OCR
RECOGNISING ACHIEVEMENT

OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS 2612

Mechanics 6

Friday **25 JUNE 2004** Morning 1 hour 20 minutes

Additional materials:
Answer booklet
Graph paper
MEI Examination Formulae and Tables (MF12)

TIME 1 hour 20 minutes

INSTRUCTIONS TO CANDIDATES

- Write your Name, Centre Number and Candidate Number in the spaces provided on the answer booklet.
- Answer any **three** questions.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The allocation of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive no marks unless you show sufficient detail of the working to indicate that a correct method is being used.
- Final answers should be given to a degree of accuracy appropriate to the context.
- Take $g = 9.8 \text{ m s}^{-2}$ unless otherwise instructed.
- The total number of marks for this paper is 60.

This question paper consists of 5 printed pages and 3 blank pages.

1 *Option 1: Rotation of a rigid body*

A trap-door at the entrance to a loft is modelled as a uniform rectangular lamina of mass M and length $2a$, smoothly hinged to a wall about a horizontal axis A. The trap-door is held in a horizontal position by a catch and is then released from rest. Fig. 1 shows a cross-section through the trap-door and the wall when the trap-door has turned through an angle θ.

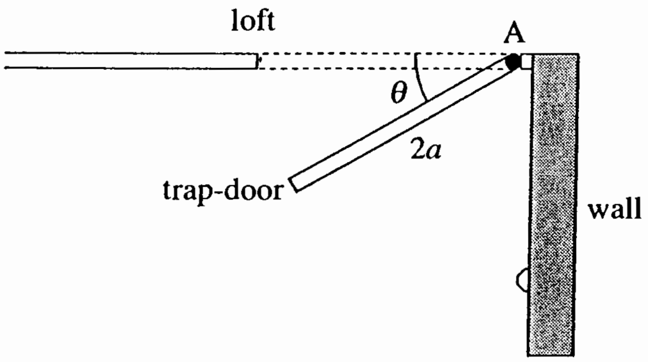


Fig. 1

(i) Find expressions for $\dot{\theta}^2$ and $\ddot{\theta}$ in terms of a, g and θ. [5]

(ii) Hence find the radial and transverse components of the reaction on the trap-door at the hinge. [6]

The trap-door is brought instantaneously to rest in a vertical position by a small doorstop which is on the wall at a distance $2a$ below A.

(iii) Show that the impulse on the hinge is $\frac{1}{6}M\sqrt{6ag}$. [5]

(iv) Show that the doorstop can be moved to a position so that the impulse on the hinge becomes zero. [4]

2 *Option 2: Vectors*

Forces $\mathbf{F_1} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \mathbf{F_2} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ and $\mathbf{F_3} = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}$ act through points with position vectors

$\mathbf{r_1} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{r_2} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ and $\mathbf{r_3} = \begin{pmatrix} a \\ 2 \\ -1 \end{pmatrix}$ respectively, where a is a constant.

(i) Find the resultant force $\mathbf{F}$. Find also the resultant torque, $\mathbf{C}$, about the origin in terms of a. [6]

A fourth force $\mathbf{F_4}$ is added, acting through a point with position vector $\mathbf{r_4}$. The system of forces is now in equilibrium.

(ii) Write down the values of $\mathbf{F_4}$ and $\mathbf{r_4} \times \mathbf{F_4}$. [2]

(iii) Explain why $\mathbf{C} . \mathbf{F_4} = 0$ and hence show that $a = \frac{43}{3}$. [4]

(iv) Given that the x-component of $\mathbf{r_4}$ is zero, calculate $\mathbf{r_4}$. [4]

(v) The force $\mathbf{F}$ is equivalent to the system of forces $\mathbf{F_1}$, $\mathbf{F_2}$ and $\mathbf{F_3}$. What is the equation of the line of action of $\mathbf{F}$? [4]

3 *Option 3: Stability and Oscillations*

A uniform rod AB of mass m and length $2a$ is smoothly hinged to the fixed point A and is free to rotate in a vertical plane. A light elastic string of natural length a and modulus $2mg$ is attached at one end to B and at the other end to a fixed point C where AC $= 2a$. The point C is on the same horizontal level as A and in the same vertical plane as the rod. The rod is at an angle θ below the horizontal as shown in Fig. 3.1.

