

**ADVANCED SUBSIDIARY GCE UNIT  
MATHEMATICS (MEI)**

Numerical Methods

**THURSDAY 25 JANUARY 2007**

**4776/01**

Morning  
Time: 1 hour 30 minutes

Additional materials:

Answer booklet (8 pages)

Graph paper

MEI Examination Formulae and Tables (MF2)

**INSTRUCTIONS TO CANDIDATES**

- Write your name, centre number and candidate number in the spaces provided on the answer booklet.
- Answer **all** the questions.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

**INFORMATION FOR CANDIDATES**

- The number of marks is given in brackets [ ] at the end of each question or part question.
- The total number of marks for this paper is 72.

**ADVICE TO CANDIDATES**

- Read each question carefully and make sure you know what you have to do before starting your answer.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.

This document consists of **4** printed pages.

## Section A (36 marks)

- 1 A calculator gives the answer to a calculation as  $1.711\,224\,5 \times 10^{98}$ , correct to 8 significant figures. Find the largest possible absolute error and the largest possible relative error in this value.

Though the calculator displays numbers such as 1.711 224 5 to 8 digit accuracy, it stores them internally to 11 digit accuracy. Explain briefly why this is done. [5]

- 2 The approximation

$$\tan x \approx x + \frac{1}{3}x^3$$

is valid for small values of  $x$  in radians.

- (i) Find the absolute and relative errors in the approximation for  $x = 0.2$ . [4]

A much more accurate approximation is given by

$$\tan x \approx x + \frac{1}{3}x^3 + kx^5,$$

where  $k$  is a constant.

- (ii) Use the first result in part (i) to estimate  $k$ , giving your answer to 2 significant figures. [3]

- 3 An equation is being solved numerically using a fixed-point iteration of the form

$$x_{r+1} = g(x_r).$$

The iteration has been used to obtain the values shown in the following table.

$r$	0	1	2	3
$x_r$	0.35	0.354767	0.356462	0.357067
Differences				
Ratio of differences				

Copy and complete the table to show the differences in successive values of  $x_r$  and the ratios of those differences. Use extrapolation to estimate the root to which this iteration is converging, giving your answer to the accuracy that appears justified. [8]

**3**

- 4** Show, graphically or otherwise, that the equation  $x^2 = \cos x$  where  $x$  is in radians has exactly one root for  $x > 0$ . Show further that the root lies in the interval  $(0.7, 0.9)$ .

Use the secant method to find the root correct to 3 decimal places. [8]

- 5** The function  $f(x)$  has the values shown in the table.

$x$	0	0.25	0.5
$f(x)$	1.1105	1.2446	1.4065

- (i)** Use the forward difference formula with  $h = 0.5$  and  $h = 0.25$  to obtain two estimates of  $f'(0)$ . Comment on the likely accuracy of these results and on the number of decimal places that it would be safe to quote. [4]
- (ii)** Obtain the best estimate you can of the value of  $f'(0.25)$ . Comment on the likely accuracy of this result in relation to those in part (i). To how many decimal places would you quote the answer? [4]

**Section B (36 marks)**

- 6** The following values of  $x$  and  $y$  were obtained in an experiment. The values of  $x$  are exact; the values of  $y$  are correct to 2 decimal places. It is required to estimate  $\alpha$ , the value of  $x$  for which  $y = 0$ .

$x$	0.9	1.1	1.2	1.4	1.5
$y$	-0.43	-0.09	0.15	0.78	1.15

- (i)** Use Lagrange's method to find the equation of the straight line joining the data points for  $x = 1.1$  and  $x = 1.2$ . Hence estimate  $\alpha$ .

By considering the maximum possible errors in the values of  $y$  obtain a range of possible values of  $\alpha$ . Hence give the value of  $\alpha$  to the accuracy that is justified. [10]

- (ii)** Obtain a further estimate of  $\alpha$  by fitting a quadratic to the data points for  $x = 1.1, 1.2$  and  $1.4$ . [8]

- 7 This question concerns the function  $f(x) = x^{-x}$ . (This can also be written as  $f(x) = \frac{1}{x^x}$ .) The table below shows some values of the function.

$x$	1	1.5	2
$f(x)$	1	0.544 331	0.25

- (i) Use the values in the table to find the Simpson's rule estimate of  $\int_1^2 f(x) dx$  with  $h = 0.5$ .  
Find the Simpson's rule estimate with  $h = 0.25$ . [7]

You are now given that the Simpson's rule estimate with  $h = 0.125$  is 0.572 344 to 6 dp. Let the three Simpson's rule estimates with  $h = 0.5, 0.25, 0.125$  be denoted by  $a, b$  and  $c$  respectively.

