OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS 2610/1

Differential Equations (Mechanics 4)

Friday **14 JANUARY 2005** Morning 1 hour 20 minutes

Additional materials:
- Answer booklet
- Graph paper
- MEI Examination Formulae and Tables (MF12)

TIME 1 hour 20 minutes

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet.
- Answer any **three** questions.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The allocation of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive no marks unless you show sufficient detail of the working to indicate that a correct method is being used.
- Final answers should be given to a degree of accuracy appropriate to the context.
- Take $g = 9.8 \text{ m s}^{-2}$ unless otherwise instructed.
- The total number of marks for this paper is 60.

This question paper consists of 4 printed pages.

1 The differential equation $\frac{dy}{dx} + 2xy = e^{-(x-2)^2}$ is to be solved for $x > 1$.

(i) Find the general solution for y in terms of x. [7]

(ii) Find the particular solution subject to the condition $y = 0$ when $x = 1$. Show that $y > 0$ for $x > 1$. Sketch the solution curve for $x \geqslant 1$. [6]

(iii) Given instead the condition that y is at its maximum value when $x = 2$, use the original differential equation to show that the maximum value of y is $\frac{1}{4}$.

Hence, or otherwise, find the particular solution. Sketch the solution curve. [7]

2 The wave function, y, of a radioactive particle satisfies the differential equations

$$\frac{d^2y}{dx^2} + 9y = 0 \qquad \text{for } x < 0,$$

$$\frac{d^2y}{dx^2} - (k-9)y = 0 \qquad \text{for } x > 0.$$

(i) In the case $k = 25$, find the general solution of each equation. [6]

The solutions found in part **(i)** satisfy the following conditions:

(1) each solution gives $y = y_0$ when $x = 0$,

(2) both solutions give the same value for $\frac{dy}{dx}$ when $x = 0$,

(3) y is bounded for all values of x.

(ii) Use these conditions to determine the arbitrary constants in the general solutions in part **(i)** in terms of y_0. [8]

(iii) Sketch the graph of y against x for the case $y_0 > 0$. [3]

(iv) In the case $k = 5$, find the general solution for $x > 0$.

State with reasons whether or not the given conditions are enough to determine the arbitrary constants in this case. [3]

3 Water is draining from a small hole near the base of a large barrel. At time t, the speed of the flow of water is $v \text{ m s}^{-1}$, the volume of water in the barrel is $V \text{ m}^3$ and the height of water above the hole is x m. The hole has a cross-section of area 0.0004 m^2.

Torricelli's law states that $v = \sqrt{2gx}$.

Initially the height of water above the hole is 2 m.

(i) Show that $\frac{\mathrm{d}V}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = -0.0004\sqrt{2gx}$. [3]

(ii) The barrel is modelled initially by taking $V = \frac{5}{3}x$. Find x in terms of t. Calculate the time for the barrel to empty. [6]

(iii) A refined model gives $V = x + x^2 - \frac{1}{3}x^3$. Calculate the time for the barrel to empty. [6]

(iv) To take account of fluctuations in the flow, the model is refined further to give

$$(1 + 2x - x^2)\frac{\mathrm{d}x}{\mathrm{d}t} = -0.0004\left(\sqrt{2gx} + 0.1\sin t\right).$$

Euler's method is used to estimate x. The algorithm is given by

$$x_{r+1} = x_r + h\dot{x}_r \qquad t_{r+1} = t_r + h,$$

where h is the step length. The following results are calculated.

t	x	$\dot{x}$
0	2	−0.00250
0.1	1.99975	
0.2		

Verify the calculations for the first step and then calculate one more step to estimate x when $t = 0.2$. [5]

[Turn over

4 The simultaneous differential equations

$$\frac{dx}{dt} - 8x + 3y = 0 \qquad (1)$$

$$\frac{dy}{dt} + 2x - 7y = 0 \qquad (2)$$

are to be solved.

(i) Eliminate y from the equations to show that $\frac{d^2x}{dt^2} - 15\frac{dx}{dt} + 50x = 0.$ [5]

(ii) Find the general solution for x. Use this solution and equation (1) to find the corresponding general solution for y. [6]

Now consider the following simultaneous differential equations.

$$\frac{dx}{dt} - 8x + 3y = e^{-t} \qquad (3)$$

$$\frac{dy}{dt} + 2x - 7y = e^{-t} \qquad (4)$$

(iii) Given that these equations have a solution of the form $x = ae^{-t}, y = be^{-t}$, calculate the values of a and b. [5]

(iv) Denoting the solutions in part **(ii)** by $x = f(t)$ and $y = g(t)$, show that

$$x = f(t) + ae^{-t}$$

$$y = g(t) + be^{-t}$$

are the general solutions of the differential equations (3) and (4). [4]

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