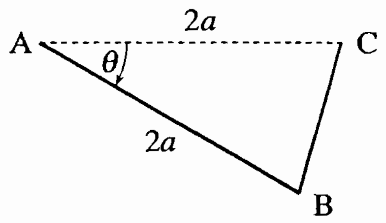


Fig. 3.1

(i) Find, in terms of θ, an expression for the potential energy of the system when the string is not slack. Hence show that the equilibrium positions occur when $f(\theta) = 0$ where

$$f(\theta) = 8\sin\theta - \cos\theta - 4\cos\tfrac{1}{2}\theta.$$ [8]

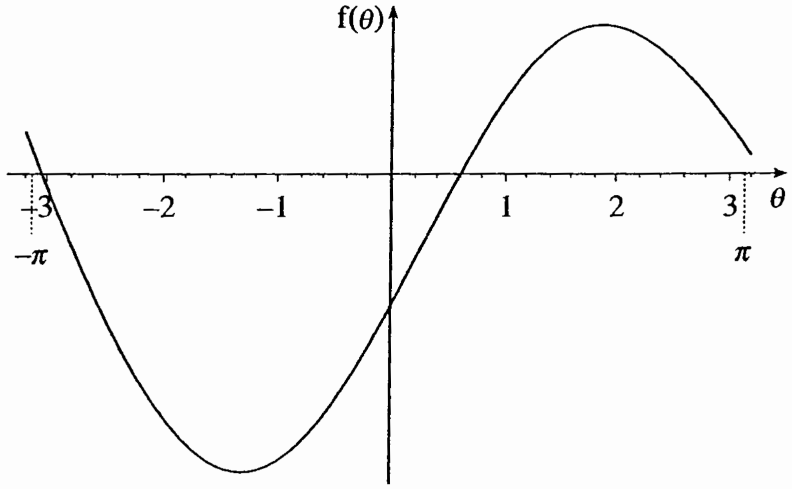


Fig. 3.2

Fig. 3.2 shows the graph of $f(\theta)$ against θ for $|\theta| \leqslant \pi$. The system has a stable equilibrium position where $\theta = \alpha$ and an unstable equilibrium position where $\theta = \beta$.

(ii) Estimate the values of α and β, explaining your reasoning with reference to the graph. Sketch the unstable position of equilibrium of the system. [6]

The rod is held slightly displaced from the position where $\theta = \alpha$ and released from rest. You are given that $f'(\alpha) \approx 7.7$, so that, for θ close to α, $f(\theta) \approx f(\alpha) + 7.7(\theta - \alpha)$.

(iii) Write down the energy equation for the system. In the case where the rod is 1 metre long, show that the period of small oscillations about the equilibrium position is approximately 0.6 seconds. [6]

4 *Option 4: Variable mass*

A small raindrop of initial mass m_0 falls from rest through a light stationary cloud. Water condenses on the raindrop so that its mass increases at a constant rate k with respect to time.

(i) Show that, if all resistance forces are neglected, the velocity of the raindrop at time t is given by

$$v = \frac{gt}{2}\left(\frac{2m_0 + kt}{m_0 + kt}\right). \qquad [6]$$

(ii) Hence find the distance fallen at time t. [6]

The model is refined to include air resistance modelled as being proportional to the speed of the raindrop.

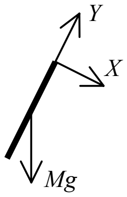
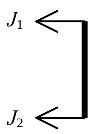
(iii) Show that the velocity, v, of the raindrop now satisfies the differential equation

$$\frac{dv}{dt} + \frac{k + \lambda}{m_0 + kt} v = g,$$

where λ is a constant.

Hence find an expression for the velocity of the raindrop at time t. [8]

