RECOGNISING ACHIEVEMENT

OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS 4758

Differential Equations

Thursday **15 JUNE 2006** Afternoon 1 hour 30 minutes

Additional materials:
- 8 page answer booklet
- Graph paper
- MEI Examination Formulae and Tables (MF2)

TIME 1 hour 30 minutes

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet.
- Answer any **three** questions.
- There is an **insert** for use in Question **3**.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The acceleration due to gravity is denoted by $g \text{ m s}^{-2}$. Unless otherwise instructed, when a numerical value is needed, use $g = 9.8$.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- The total number of marks for this paper is 72.

This question paper consists of 3 printed pages, 1 blank page and an insert.

1 The displacement x at time t of an oscillating system from a fixed point is given by

$$\ddot{x} + 2\lambda\dot{x} + 5x = 0,$$

where $\lambda \geqslant 0$.

(i) For what value of λ is the motion simple harmonic? State the general solution in this case. [3]

(ii) Find the range of values of λ for which the system is under-damped. [3]

Consider the case $\lambda = 1$.

(iii) Find the general solution of the differential equation. [3]

When $t = 0, x = x_0$ and $\dot{x} = 0$, where x_0 is a positive constant.

(iv) Find the particular solution. [4]

(v) Find the least positive value of t for which $x = 0$. [3]

Now consider the case $\lambda = 3$ with the same initial conditions.

(vi) Find the particular solution and show that it is never zero for $t > 0$. [8]

2 The positive quantities x, y and z are related and vary with time t, where $t \geqslant 0$. The value of x is described by the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} + 2x = t + 1.$$

When $t = 0, x = 1$.

(i) Solve the equation to find x in terms of t. [9]

The quantity y is related to x by the differential equation $2x\frac{\mathrm{d}y}{\mathrm{d}x} = y$. When $t = 0, y = 4$.

(ii) Solve the equation to find y in terms of x. Hence express y in terms of t. [5]

The quantity z is related to x by the differential equation $x\frac{\mathrm{d}z}{\mathrm{d}x} + 2z = 6x$. When $t = 0, z = 3$.

(iii) Solve this equation for z in terms of x. Calculate the values of x, y and z when $t = 1$, giving your answers correct to 3 significant figures. [10]

3 **Answer parts (i) and (ii) on the insert provided.**

Two spherical bodies, Alpha and Beta, each of radius 1000 km, are in deep space. The point A is on the surface of Alpha, and the point B is on the surface of Beta. These points are the closest points on the two bodies and the distance AB has the constant value of 8000 km.

A probe is fired from A at a speed of V_0 km s^{-1} in an attempt to reach B, travelling in a straight line. At time t seconds after firing, the displacement of the probe from A is x km, and the velocity of the probe is v km s^{-1}.

The equation of motion for the probe is

$$v\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1}{(9000 - x)^2} - \frac{1}{(1000 + x)^2}.$$

This differential equation is to be investigated first by means of a tangent field, shown on the insert.

(i) Show that the direction indicators are parallel to the v-axis when $v = 0$ $(x \neq 4000)$. Show also that the direction indicators are parallel to the x-axis when $x = 4000$ $(v \neq 0)$. Hence complete the tangent field on the insert, excluding the point $(4000, 0)$. [6]

(ii) Sketch the solution curve through $(0, 0.025)$ and the solution curve through $(0, 0.05)$. Hence state what happens to the probe when the speed of projection is

(A) 0.025 km s^{-1},

(B) 0.05 km s^{-1}. [6]

(iii) Solve the differential equation to find v^2 in terms of x and V_0. [6]

(iv) Given that the probe reaches B, state the value of x at which v^2 is least. Hence find from your solution in part **(iii)** the range of values of V_0 for which the probe reaches B. [6]

4 The simultaneous differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2x - y + 3$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = 5x - 4y + 18$$

are to be solved for $t \geqslant 0$.

(i) Show that $\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\dfrac{\mathrm{d}x}{\mathrm{d}t} - 3x = -6$. [6]

(ii) Find the general solution for x in terms of t. Hence obtain the corresponding general solution for y. [9]

(iii) Given that $x = 4$, $y = 17$ when $t = 0$, find the particular solutions for x and y and sketch a graph of each solution. [9]

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