RECOGNISING ACHIEVEMENT

OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS **2625**

Numerical Computation

Friday **27 JANUARY 2006** Afternoon 2 hours 30 minutes

Additional materials:
Answer booklet
Graph paper
MEI Examination Formulae and Tables (MF12)

TIME 2 hours 30 minutes

INSTRUCTIONS TO CANDIDATES

- Write your Name, Centre Number and Candidate Number in the spaces provided on the answer booklet.
- Answer any **three** questions.
- Additional sheets, including computer print-outs, should be fastened securely to the answer booklet.

COMPUTING RESOURCES

- Candidates will require access to a computer with a spreadsheet program and suitable printing facilities throughout the examination.

INFORMATION FOR CANDIDATES

- The allocation of marks is given in brackets [ ] at the end of each question or part question.
- In each of the questions you are required to write spreadsheet routines to carry out various numerical analysis processes. You should note the following points.
- You will not receive credit for using any numerical analysis functions which are provided within the spreadsheet. For example, many spreadsheets provide a solver routine; you will not receive credit for using this routine when asked to write your own procedure for solving an equation.

  You may use the following built-in mathematical functions: square root, sin, cos tan, arcsin, arccos, arctan, ln, exp.
- For each question you attempt, you should submit print-outs showing the spreadsheet routine you have written and the output it generates. It will be necessary to print out the formulae in the cells as well as the values in the cells.

  You are not expected to print out and submit everything your routine produces, but you are required to submit sufficient evidence to convince the examiner that a correct procedure has been used.
- The total number of marks for this paper is 60.

This question paper consists of 3 printed pages and 1 blank page.

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1 A sequence of iterates $x_0, x_1, x_2, \ldots$ converges to a value α. The errors in the iterates are $e_0, e_1, e_2, \ldots$ where $e_{r+1} \approx ke_r$ for some constant k.

(i) Use the first three iterates to show that k may be estimated as $\dfrac{x_2 - x_1}{x_1 - x_0}$, and that α may be estimated as $\dfrac{x_2 - kx_1}{1 - k}$. [6]

(ii) Show that the equation $2x = 3e^{-x}$ has exactly one root. Write a spreadsheet routine to show that the iteration based on the rearrangement

$$x = 1.5e^{-x}$$

converges slowly. Show how to speed up the convergence using the technique in part **(i)**, and hence find α correct to 6 decimal places. [7]

(iii) Show that an alternative rearrangement of the equation $2x = 3e^{-x}$ is

$$x = -\ln(ax)$$

for some constant a which you should find. Determine whether or not the iteration based on this rearrangement converges. Investigate whether or not the technique in part **(i)** can usefully be applied in this case. [5]

(iv) Comment briefly on what you have found in part **(iii)** and the statement made at the beginning of the question that the sequence of iterates $x_0, x_1, x_2, \ldots$ converges. [2]

2 In the table, the values of x_i are exact but the values of y_i are subject to error.

x_i	0	1	2	3	4	5	6
y_i	0.2	2.3	4.1	6.1	7.9	9.9	12.0

A model is proposed of the form

$$y = ax^3 + bx^2 + cx + \varepsilon,$$

where ε is the error, and there is no constant term. The values of a, b, c are to be found by using the least squares method.

(i) By considering the sum of the squares of the errors, show that one of the normal equations is

$$\Sigma x_i^3 y_i = a\Sigma x_i^6 + b\Sigma x_i^5 + c\Sigma x_i^4.$$

Write down the other two normal equations. [6]

(ii) Use a spreadsheet to obtain

- the sums required in the normal equations,
- the solution of the normal equations,
- the values of y given by the model,
- the residual sum of squares. [12]

(iii) Now suppose that the model is modified to include a constant term. State, without doing any further calculations, what effect this would have on the residual sum of squares and explain why. [2]

3 **(i)** Derive the Gaussian two point rule,

$$\int_{-h}^{h} \mathrm{f}(x)\,\mathrm{d}x \approx h\left(\mathrm{f}\left(-\frac{h}{\sqrt{3}}\right) + \mathrm{f}\left(\frac{h}{\sqrt{3}}\right)\right).$$

Describe briefly how the Gaussian two point rule compares for accuracy and efficiency with the trapezium rule and Simpson's rule. [9]

(ii) Obtain a sequence of Gaussian two point rule estimates for the integral

$$\int_{\frac{1}{6}\pi}^{\frac{5}{6}\pi} \sqrt{\sin x}\,\mathrm{d}x,$$

beginning with $h = \frac{1}{3}\pi$ and successively halving h.

By considering the differences in these estimates obtain the value of the integral correct to 6 significant figures. [11]

4 The second order differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\sin y \qquad (1)$$

may be approximated by the equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -y \qquad (2)$$

when y is small.

(i) Verify that the function $y = A\cos x + B\sin x$, where A and B are arbitrary constants, satisfies (2).

Find the values of A and B for which $y = 0$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 0.8$ when $x = 0$. Write down the maximum value of y in this case, and give the smallest positive value of x at which this maximum occurs. [6]

(ii) Equation (1) is to be solved numerically in the case for which $y = 0$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 0.8$ when $x = 0$.

Show that the finite difference equations for the numerical solution can be written

$$y_{r+1} = 2y_r - y_{r-1} - h^2 \sin y_r,$$
$$y_1 = 0.8h.$$

Use a spreadsheet to develop an approximate solution, taking $h = 0.1$. Hence estimate the maximum value of y and the smallest positive value of x at which this maximum occurs.

By reducing the value of h appropriately, determine these values of y and x correct to 3 significant figures. [12]

(iii) Comment briefly on the values of x and y found in parts **(i)** and **(ii)**. [2]

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