Mark Scheme

1(i)	$I = \frac{4}{3}Ma^2$	B1	
	$\frac{1}{2}I\dot{\theta}^2 - Mga\sin\theta = 0$	M1	using energy
	$\dot{\theta}^2 = \dfrac{3g}{2a}\sin\theta$	F1	follow their I
	$I\ddot{\theta} = Mga\cos\theta$	M1	equation of motion or differentiate energy
	$\ddot{\theta} = \dfrac{3g}{4a}\cos\theta$	F1	follow their I
			5
(ii)	 $Y - Mg\sin\theta = Ma\dot{\theta}^2$	M1	radial N2L equation
	$= \frac{3}{2}Mg\sin\theta$	M1	substitute for $\dot{\theta}^2$
	$\Rightarrow Y = \frac{5}{2}Mg\sin\theta$	A1	
	$X + Mg\cos\theta = Ma\ddot{\theta}$	M1	transverse N2L equation
	$= \frac{3}{4}Mg\cos\theta$	M1	substitute for $\ddot{\theta}$
	$\Rightarrow X = -\frac{1}{4}Mg\cos\theta$	A1	
			6
(iii)	 $J_2 \cdot 2a = \frac{4}{3}Ma^2\dot{\theta}$	M1	angular momentum about A
	$J_1 + J_2 = Mv = Ma\dot{\theta}$	M1	linear momentum for centre of mass
	$\Rightarrow J_1 = \frac{1}{3}Ma\dot{\theta}$	dM1	dependent on both previous M marks
	$\theta = \frac{1}{2}\pi \Rightarrow \dot{\theta}^2 = \dfrac{3g}{2a}$	M1	
	$\Rightarrow J_1 = \frac{1}{3}Ma\sqrt{\dfrac{3g}{2a}} = \frac{1}{6}M\sqrt{6ag}$	E1	
			5
(iv)	If possible at distance x below A		
	$Jx = \frac{4}{3}Ma^2\dot{\theta}$	B1	follow their I
	$J = Ma\dot{\theta}$	B1	
	$\Rightarrow x = \frac{4}{3}a$	M1	eliminate J
	$\frac{4}{3}a < 2a$ so possible if doorstop is $\frac{4}{3}a$ below A	A1	for $\frac{4}{3}a$
			4

2(i)	$\mathbf{F} = \sum \mathbf{F}_i$	M1		
	$= 6\mathbf{i} + \mathbf{j} - 3\mathbf{k}$	A1		
	$\mathbf{C} = \sum \mathbf{r}_i \times \mathbf{F}_i$	M1	use of $\mathbf{r}\times\mathbf{F}$ (or $\mathbf{F}\times\mathbf{r}$)	
	$= \begin{vmatrix} \mathbf{i} & 1 & 2 \\ \mathbf{j} & 1 & -1 \\ \mathbf{k} & 0 & 1 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & 1 & 3 \\ \mathbf{j} & -1 & 2 \\ \mathbf{k} & 2 & -1 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & a & 1 \\ \mathbf{j} & 2 & 0 \\ \mathbf{k} & -1 & -3 \end{vmatrix}$	M1	attempt all vector products	
	$= (\mathbf{i} - \mathbf{j} - 3\mathbf{k}) + (-3\mathbf{i} + 7\mathbf{j} + 5\mathbf{k}) + (-6\mathbf{i} + (3a-1)\mathbf{j} - 2\mathbf{k})$	B1	one correct product (accept $\mathbf{F}\times\mathbf{r}$)	
	$= -8\mathbf{i} + (3a+5)\mathbf{j}$	A1		
				6
(ii)	$\mathbf{F} + \mathbf{F}_4 = \mathbf{0} \Rightarrow \mathbf{F}_4 = -6\mathbf{i} - \mathbf{j} + 3\mathbf{k}$	B1	– (their **F**)	
	$\mathbf{C} + \mathbf{r}_4 \times \mathbf{F}_4 = \mathbf{0} \Rightarrow \mathbf{r}_4 \times \mathbf{F}_4 = 8\mathbf{i} - (3a+5)\mathbf{j}$	B1	– (their **C**)	
				2
(iii)	$\mathbf{r}_4 \times \mathbf{F}_4 \perp \mathbf{F}_4 \Rightarrow \mathbf{C} \perp \mathbf{F}_4$	M1		
	$\Rightarrow \mathbf{C}.\mathbf{F}_4 = 0$	E1		
	$48 - (3a+5) = 0$	M1		
	$a = \frac{43}{3}$	E1		
				4
(iv)	$\mathbf{r}_4 \times \mathbf{F}_4 = -\mathbf{C}$			
	$\begin{pmatrix} 0 \\ y \\ z \end{pmatrix} \times \begin{pmatrix} -6 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 \\ -48 \\ 0 \end{pmatrix}$	M1	set up equation with vector product	
	$\begin{pmatrix} 3y+z \\ -6z \\ 6y \end{pmatrix} = \begin{pmatrix} 8 \\ -48 \\ 0 \end{pmatrix}$	M1	calculate vector product	
	$\Rightarrow y = 0, z = 8$	M1	solving	
	$\mathbf{r}_4 = 8\mathbf{k}$	A1	must state vector	
				4
(v)	line of action of $\mathbf{F}_4$ is $\mathbf{r} = \mathbf{r}_4 + \lambda\mathbf{F}_4$	B1	direction vector a multiple of $\mathbf{F}_4$	
	i.e. $\mathbf{r} = 8\mathbf{k} + \lambda(-6\mathbf{i} - \mathbf{j} + 3\mathbf{k})$	M1	reasonable attempt at line of action	
	$\mathbf{F} = -\mathbf{F}_4$ along same line of action	M1		
	so line of action of **F** is $\mathbf{r} = 8\mathbf{k} + \lambda(-6\mathbf{i} - \mathbf{j} + 3\mathbf{k})$	A1	or any equivalent form	
	Alternatively, $\mathbf{r} = \mathbf{b} + \lambda\mathbf{F}$	B1	direction vector a multiple of their **F**	
	$\mathbf{b} \times \mathbf{F} = \mathbf{C}$	M1	set up equation	
		M1	solve for **b**	
	so line of action of **F** is $\mathbf{r} = \mathbf{b} + \lambda(6\mathbf{i} + \mathbf{j} - 3\mathbf{k})$	A1	with a correct explicit vector **b**	
				4

