OCR
RECOGNISING ACHIEVEMENT

OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS 2605

Pure Mathematics 5

Tuesday **29 JUNE 2004** Afternoon 1 hour 20 minutes

Additional materials:
Answer booklet
Graph paper
MEI Examination Formulae and Tables (MF12)

TIME 1 hour 20 minutes

INSTRUCTIONS TO CANDIDATES

- Write your Name, Centre Number and Candidate Number in the spaces provided on the answer booklet.
- Answer any **three** questions.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The allocation of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive no marks unless you show sufficient detail of the working to indicate that a correct method is being used.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The total number of marks for this paper is 60.

This question paper consists of 3 printed pages and 1 blank page.

1 **(a)** The equation $x^3 + px^2 + qx + r = 0$ has roots α, $2\alpha + \lambda$ and $2\alpha - \lambda$.

(i) Express p, q and r in terms of α and λ. [4]

(ii) Show that $r = \dfrac{p(25q - 4p^2)}{125}$. [3]

(iii) Given that $p = -5\sqrt{2}$ and $q = 9$, find the three roots in an exact form. [5]

(b) When the polynomial $f(x)$ is divided by $(x - 4)$, the remainder is 29.

When $f(x)$ is divided by $(x + 2)^2$, the remainder is $5x - 3$.

Find the remainder when $f(x)$ is divided by $(x + 2)(x - 4)$. [8]

2 **(a)** Find the exact value of $\displaystyle\int_{-1.5}^{1.5} \frac{1}{\sqrt{9 - 2x^2}}\,dx$. [5]

(b) **(i)** Find $\displaystyle\int_{0}^{\ln a} (12\cosh x - 8\sinh x)\,dx$ in terms of a, simplifying your answer. [4]

(ii) Solve the equation $12\cosh x - 8\sinh x = 9$, giving the answers in logarithmic form. [6]

(iii) Show that $12\cosh x - 8\sinh x \geqslant 4\sqrt{5}$. [5]

3 **(a)** Using de Moivre's theorem, express $\tan 4\theta$ in terms of $\tan\theta$. [5]

(b) In this question, give all answers in an exact form, with arguments in radians between $-\pi$ and π.

(i) Find the modulus and argument of $2 - 2j$. [2]

(ii) Hence find the modulus and argument of each of the three cube roots of $2 - 2j$. Illustrate these cube roots on an Argand diagram. [5]

The points representing the cube roots are the vertices of a triangle T.

(iii) Find the modulus and argument of each of the three complex numbers which are represented by the midpoints of the sides of T. [5]

The three complex numbers in part **(iii)** are the cube roots of w.

(iv) Find w, in the form $a + bj$. [3]

4 The point $P(a\sec\theta,\ b\tan\theta)$, where $0 < \theta < \frac{1}{2}\pi$, lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

(i) Show that the equation of the tangent to the hyperbola at P is

$$(b\sec\theta)x - (a\tan\theta)y = ab.$$ [4]

This tangent crosses the asymptotes at A and B, and crosses the x-axis at C. The asymptote through A has positive slope, and the asymptote through B has negative slope.

(ii) Find the coordinates of A, B and C. [6]

(iii) Show that P is the midpoint of AB. [3]

(iv) Show that the area of the triangle OCA is $\frac{ab\cos\theta}{2(\sec\theta - \tan\theta)}$, and find the area of the triangle OCB (where O is the origin). [3]

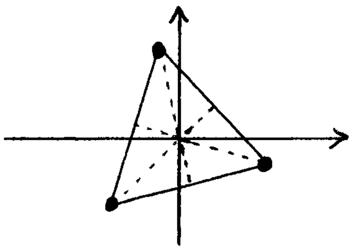
(v) Find the area of the triangle OAB, and show that this area is independent of the position of P on the hyperbola. [4]

Mark Scheme

1(a)(i)	$p = -5\alpha$ $q = \alpha(2\alpha+\lambda) + \alpha(2\alpha-\lambda) + (2\alpha+\lambda)(2\alpha-\lambda)$ $= 8\alpha^2 - \lambda^2$ $r = -\alpha(2\alpha+\lambda)(2\alpha-\lambda)$ $= -\alpha(4\alpha^2 - \lambda^2)$	B1 M1A1 B1 **4**	
(ii)	$r = -\alpha(4\alpha^2 - 8\alpha^2 + q)$ $= \frac{1}{5}p(q - \frac{4}{25}p^2)$ $= \dfrac{p(25q-4p^2)}{125}$	M1 M1 A1 (ag) **3**	Eliminating λ Eliminating α Fully correct working only
	OR $\dfrac{p(25q-4p^2)}{125} = \dfrac{-5\alpha(200\alpha^2 - 25\lambda^2 - 100\alpha^2)}{125}$ M1 $= -\alpha(4\alpha^2 - \lambda^2)$ M1 $= r$ A1		 *Dependent on previous M1* Fully correct working only
(iii)	$\alpha = -\frac{1}{5}p = \sqrt{2}$ $\lambda^2 = 8\alpha^2 - q = 16 - 9 = 7$ Other roots are $2\sqrt{2}+\sqrt{7}$, $2\sqrt{2}-\sqrt{7}$	B1 M1A1 A1A1 ft **5**	
(b)	$\mathrm{f}(4) = 29$ $\mathrm{f}(-2) = -13$ $\mathrm{f}(x) = (x+2)(x-4)\,\mathrm{g}(x) + ax + b$ Putting $x = 4$, $29 = 4a + b$ Putting $x = -2$, $-13 = -2a + b$ $a = 7$, $b = 1$ Remainder is $7x + 1$	B1 B1 B1 M1 A1 A1 M1 A1 **8**	 Substituting $x = 4$ *or* $x = -2$ Solving to obtain a or b

