When finding *y*, many candidates correctly used the first equation. Pleasingly, fewer candidates than in past examinations attempted to set up and solve a new differential equation.

(iv) The particular solutions were often done well. When calculating the initial gradients, many differentiated their solutions, rather than the simpler approach of using the differential equations. The sketches were often done well, but candidates were expected to show the initial conditions and the initial gradient should have been consistent with their values.