3(i)	$V(\theta) = -mga\sin\theta + \dfrac{\lambda x^2}{2l_0}$	M1	attempt at potential energy (two terms required)
	$= -mga\sin\theta + \dfrac{2mg}{2a}(BC-a)^2$	A1	EPE term correct
	$= -mga\sin\theta + \dfrac{2mg}{2a}(4a\sin\frac{1}{2}\theta - a)^2$	M1	BC in terms of θ
	$= mga\left(-\sin\theta + (4\sin\frac{1}{2}\theta - 1)^2\right)$	A1	
	$V'(\theta) = mga\left(-\cos\theta + 2(4\sin\frac{1}{2}\theta - 1)\cdot 2\cos\frac{1}{2}\theta\right)$	M1	differentiate
	$= mga\left(-\cos\theta + 16\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta - 4\cos\frac{1}{2}\theta\right)$		
	$= mga\left(8\sin\theta - \cos\theta - 4\cos\frac{1}{2}\theta\right)$	M1	use trig. identity
	$= mga\,\mathrm{f}(\theta)$		
	equilibrium $\Leftrightarrow V'(\theta) = 0$	M1	
	$\Leftrightarrow \mathrm{f}(\theta) = 0$	E1	condition for equilibrium must be referred to
			8
(ii)	Equilibrium at $\theta \approx 0.6, -3$	B1	identify one correct value
	for stable equilibrium, $\mathrm{f}(\alpha) = 0, \mathrm{f}'(\alpha) > 0$	B1	reason for choosing α
	for unstable equilibrium, $\mathrm{f}(\beta) = 0, \mathrm{f}'(\beta) < 0$	B1	reason for choosing β
	so from graph, $\alpha \approx 0.6$, $\beta \approx -3$	B1	both correctly identified
		B2	diagram of system
			Special case if $\mathrm{f}(\theta)$ interpreted as $V(\theta)$: SC1 for equilibrium at θ at max and min of f SC1 for justifying stability for both SC1 for diagram for their β
			6
(iii)	$\frac{1}{2}I\dot{\theta}^2 + V(\theta) = \text{constant}$	M1	energy
	$\frac{1}{2}\left(\frac{4}{3}ma^2\right)\dot{\theta}^2 + V(\theta) = \text{constant}$	B1	correct moment of inertia
	$\frac{4}{3}ma^2\ddot{\theta}\dot{\theta} + V'(\theta)\dot{\theta} = 0$	M1	differentiating an energy equation
	$\ddot{\theta} + \dfrac{3g}{4a}\mathrm{f}(\theta) = 0$	A1	
	$\mathrm{f}(\theta) \approx \mathrm{f}(\alpha) + 7.7(\theta - \alpha) = 7.7(\theta - \alpha)$		
	$\ddot{\theta} + 7.7\left(\dfrac{3g}{4a}\right)(\theta - \alpha) \approx 0$	M1	use approximation to get equation of motion
	$a = 0.5 \Rightarrow T \approx 2\pi\sqrt{\dfrac{4\times 0.5}{7.7\times 3g}} \approx 0.6$	E1	
			6