- (ii) Find the value of the ratio of differences  $\frac{c - b}{b - a}$ . State the theoretical value of this ratio and comment. [5]
- (iii) Extrapolate from  $b$  and  $c$  to obtain a further estimate of the integral.

Give the value of the integral to the accuracy that appears to be justified, explaining your reasoning. [6]

**Mark Scheme 4776  
January 2007**

MEI Numerical Methods (4776) January 2007						Mark scheme
1	mpe:	$0.000\ 000\ 05 \times 10^{98} = 5 \times 10^{90}$				[M1A1]
	mpre:	$0.000\ 000\ 05 / 1.7112245 = 2.92 \times 10^{-8}$				[M1A1]
	Extra digits are used internally so that rounding errors will not (usually) show in the displayed answer					[E1]
<b>[TOTAL 5]</b>						
2	(i)	tan 0.2=	0.20271	approx =	0.20266	[A1A1]
		error:	-4.3E-05	rel error:	-0.00021	[A1A1]
	(ii)	k 0.2^5=	4.34E-05	hence k=	0.13552 8	accept 0.13 or 0.14
<b>[subtotal 4]</b>						
<b>[subtotal 3]</b>						
<b>[TOTAL 7]</b>						
3	r	0	1	2	3	
	$x_r$	0.35	0.354767	0.35646	0.35706	
	Differences		0.004767	0.00169	0.00060	
	Ratio of differences			0.35557	0.35693	
	root =	0.35706	+0.000605 (0.356932 + 0.356932 <sup>2</sup> + ...)			
	=	0.35740				
		0.3574 seems justified				
<b>[M1A1]</b>						
<b>[M1A1]</b>						
<b>[M1A1]</b>						
<b>[TOTAL 8]</b>						
4	Graph of $y = \cos x$ and $y = x^2$ showing one intersection for $x > 0$ . (Or equivalent.)					[G2]
	x	0.7	0.9	change of sign so root		
	$\cos x - x^2$	0.27484	-0.18839			[M1A1]
	r	0	1	2	3	4
	$x_r$	0.7	0.9	0.81866	0.82390	0.82413
	f(x)	0.27484	-0.18839	0.01298	0.00053	0.824
		2	9	9	1	-1.6E-06
						to 3dp
<b>[M1A1A1]</b>						
<b>[A1]</b>						
<b>[TOTAL 8]</b>						
5	x	0	0.25	0.5		
	f(x)	1.1105	1.2446	1.4065		
	h	0.5	0.25			
	f'(0)	0.5920	0.5364			
poor accuracy: estimates very different, at most 1 dp reliable					[M1A1A1]	[E1]
<b>[subtotal 4]</b>						
	h	0.25				
	f'(0.25)	0.5920				
accuracy likely to be better because of the use of central difference formula;					[M1A1]	[E1]

nothing more than 1 dp because there is nothing to compare the answer with.

[E1]  
[subtotal 4]  
[TOTAL 8]

<b>6</b>	x	0.9	1.1	1.2	1.4	1.5
	f(x)	-0.43	-0.09	0.15	0.78	1.15

(i)  $y = -0.09(x - 1.2) / (1.1 - 1.2) + 0.15(x - 1.1) / (1.2 - 1.1) = 2.4x - 2.73$  [M1A1A1]  
 Estimate of  $\alpha$ : 1.1375 [A1]

Using values -0.085 and 0.155 gives  $\alpha$  as 1.1354 [M1A1]  
 Using values -0.095 and 0.145 gives  $\alpha$  as 1.1396 [M1A1]  
 Hence quote 1.14 [A1]  
 [subtotal 10]

(ii)  $y = -0.09(x - 1.2)(x - 1.4) / (1.1 - 1.2)(1.1 - 1.4) +$  two similar terms [M1A1A1A1]  
 $y = -3(x^2 - 2.6x + 1.68) - 7.5(x^2 - 2.5x + 1.54) + 13(x^2 - 2.3x + 1.32)$  [A1]  
 $= 2.5x^2 - 3.35x + 0.57$  [A1]  
 $y = 0$  gives  $\alpha = 1.14$  (reject other root) [M1A1]  
 [subtotal 8]

[TOTAL 18]

<b>7</b>						
<b>(i)</b>	x	$x^h - x$	M	T	S	
	1	1				
	2	0.25		0.625		
		0.54433		0.58466	<b>0.57122</b>	
	1.5	1	0.544331	6	1	(h=0.5)
		<b>0.75659</b>				
	1.25	3				
		<b>0.37556</b>		0.57537	<b>0.57227</b>	(h=0.25)
	1.75	4	0.566078	2	4	

