RECOGNISING ACHIEVEMENT

OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS 2606

Pure Mathematics 6

Wednesday **25 JANUARY 2006** Morning 1 hour 20 minutes

Additional materials:
Answer booklet
Graph paper
MEI Examination Formulae and Tables (MF12)

TIME 1 hour 20 minutes

INSTRUCTIONS TO CANDIDATES

- Write your Name, Centre Number and Candidate Number in the spaces provided on the answer booklet.
- Answer any **three** questions.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The allocation of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive no marks unless you show sufficient detail of the working to indicate that a correct method is being used.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The total number of marks for this paper is 60.

This question paper consists of 4 printed pages.

 [Turn over

Option 1: Vectors and Matrices

1 You are given the matrix $\mathbf{M} = \begin{pmatrix} k & 2 \\ 0 & 3 \end{pmatrix}$, where $k \neq 3$.

(i) Find the eigenvalues of **M**, and the corresponding eigenvectors. [7]

(ii) Write down a matrix **P** for which $\mathbf{P}^{-1}\mathbf{M}\mathbf{P}$ is a diagonal matrix. [2]

(iii) Hence find the matrix $\mathbf{M}^n$. [7]

(iv) For the case $k = 1$, use the Cayley-Hamilton theorem to find integers p and q such that

$$\mathbf{M}^9 = p\mathbf{M}^8 + q\mathbf{M}^7.$$ [4]

Option 2: Limiting Processes

2 **(i)** State the behaviour of $x^n \ln x$, as x tends to zero through positive values, in each of the cases

(A) $n > 0$,

(B) $n = 0$,

(C) $n < 0$. [3]

(ii) For the curve $y = -x \ln x$, find the coordinates of the stationary point, and determine the gradient close to $x = 0$. Hence sketch the curve. [5]

(iii) Find $\int_a^1 (-x \ln x)\,dx$ in terms of a (where $0 < a < 1$), and hence find $\int_0^1 (-x \ln x)\,dx$. [4]

(iv) Explain in detail how $\sum_{r=1}^{n} \frac{r}{n^2} \ln\left(\frac{n}{r}\right)$ is related to the area under the curve $y = -x \ln x$, and hence find the limit of $\sum_{r=1}^{n} \frac{r}{n^2} \ln\left(\frac{n}{r}\right)$ as $n \to \infty$. [4]

(v) Use the result in part **(iv)** to show that $\sum_{r=1}^{200} r \ln r \approx 96\,500$. [4]

Option 3: Multi-Variable Calculus

3 A surface has equation $z = e^x(8 + 2xy + y^2)$.

(i) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. [3]

(ii) Show that $(2, -2, 4e^2)$ is a stationary point on the surface, and find the coordinates of the other stationary point. [6]

(iii) Sketch the section of the surface given by $x = 2$, and, on a separate diagram, sketch the section of the surface given by $y = -2$.

What can you deduce about the nature of the stationary point $(2, -2, 4e^2)$? [6]

(iv) Find, in the form $ax + by + cz + d = 0$, the equation of the tangent plane to the surface at the point P (0, 2, 12). [3]

(v) The point $(h, 2 - h, 12 + k)$, where h and k are small, is a point on the surface close to P. Find an approximate expression for k in terms of h. [2]

Option 4: Differential Geometry

4 **(a)** Find the arc length of the polar curve $r = e^{k\theta}$ for $0 \leqslant \theta \leqslant \pi$, where k is a positive constant. [5]

(b) A curve C has parametric equations

$$x = 3at, \quad y = 2at^3,$$

where a is a positive constant.

(i) Find the equation of the normal to C at the general point $(3at, 2at^3)$. [3]

(ii) Hence or otherwise find parametric equations for the evolute of C. [6]

(iii) Find the curved surface area formed when the arc of C for which $0 \leqslant t \leqslant 1$ is rotated through 2π radians about the x-axis. [6]

[Turn over

Option 5: Abstract Algebra

5 The real vector space V consists of all vectors $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$, where a, b and c are real numbers.

Three particular vectors are $e_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$, $e_2 = \begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix}$ and $e_3 = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$.

(i) Express $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ as a linear combination of e_1, e_2 and e_3.

Deduce that $\{e_1, e_2, e_3\}$ is a basis for the vector space V. [6]

A linear mapping $T : V \quad V$ is defined by $T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2b + c \\ -2a - 3c \\ a - 3b \end{pmatrix}$.

(ii) Find the matrix of T with respect to the basis $\{e_1, e_2, e_3\}$. [5]

The subspace K of V consists of all vectors x for which $Tx = 0$.

(iii) Prove that $\{e_3\}$ is a basis for K. [3]

(iv) Simplify $T(e_1 + \lambda e_3)$, where λ is a real number. [2]

(v) Hence find the vector v such that $Tv = e_2$ and v is perpendicular to e_2. [4]