RECOGNISING ACHIEVEMENT

OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS **2606**

Pure Mathematics 6

Wednesday **16 JUNE 2004** Afternoon 1 hour 20 minutes

Additional materials:
Answer booklet
Graph paper
MEI Examination Formulae and Tables (MF12)

TIME 1 hour 20 minutes

INSTRUCTIONS TO CANDIDATES

- Write your Name, Centre Number and Candidate Number in the spaces provided on the answer booklet.
- Answer any **three** questions.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The allocation of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive no marks unless you show sufficient detail of the working to indicate that a correct method is being used.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The total number of marks for this paper is 60.

This question paper consists of 4 printed pages.

[Turn over

Option 1: Vectors and Matrices

1 (a) (i) Find the inverse of the matrix $\begin{pmatrix} 3 & -7 & k \\ 2 & 2 & 5 \\ 1 & 3 & 4 \end{pmatrix}$ where $k \neq 0$. [6]

The matrix $\mathbf{A}$ has eigenvectors $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -7 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix}$ with corresponding eigenvalues -1, 1, 0 respectively.

(ii) Express $\mathbf{A}$ as the product of three matrices, and hence find $\mathbf{A}$. [5]

(b) Four points have coordinates $A(1, -8, 12)$, $B(-5, 7, 24)$, $C(3, 5, 8)$ and $D(14, 2, 0)$.

(i) Find the shortest distance from C to the line AB. [4]

(ii) Find the shortest distance between the skew lines AB and CD. [5]

Option 2: Limiting Processes

2 (i) State the behaviour of $x^n \ln x$, as x tends to zero through positive values, in each of the cases

(*A*) $n > 0$,

(*B*) $n = 0$,

(*C*) $n < 0$. [3]

(ii) For the curve $y = -x \ln x$, find the coordinates of the stationary point, and determine the gradient close to $x = 0$. Hence sketch the curve. [5]

(iii) Find $\int_a^1 (-x \ln x)\, dx$ in terms of a (where $0 < a < 1$), and hence find $\int_0^1 (-x \ln x)\, dx$. [4]

(iv) Explain in detail how $\sum_{r=1}^{n} \frac{r}{n^2} \ln\left(\frac{n}{r}\right)$ is related to the area under the curve $y = -x \ln x$, and hence find the limit of $\sum_{r=1}^{n} \frac{r}{n^2} \ln\left(\frac{n}{r}\right)$ as $n \to \infty$. [4]

(v) Use the result in part (iv) to show that $\sum_{r=1}^{100} r \ln r \approx 20\,760$. [4]

Option 3: Multi-Variable Calculus

3 A surface has equation $g(x, y, z) = 0$, where $g(x, y, z) = xz^2 - 4xz + 4xy + x^2 + y^2 + 15$.

(i) Find $\dfrac{\partial g}{\partial x}$, $\dfrac{\partial g}{\partial y}$ and $\dfrac{\partial g}{\partial z}$. [4]

(ii) Find the equation of the normal line to the surface at the point $P(-1, 5, 7)$. [3]

(iii) The point $(-1 + \delta x, 5 + \delta y, 7 + \delta z)$, where δx, δy and δz are small, is a point on the surface close to P.

Find an approximate expression for δz in terms of δx and δy. [4]

(iv) Find the coordinates of the points on the surface at which the normal line is parallel to the x-axis. [9]

Option 4: Differential Geometry

4 A curve has parametric equations $x = a(1 - \cos\theta), y = a(\theta - \sin\theta)$, for $0 \leqslant \theta \leqslant \pi$.

(i) Show that $\dfrac{dy}{dx} = \tan\frac{1}{2}\theta$. [3]

(ii) The arc length from the origin to a general point on the curve is s, and $\dfrac{dy}{dx} = \tan\psi$.

Express s in terms of θ, and hence show that the intrinsic equation of the curve is

$$s = 4a(1 - \cos\psi).$$ [6]

(iii) Find the radius of curvature in terms of θ. [3]

(iv) Show that the centre of curvature corresponding to a general point on the curve is

$$\left(-a(1 - \cos\theta), a(\theta + \sin\theta)\right).$$ [8]

Option 5: Abstract Algebra

5 The set G consists of all real numbers not equal to 2.

A binary operation $*$ is defined on real numbers x, y by $x * y = xy - 2x - 2y + 6$.