2 (a)	$\int_{-1.5}^{1.5} \frac{1}{\sqrt{9-2x^2}}\,dx = \left[\frac{1}{\sqrt{2}} \arcsin \frac{\sqrt{2}x}{3} \right]_{-1.5}^{1.5}$ $= \frac{1}{\sqrt{2}}\frac{\pi}{4} - \left(-\frac{1}{\sqrt{2}}\frac{\pi}{4} \right)$ $= \frac{\pi}{2\sqrt{2}}$	M1 A1 A1 M1 A1 **5**	For arcsin *or* any sine substitution For $\frac{\sqrt{2}x}{3}$ *or* $\sqrt{2}x = 3\sin\theta$ For $\frac{1}{\sqrt{2}}$ *or* $\left[\frac{1}{\sqrt{2}}\theta \right]$ Substitution of limits *(Dep on 1st M1)* *Must be θ limits if appropriate* (allow $\arcsin \frac{1}{2}\sqrt{2}$ or decimals for M1)
(b)(i)	$\int_0^{\ln a} (12\cosh x - 8\sinh x)\,dx$ $= \left[12\sinh x - 8\cosh x \right]_0^{\ln a}$ $= 12\sinh(\ln a) - 8\cosh(\ln a) - (0-8)$ $= 6\left(a - \frac{1}{a} \right) - 4\left(a + \frac{1}{a} \right) + 8$ $= 2a - \frac{10}{a} + 8 \quad \left(= \frac{2(a+5)(a-1)}{a} \right)$	 B1 M1 M1 A1 **4**	 Or $\left[2e^x - 10e^{-x} \right]_0^{\ln a}$ substituting $x = \ln a$ *and* $x = 0$ use of $e^{\ln a} = a$ *and* $e^{-\ln a} = \frac{1}{a}$
(ii)	$6(e^x + e^{-x}) - 4(e^x - e^{-x}) = 9$ $2e^{2x} - 9e^x + 10 = 0$ $e^x = 2, \ \frac{5}{2}$ $x = \ln 2, \ \ln \frac{5}{2}$	M1 M1 M1 A1 A1A1 **6**	 Obtaining a value of e^x *SR* B2B2 for $\ln 2$ and $\ln 2.5$ with no working
	OR $144(1+s^2) = (8s+9)^2$ M1 $80s^2 - 144s + 63 = 0$ M1 M1 $s = 0.75, \ 1.05$ A1 $x = \ln 2, \ \ln \frac{5}{2}$ A1A1		Or $(12c-9)^2 = 64(c^2-1)$ Or $80c^2 - 216c + 145 = 0$ Obtaining a value of s (or c) Or $c = 1.25, \ 1.45$ Max A1 if any 'extras' given

2 (b) (iii)	Let $y = 12\cosh x - 8\sinh x = 2e^{x} + 10e^{-x}$ $\dfrac{dy}{dx} = 2e^{x} - 10e^{-x}$ $= 0$ when $e^{x} = \sqrt{5}$ Then $y = 2\sqrt{5} + \dfrac{10}{\sqrt{5}} = 2\sqrt{5} + 2\sqrt{5} = 4\sqrt{5}$ $\dfrac{d^2y}{dx^2} = y = 4\sqrt{5} > 0$, so this is a minimum value (y is continuous and no other turning points) Hence $y \geq 4\sqrt{5}$ for all x	B1 M1A1 M1 A1 **5**	Or $12\sinh x - 8\cosh x$ Or M1 for $\tanh x = \frac{2}{3}$ A1 for $\cosh x = \frac{3}{\sqrt{5}}$ *or* $\sinh x = \frac{2}{\sqrt{5}}$
	OR $y = 2\left(e^{\frac{1}{2}x} - \sqrt{5}e^{-\frac{1}{2}x}\right)^2 + 4\sqrt{5}$ M2A1A1 $\geq 4\sqrt{5}$ A1		Or $y = 2e^{-x}(e^{x} - \sqrt{5})^2 + 4\sqrt{5}$ *SR* Showing that $2e^{x} + 10e^{-x} \geq 4\sqrt{5}$ leads to $(e^{x} - \sqrt{5})^2 \geq 0$ earns M2A1A1. Plus A1 for stating that the argument is reversible
	OR $2e^{2x} - ye^{x} + 10 = 0$ $y^2 - 80 \geq 0$ M2A1 $y \geq 4\sqrt{5}$ (or $y \leq -4\sqrt{5}$) A1 $y > 0$, so $y \geq 4\sqrt{5}$ A1		
	OR $y = R\cosh(x - \alpha)$ where $R\cosh\alpha = 12$, $R\sinh\alpha = 8$ $R^2 = 12^2 - 8^2$, $\tanh\alpha = \frac{2}{3}$ M2A1 $y = 4\sqrt{5}\cosh(x - \alpha) \geq 4\sqrt{5}$ A1 $R > 0$ since $R\cosh\alpha = 12$ A1		