[M1A1A1]  
[M1A1A1A1]  
[subtotal 7]

(ii) a = 0.57122 [M1A1A1]  
 b = 0.57227 [E1E1]  
 c = 0.57234 [M1A1A1]  
 $b - a = 0.00105$   
 $c - b = 7E-05$   
 ratio =  $0.06647$   
 Theoretically  $1/16 (= 0.0625)$ : good agreement with theory. [E1E1]  
 [subtotal 5]

(iii)  $0.572344 + 0.0000699(1/16 + 1/16^2 + \dots)$  [M1A1]  
 $= 0.572349$  [A1]

0.57235 appears completely secure from the rate of convergence [A1E1]  
 but there may be rounding errors in the 6th dp [E1]  
 [subtotal 6]

[TOTAL 18]

## 4776 - Numerical Methods

### General Comments

There was, as usual, considerable variation in the level of preparation shown by candidates, but there seemed to be fewer than usual who were completely out of their depth. Many candidates are good at applying routine techniques accurately, though very often work is not presented concisely and logically. It is very difficult (for the candidate and for the examiner) to see what is and is not correct in a jumble of numbers. Setting down the numerical work systematically helps towards getting it right. The analysis and interpretation of results still presents challenges to some of those who can cope easily with the numerical work. It should be remembered that the numbers themselves, without analysis and interpretation, are almost meaningless.

### Comments on Individual Questions

#### 1 Error analysis

This was a routine exercise in finding absolute and relative error and it was frequently well done. The second part asked why a calculator displaying 8 digits might work to 11 digits internally. Most answered that the extra digits gave greater accuracy: true, but a rather poor answer. The aim is (as far as possible) to make the displayed answer correct to 8 significant figures.

#### 2 Approximation and errors

Most candidates had no difficulty with this question. The second part, finding the constant  $k$ , sometimes produced sign errors.

#### 3 Solution of equation; extrapolation

This question was frequently well answered, but there were two common mistakes. Firstly, the ratio of differences was calculated as the reciprocal of the common form: this is not a problem if it is handled correctly subsequently. Secondly, there was a tendency to suppose that the ratio of differences should be a 'neat' number: in this case  $\frac{1}{3}$  was popular. Though this has little effect on the final answer, it is faulty reasoning. The rate of convergence of a first order process might be any number at all between  $-1$  and  $1$ .

#### 4 Solution of equation: secant method

The graphs to show that there is only one root were of variable quality. Inevitably the least convincing were those taken unthinkingly from a graphical calculator with its domain set inappropriately. Almost all could locate the root by means of a sign change and the secant method was well done by many. (In some cases there was a numerical error in the secant process but it was followed by a recovery. This attracted most of the credit.)



5 **Numerical differentiation**

The numerical values of the various estimates were generally found correctly, The comments on the forward difference method were usually appropriately cautious and the majority were aware that the central difference formula is more accurate than the forward difference method. Some candidates tried to make something of the fact that one of the numerical answers in part (i) is identical to the numerical answer in part (ii).

6 **Interpolation: Lagrange's method**

In part (i), the equation of the straight line and the estimate of the root,  $\alpha$ , were generally found accurately. (A few, however, were thrown by the use of Lagrange's method for a straight line.) Finding the range of values of  $\alpha$  caused problems, however: the  $x$  values pair off as  $-0.085$  with  $0.155$  and  $-0.095$  with  $0.145$ . Many had these pairings wrong.

7 **Numerical integration**

This question was the least well done. This is both surprising and disappointing as the topic is straightforward and (one would have thought) familiar. The function,  $x^{-x}$ , was unusual but with a careful explanation and with some given values that did not seem to be the problem. Rather, it seemed that candidates were just not familiar with the basics of Simpson's rule.

The first request was to 'use the values in the table to find the Simpson's rule estimate ... with  $h = 0.5$ '. The range of integration was from 1 to 2, and the values in the table were  $x = 1, 1.5, 2$ . The standard form of Simpson's rule to integrate from  $a$  to  $b$  has  $h = (b - a)/2$ . It was therefore deeply puzzling to find candidates calculating  $f(1.25)$  and  $f(1.75)$  and then working with  $h = 0.25$ . These candidates then went on to find the Simpson's rule estimate with  $h = 0.125$  when asked to use  $h = 0.25$ , and were surprised (or not) to find that their answer coincided (or didn't) with the given answer for  $h = 0.125$ . As much credit as possible was given for the ability to calculate a Simpson's rule estimate, but inevitably these candidates lost marks.