(i) Prove that G, with the binary operation $*$, is a group. [9]

(ii) Find an element of G of order 2. [2]

The set $H = \{3, 5, 9, 11\}$ has a binary operation $\circ$ defined by

$x \circ y$ is the remainder when $x * y$ is divided by 20.

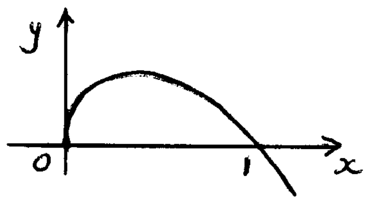
(iii) Give the combination table for H, and hence prove that H is a group. [5]

(iv) Determine whether H is a cyclic group or not. [2]

(v) Explain why H is not a subgroup of G. [2]

Mark Scheme

1(a)(i)	$\mathbf{P}_k^{-1} = \frac{1}{4k}\begin{pmatrix} -7 & 28+3k & -35-2k \\ -3 & 12-k & -15+2k \\ 4 & -16 & 20 \end{pmatrix}$	M1A1 M1 A1 M1 A1 **6**	Calculation of determinant Finding at least 3 cofactors 6 signed cofactors correct Fully correct method for inverse Inverse correct
(ii)	$\mathbf{P}_1^{-1}\mathbf{A}\mathbf{P}_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	M1	
	$\mathbf{A} = \mathbf{P}_1\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\mathbf{P}_1^{-1}$	A1	
	$= \begin{pmatrix} 3 & -7 & 1 \\ 2 & 2 & 5 \\ 1 & 3 & 4 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\frac{1}{4}\begin{pmatrix} -7 & 31 & -37 \\ -3 & 11 & -13 \\ 4 & -16 & 20 \end{pmatrix}$	M1 A1 ft	Substituting $k = 1$ in $\mathbf{P}^{-1}$ 3 numerical matrices in correct order
	$= \begin{pmatrix} 10.5 & -42.5 & 50.5 \\ 2 & -10 & 12 \\ -0.5 & 0.5 & -0.5 \end{pmatrix}$	A1 **5**	
(b)(i)	$\overrightarrow{AC} \times \overrightarrow{AB} = \begin{pmatrix} 2 \\ 13 \\ -4 \end{pmatrix} \times \begin{pmatrix} -6 \\ 15 \\ 12 \end{pmatrix} = 108\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$	M1A1	
	Distance is $\frac{\lvert \overrightarrow{AC} \times \overrightarrow{AB} \rvert}{\lvert \overrightarrow{AB} \rvert} = \frac{108\sqrt{2^2+1^2}}{\sqrt{6^2+15^2+12^2}}$	M1	
	$= 12$	A1 **4**	
	OR If N is $(1-6\lambda,\ -8+15\lambda,\ 12+12\lambda)$ $\overrightarrow{CN} \cdot \overrightarrow{AB} = \begin{pmatrix} 1-6\lambda-3 \\ -8+15\lambda-5 \\ 12+12\lambda-8 \end{pmatrix} \cdot \begin{pmatrix} -6 \\ 15 \\ 12 \end{pmatrix} = 0$ M1 $\lambda = \frac{1}{3}$, N is $(-1,\ -3,\ 16)$ M1A1 $\mathrm{CN} = \sqrt{4^2+8^2+8^2} = 12$ A1		
(ii)	$\overrightarrow{AB} \times \overrightarrow{CD} = \begin{pmatrix} -6 \\ 15 \\ 12 \end{pmatrix} \times \begin{pmatrix} 11 \\ -3 \\ -8 \end{pmatrix} = 21\begin{pmatrix} -4 \\ 4 \\ -7 \end{pmatrix}$	M1A1	
	Distance is $\frac{\begin{pmatrix} 2 \\ 13 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} -4 \\ 4 \\ -7 \end{pmatrix}}{\sqrt{4^2+4^2+7^2}}$	M1A1 ft	
	$= \frac{72}{9} = 8$	A1 **5**	