3 (a)	$(c + \mathrm{j}s)^4 = c^4 + 4\mathrm{j}c^3s - 6c^2s^2 - 4\mathrm{j}cs^3 + s^4$	M1	*(M0 if binomial coefficients omitted)*
	$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$	M1 A1	$\cos 4\theta =$ real part *or* $\sin 4\theta =$ im part
	$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$	A1	
	$\tan 4\theta = \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$	A1 **5**	*No marks if result obtained without using DMT*
(b)(i)	$\lvert 2 - 2\mathrm{j} \rvert = \sqrt{8}$ $\arg(2 - 2\mathrm{j}) = -\frac{1}{4}\pi$	B1 B1 **2**	
(ii)	All the cube roots have modulus $\sqrt{2}$ Arguments are $-\frac{1}{12}\pi,\ -\frac{3}{4}\pi,\ \frac{7}{12}\pi$ 	B1 B2 B2 **5**	Allow $8^{\frac{1}{6}}$ Give B1 ft for one correct Give B1 ft for two points correct
(iii)	The three numbers all have modulus $\frac{1}{2}\sqrt{2}$ Arguments are $\frac{1}{4}\pi,\ \frac{11}{12}\pi,\ -\frac{5}{12}\pi$	M1 A1 ft M1 A2 **5**	Correct method for finding modulus of one midpoint Averaging 2 arguments (or adding π) (or adding $\frac{1}{3}\pi$) Give A1 ft for one correct
(iv)	$\lvert w \rvert = \left(\frac{1}{2}\sqrt{2}\right)^3 = \frac{1}{4}\sqrt{2}$ $\arg w = \frac{3}{4}\pi$ $w = \frac{1}{4}\sqrt{2}\left(-\frac{1}{\sqrt{2}} + \mathrm{j}\frac{1}{\sqrt{2}}\right) = -\frac{1}{4} + \frac{1}{4}\mathrm{j}$	M1 M1 A1 **3**	
	OR $w = \left(\frac{1}{2} + \frac{1}{2}\mathrm{j}\right)^3$ M1 $= \frac{1}{8} + \frac{3}{8}\mathrm{j} - \frac{3}{8} - \frac{1}{8}\mathrm{j}$ M1 $= -\frac{1}{4} + \frac{1}{4}\mathrm{j}$ A1		
	OR $w = (-\frac{1}{2})^3(2 - 2\mathrm{j})$ M2 $= -\frac{1}{4} + \frac{1}{4}\mathrm{j}$ A1		