2 (i)	$(A)\ \ x^n \ln x \to 0$ $(B)\ \ \ln x \to -\infty, \quad (C)\ \ x^n \ln x \to -\infty$	B1 B1B1 **3**	 Give B1 if both given as ∞
(ii)	$\frac{\mathrm{d}y}{\mathrm{d}x} = -\ln x - 1$ Stationary point at $(\mathrm{e}^{-1},\ \mathrm{e}^{-1})$ When $x \approx 0$, $\frac{\mathrm{d}y}{\mathrm{d}x}$ is large 	M1 A1 A1 B1 B1 **5**	Use of produce rule Infinite gradient at $(0,\ 0)$ Correct shape; crossing x-axis at 1
(iii)	$\int_a^1 (-x\ln x)\,\mathrm{d}x = \left[-\tfrac{1}{2}x^2\ln x + \tfrac{1}{4}x^2\right]_a^1$ $= \tfrac{1}{4} + \tfrac{1}{2}a^2\ln a - \tfrac{1}{4}a^2$ As $a \to 0$, $a^2\ln a \to 0$ Hence $\int_0^1 (-x\ln x)\,\mathrm{d}x = \tfrac{1}{4}$	M1 A1 M1 A1 **4**	Integration by parts
(iv)	$\sum_{r=1}^{n} \frac{r}{n^2}\ln\frac{n}{r} = \sum_{r=1}^{n} \frac{1}{n}\left(-\frac{r}{n}\ln\frac{r}{n}\right)$ This is the area of rectangles of width $\frac{1}{n}$ with their (top RH) corners on the curve $y = -x\ln x$ between $x = 0$ and $x = 1$ Hence $\lim_{n\to\infty}\left(\sum_{r=1}^{n} \frac{r}{n^2}\ln\frac{n}{r}\right) = \frac{1}{4}$	B1 B1 B1 B1 ft **4**	
(v)	$\sum_{r=1}^{100} \frac{r}{100^2}\ln\frac{100}{r} \approx \frac{1}{4}$ $\sum_{r=1}^{100} r(\ln 100 - \ln r) \approx 2500$ $(\ln 100)\sum_{r=1}^{100} r - \sum_{r=1}^{100} r\ln r \approx 2500$ $\sum_{r=1}^{100} r\ln r \approx (\ln 100)(\tfrac{1}{2}\times 100\times 101) - 2500$ $= 20756.1 \approx 20760$	M1 M1 M1 A1 (ag) **4**	 *SR* M1 for $(\pm)\int_a^b x\ln x\,\mathrm{d}x$ with $a = 0$ or 1 and $b = 100$ or 101 A1 for $\left[\tfrac{1}{2}x^2\ln x - \tfrac{1}{4}x^2\right]_a^b$ (= 20526 or 20989)

3 (i)	$\frac{\partial g}{\partial x} = z^2 - 4z + 4y + 2x$ $\frac{\partial g}{\partial y} = 4x + 2y, \quad \frac{\partial g}{\partial z} = 2xz - 4x$	B2 B1B1 **4**	Give B1 for two terms correct
(ii)	At P, $\frac{\partial g}{\partial x} = 39, \ \frac{\partial g}{\partial y} = 6, \ \frac{\partial g}{\partial z} = -10$ Direction of the normal line is $\begin{pmatrix} 39 \\ 6 \\ -10 \end{pmatrix}$ Equation is $\mathbf{r} = \begin{pmatrix} -1 \\ 5 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 39 \\ 6 \\ -10 \end{pmatrix}$	M1 M1 A1 **3**	
(iii)	$\delta g \approx 39\delta x + 6\delta y - 10\delta z$ and $\delta g = 0$	M1A1 ft M1	
	OR Tangent plane is $39x + 6y - 10z = -79$ M1 $39(-1 + \delta x) + 6(5 + \delta y) - 10(7 + \delta z) \approx -79$ M1A1 ft		
	$\delta z \approx 3.9\delta x + 0.6\delta y$	A1 **4**	*Notes* $\delta z \approx 39\delta x + 6\delta y$ gets M0 $\frac{\partial z}{\partial x} = \frac{\partial g}{\partial x} \div \frac{\partial g}{\partial z} = -3.9$ leading to $\delta z \approx -3.9\delta x - 0.6\delta y$ earns M1A0M1A0 If x, y, z retained, M1A0M1A0
(iv)	Require $4x + 2y = 0$ and $2xz - 4x = 0$ $x = 0 \Rightarrow y = 0 \Rightarrow g(x, y, z) = 15 \neq 0$, not possible $y = -2x, \quad z = 2$ $4x - 8x - 8x^2 + x^2 + 4x^2 + 15 = 0$ $3x^2 + 4x - 15 = 0$ $x = -3, \ \frac{5}{3}$ Points are $(-3, \ 6, \ 2)$ and $(\frac{5}{3}, \ -\frac{10}{3}, \ 2)$	M1M1 A1 A1A1 M1 M1 A1A1 **9**	 Substituting into $g(x, y, z) = 0$ Solving quadratic equation

4 (i)	$\dfrac{dy}{dx} = \dfrac{a(1-\cos\theta)}{a\sin\theta}$	B1	
	$= \dfrac{2a\sin^2\frac{1}{2}\theta}{2a\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}$	M1	
	$= \tan\frac{1}{2}\theta$	A1 (ag) **3**	
(ii)	$s = \int\sqrt{(2a\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta)^2 + (2a\sin^2\frac{1}{2}\theta)^2}\,d\theta$	M1A1	Any correct form
	$= \int 2a\sin\frac{1}{2}\theta\,d\theta$		
	$= -4a\cos\frac{1}{2}\theta + C$	A1	$+C$ not required
	$s = 0$ when $\theta = 0 \Rightarrow C = 4a$	M1	
	$s = 4a(1-\cos\frac{1}{2}\theta)$		
	$\psi = \frac{1}{2}\theta$, so $s = 4a(1-\cos\psi)$	M1A1 (ag) **6**	
(iii)	$\rho = \dfrac{ds}{d\psi} = 4a\sin\psi$	M1A1	*SR* M1A0 for $\rho = \dfrac{1}{4a\sin\psi}$
	$= 4a\sin\frac{1}{2}\theta$	A1 **3**	
	OR $\dfrac{d^2y}{dx^2} = \dfrac{\frac{1}{2}\sec^2\frac{1}{2}\theta}{a\sin\theta}$ M1		Obtained and used in formula for ρ or κ
	$\rho = \dfrac{2a\sin\theta\left(1+\tan^2\frac{1}{2}\theta\right)^{\frac{3}{2}}}{\sec^2\frac{1}{2}\theta}$ A2		Any correct form
	OR $\ddot{x} = a\cos\theta$, $\ddot{y} = a\sin\theta$ M1		Obtained and used in formula for ρ or κ
	$\rho = \dfrac{a(2-2\cos\theta)^{\frac{3}{2}}}{1-\cos\theta}$ A2		
(iv)	$\mathbf{c} = \begin{pmatrix} a(1-\cos\theta) \\ a(\theta-\sin\theta) \end{pmatrix} + \rho\begin{pmatrix} -\sin\psi \\ \cos\psi \end{pmatrix}$	M1A1	For $\hat{\mathbf{n}} = \begin{pmatrix} -\sin\psi \\ \cos\psi \end{pmatrix}$ or equivalent
	$= \begin{pmatrix} a(1-\cos\theta) + 4a\sin\frac{1}{2}\theta(-\sin\frac{1}{2}\theta) \\ a(\theta-\sin\theta) + 4a\sin\frac{1}{2}\theta(\cos\frac{1}{2}\theta) \end{pmatrix}$	M1A1 ft A1 ft	Ft only from incorrect ρ
	$= \begin{pmatrix} a(1-\cos\theta) - 2a(1-\cos\theta) \\ a(\theta-\sin\theta) + 2a\sin\theta \end{pmatrix}$	M1 M1	
	$= \begin{pmatrix} -a(1-\cos\theta) \\ a(\theta+\sin\theta) \end{pmatrix}$	A1 (ag) **8**	

<table>
<tr><td>5 (i)</td><td>G is closed if $x * y \neq 2$
$x * y = 2 \Rightarrow (x-2)(y-2) = 0$
As $x \neq 2$ and $y \neq 2$, we have $x * y \neq 2$

$(x * y) * z = (xy - 2x - 2y + 6) * z$
$= (xy - 2x - 2y + 6)z - 2(xy - 2x - 2y + 6) - 2z + 6$
$= xyz - 2xz - 2yz - 2xy + 4x + 4y + 4z - 6$
$x * (y * z)$
$= x(yz - 2y - 2z + 6) - 2x - 2(yz - 2y - 2z + 6) + 6$
$= xyz - 2xy - 2xz - 2yz + 4x + 4y + 4z - 6$
Hence G is associative

The identity element is 3
(since $3 * x = 3x - 6 - 2x + 6 = x$)

$x * y = 3 \Leftrightarrow y = \dfrac{2x-3}{x-2}$
Since $x \neq 2$ and $\dfrac{2x-3}{x-2} \neq 2$, every element of G has an inverse in G</td><td>M1

A1

M1

A1

A1

B1

M1A1

A1
9</td><td>

M0 if only particular example(s) given</td></tr>
<tr><td>(ii)</td><td>$x * x = 3 \Leftrightarrow x^2 - 4x + 6 = 3$
$\Leftrightarrow x = 1, 3$
The only element with order 2 is 1</td><td>M1