4 (i)	$\dfrac{dy}{dx} = \dfrac{b\sec^2\theta}{a\sec\theta\tan\theta}$	M1	Or $\dfrac{2x}{a^2} - \dfrac{2y}{b^2}\dfrac{dy}{dx} = 0$
	$= \dfrac{b\sec\theta}{a\tan\theta}$	A1	Any correct form in terms of θ
	Tangent is $y - b\tan\theta = \dfrac{b\sec\theta}{a\tan\theta}(x - a\sec\theta)$	M1	*SR* B1 if result obtained by quoting
	$(b\sec\theta)x - (a\tan\theta)y = ab(\sec^2\theta - \tan^2\theta) = ab$	A1 (ag) **4**	$\dfrac{xx_1}{a^2} - \dfrac{yy_1}{b^2} = 1$
(ii)	Asymptotes are $y = \dfrac{b}{a}x$ and $y = -\dfrac{b}{a}x$	B1	For either
	For A, $(b\sec\theta)x - (b\tan\theta)x = ab$	M1	
	A is $\left(\dfrac{a}{\sec\theta - \tan\theta},\ \dfrac{b}{\sec\theta - \tan\theta}\right)$	A1	
	For B, $(b\sec\theta)x + (b\tan\theta)x = ab$	M1	
	B is $\left(\dfrac{a}{\sec\theta + \tan\theta},\ \dfrac{-b}{\sec\theta + \tan\theta}\right)$	A1	*Condone A and B interchanged, etc*
	C is $(a\cos\theta,\ 0)$	B1 **6**	
(iii)	x-coordinate of midpoint of AB is		
	$\tfrac{1}{2}\left(\dfrac{a}{\sec\theta - \tan\theta} + \dfrac{a}{\sec\theta + \tan\theta}\right)$	M1	*Or* expressions for PA^2 and PB^2
	$= \dfrac{a\sec\theta}{\sec^2\theta - \tan^2\theta} = a\sec\theta$	M1	Relevant simplification
	Hence the midpoint of AB is P	A1 **3**	
(iv)	Area OCA is $\tfrac{1}{2}(a\cos\theta)\left(\dfrac{b}{\sec\theta - \tan\theta}\right)$	M1	
	$= \dfrac{ab\cos\theta}{2(\sec\theta - \tan\theta)}$	A1 (ag)	
	OCB is $\tfrac{1}{2}(a\cos\theta)\left(\dfrac{b}{\sec\theta + \tan\theta}\right) = \dfrac{ab\cos\theta}{2(\sec\theta + \tan\theta)}$	B1 **3**	Any correct form
(v)	Area OAB is OCA + OCB	M1	
	$= (\tfrac{1}{2}ab\cos\theta)\left(\dfrac{2\sec\theta}{\sec^2\theta - \tan^2\theta}\right)$	M1	
	$= ab$ (which is independent of θ)	M1A1 **4**	

Examiner’s Report

2605 Pure Mathematics 5

General Comments

The general standard of work was good, and most candidates were able to use their knowledge of the topics in this unit, together with manipulative skills, to write some good solutions. There was a wide range of performance, with about a quarter of candidates scoring 50 marks or more (out of 60), and about a quarter scoring less than 30 marks. Q.3 was far less popular that the other three questions.

Comments on Individual Questions

Q.1 This was the best answered question, with half the attempts scoring 15 marks or more (out of 20) and about a quarter scoring full marks.

(a) This was answered well. In part (i) the signs of p and r were quite often incorrect. Part (ii) was tackled confidently, either by eliminating α and λ from the equations in part (i) or by substituting their expressions for p, q and r into both sides. Almost all candidates knew how to find the roots in part (iii).

(b) This question gave no guidance on how to proceed, but very many candidates produced excellent solutions, writing down appropriate identities and substituting $x = 4$ and $x = -2$. However, a substantial number were unable to do anything of value beyond writing f(4) = 29.

Q.2 This question was also well answered, with about a quarter of the attempts scoring 19 or 20 marks.

(a) The great majority of candidates realised that the integral should involve arcsin, although the factor $\frac{1}{\sqrt{2}}$ was often incorrect. The definite integral was very often found correctly, with just the occasional use of degrees instead of radians.

(b) The integration in part (i) was done well, although many candidates did not include the contribution from $x = 0$, and in the simplification of sinh(ln$a$) it was often thought that $e^{-\ln a} = -a$.

Most candidates rewrote the equation in part (ii) in terms of exponentials, and were able to solve it efficiently and accurately. Those who tried squaring the equation, or writing the left-hand side as $R\cosh(x - \alpha)$, were usually unsuccessful.

The inequality in part (iii) was usually derived by finding the stationary point on the curve $y = 12\cosh x - 8\sinh x$. This was done well, although 1 mark was often lost because the point was not shown to be a minimum. Another approach, used quite frequently, was to write $12\cosh x - 8\sinh x = k$ as a quadratic equation in e^x and consider the discriminant, which gives $k^2 - 80 > 0$. Here 1 mark was usually lost for concluding that $k > 4\sqrt{5}$ without stating that k must be positive.

Q.3 In part (a) most candidates could find $\sin 4\theta$ and $\cos 4\theta$ in terms of $\sin\theta$ and $\cos\theta$, but the final step, expressing $\tan 4\theta$ in terms of $\tan\theta$, defeated many.

In part (b), the modulus and argument were usually given correctly in part (i), but finding the cube roots in part (ii) caused a surprising amount of trouble; the argument was very often left unchanged, and when it was divided by 3 its negative sign was often lost. In part (iii) most candidates made a good attempt to find the midpoints, and the method for finding the cube in part (iv) was well understood.

Q.4 This question was attempted by only about 20% of candidates, and these attempts were often incomplete. The average mark was about 11, making this the worst answered question.

In part (i) the equation of the tangent was usually derived correctly, although a substantial number left the gradient in terms of x and y.

The remaining parts (ii) to (v) were often omitted, but they were well understood and often answered correctly by those who attempted them.