A1
2</td><td></td></tr>
<tr><td>(iii)</td><td><table><tr><td></td><td>3</td><td>5</td><td>9</td><td>11</td></tr><tr><td>3</td><td>3</td><td>5</td><td>9</td><td>11</td></tr><tr><td>5</td><td>5</td><td>11</td><td>3</td><td>9</td></tr><tr><td>9</td><td>9</td><td>3</td><td>11</td><td>5</td></tr><tr><td>11</td><td>11</td><td>9</td><td>5</td><td>3</td></tr></table>Table shows H is closed
$\circ$ is associative since $*$ is associative
The identity element is 3

3, 5, 9, 11 have inverses 3, 9, 5, 11</td><td>B2

B1

B2
5</td><td>Give B1 for one bold value correct

For any two of these statements

Give B1 for two correct</td></tr>
<tr><td>(iv)</td><td>H is cyclic since it has an element of order 4
The element 5 (or 9) has order 4</td><td>M1
A1
2</td><td></td></tr>
<tr><td>(v)</td><td>H is not a subgroup of G since the binary operation is different;
e.g. in G, $5 * 9 = 23$; but in H, $5 \circ 9 = 3$</td><td>M1
A1
2</td><td></td></tr>
</table>

Examiner’s Report

2606 Pure Mathematics 6

General Comments

Most candidates presented their work clearly, and produced a good solution to at least one question. There was a wide range of performance, with 20% of candidates scoring more than 50 marks (out of 60) and about a quarter scoring 30 marks or less. The most popular choice of questions was 1, 3, 4 followed by 1, 2, 3 and 1, 3, 5. These three combinations accounted for over 90% of the candidates.

Comments on Individual Questions

Q.1 This question was attempted by most candidates, and the average mark was about 14 (out of 20).

In part (a) the inverse matrix was very often found correctly, and the method for finding **A** was quite well understood. Sometimes the three matrices were multiplied in the opposite order, and several candidates left the answer in terms of *k*, without substituting $k = 1$ into the inverse matrix.

Part (b)(i) was found to be the hardest part of the question. The formula $\frac{|(\mathbf{p}-\mathbf{a})\times\mathbf{d}|}{|\mathbf{d}|}$ for the perpendicular distance from a point to a line was often used successfully, although it was frequently mis-remembered (or misapplied) with a scalar product instead of the vector product. Quite a few attempted to use the formula for the distance of a point from a plane. An alternative approach, finding the foot of the perpendicular, was used successfully by some candidates.

Finding the shortest distance between two skew lines in part (b)(ii) was understood much better.

Q.2 This question was attempted by 40% of the candidates, and the average mark was about 12.

In part (i) the limiting values of $x^n \ln x$ were not universally known, and most candidates got one or more of these wrong. This had consequences for the curve in part (ii) which often had $x = 0$ as an asymptote, and for the limiting value of the integral in part (iii).

There were many good explanations in part (iv), although several candidates considered strips of width 1 instead of $\frac{1}{n}$.

Few candidates could use their result from part (iv) to make progress in part (v). A fair number used an alternative method, approximating the sum of the series by $\int_1^{100} x \ln x \, dx$, which gives a much less accurate value.

Q.3 This question was attempted by almost every candidate, and the average mark was about 14.

Parts (i), (ii) and (iii) were very often answered correctly.

In part (iv), most candidates knew that the condition for the normal line to be parallel to the *x*-axis was $\frac{\partial \mathbf{g}}{\partial y} = \frac{\partial \mathbf{g}}{\partial z} = 0$ (although some thought it was $\frac{\partial \mathbf{g}}{\partial x} = 0$). However, very many candidates combined these equations with $\frac{\partial \mathbf{g}}{\partial x}=1$, instead of using the equation of the surface, to find the coordinates of the points.

Q.4 This question was attempted by half the candidates, and the average was about 12.

The principles involved in all parts of the question were well understood, but many candidates were prevented from obtaining the given results because of their lack of competence with the half-angle formulae. For example, a surprising number of candidates were unable to do part (i).

Q.5 This question was attempted by less than 20% of candidates, and the average mark was about 12.

Few candidates earned full marks in part (i). Very many did not treat the condition $x \neq 2$ sufficiently rigorously when considering closure or inverses, and some confused associativity with commutativity. Others ‘proved’ associativity by giving one or two particular examples.

The remaining parts (ii) to (v) were generally well answered.