4(i)	$\frac{\mathrm{d}m}{\mathrm{d}t} = k \Rightarrow m = m_0 + kt$	B1		
	$\frac{\mathrm{d}}{\mathrm{d}t}(mv) = mg$	M1	using N2L in variable mass form	
	$mv = \int mg\mathrm{d}t = \int (m_0 + kt)g\mathrm{d}t$	M1	integrating	
	$= m_0 gt + \frac{1}{2}gkt^2 + A$	A1	including constant	
	$t = 0, v = 0 \Rightarrow A = 0$	M1	use condition on v	
	$v = \frac{gt(2m_0 + kt)}{2(m_0 + kt)}$	E1		
				6
(ii)	$x = \int \frac{gt(2m_0 + kt)}{2(m_0 + kt)}\mathrm{d}t$	M1		
	$= \int \frac{gt}{2}\left(1 + \frac{m_0}{m_0 + kt}\right)\mathrm{d}t = \int \frac{g}{2}\left(t + \frac{m_0}{k} - \frac{m_0{}^2}{k}\left(\frac{1}{m_0 + kt}\right)\right)\mathrm{d}t$	M1	rearranging into suitable form for integration	
	$= \frac{g}{2}\left(\frac{1}{2}t^2 + \frac{m_0}{k}t - \frac{m_0}{k^2}\ln(m_0 + kt)\right) + B$	M1 A1	integrating	
	$t = 0, x = 0 \Rightarrow B = \frac{g}{2}\left(\frac{m_0^2}{k^2}\ln m_0\right)$	M1	use condition on x	
	$x = \frac{g}{2}\left(\frac{1}{2}t^2 + \frac{m_0}{k}t - \frac{m_0}{k^2}\ln\left(1 + \frac{kt}{m_0}\right)\right)$	A1		
				6
(iii)	$\frac{\mathrm{d}}{\mathrm{d}t}(mv) = mg - \lambda v$	M1	using N2L in variable mass form	
	$\Rightarrow m\frac{\mathrm{d}v}{\mathrm{d}t} + vk = mg - \lambda v$	M1	using product rule and rearranging	
	$\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{k + \lambda}{m_0 + kt}v = g$	E1		
	$I = \exp\left(\int \frac{k + \lambda}{m_0 + kt}\mathrm{d}t\right) = \exp\left(\left(1 + \frac{\lambda}{k}\right)\ln(m_0 + kt)\right)$	M1		
	$= (m_0 + kt)^{1 + \lambda/k}$	A1	simplified	
	$\frac{\mathrm{d}}{\mathrm{d}t}\left((m_0 + kt)^{1 + \lambda/k}v\right) = (m_0 + kt)^{1 + \lambda/k}g$			
	$(m_0 + kt)^{1 + \lambda/k}v = \frac{1}{2k + \lambda}(m_0 + kt)^{2 + \lambda/k}g + C$	M1	integrating	
	$t = 0, v = 0 \Rightarrow C = -\frac{m_0{}^{2 + \lambda/k}}{2k + \lambda}g$	M1	use condition on v	
	$v = \frac{g}{2k + \lambda}\left(m_0 + kt - \frac{m_0{}^{2 + \lambda/k}}{(m_0 + kt)^{1 + \lambda/k}}\right)$	A1		
				8

Examiner’s Report

2612 Mechanics 6

General Comments

There were some good scripts, although virtually all candidates ran into difficulties at some point. Question 3 was the most popular with all but 2 candidates attempting it. Questions 2 and 4 were also very popular, but only 15 of the 40 candidates attempted question 1. Question 2 attracted the most successful responses.

Comments on Individual Questions

Q.1 The early parts of the question were well done, but candidates often found the use of momentum and angular momentum difficult in the last two parts of the question. Most responses omitted the impulse at the hinge in part (iii) where it was necessary, and then included it in part (iv) when it was zero! Most responses did not appreciate the need for two separate impulse-momentum equations, i.e. linear and angular.

Q.2 This question was usually done very well. Most candidates were very confident in their use of vector product. The main difficulty was found in justifying $\mathbf{C.F}_4 = 0$. Many candidates thought that stating that they were perpendicular was sufficient, but showing that they were perpendicular was the key step! The last part of the question was often done well, with some candidates using the line of action of $\mathbf{F}_4$, as led by the question, and others calculating it from scratch.

Q.3 Candidates understood the general principles of this topic, but many had difficulties with the technicalities. A particular problem arose with many candidates confusing f(θ) with the potential energy and therefore misreading the graph. Some allowance for this was made in the mark scheme. In the final part of the question, some candidates did not realise that they needed to differentiate the energy equation to obtain the equation of motion.

Q.4 The first part of the question was often well done, but many candidates unnecessarily expanded $\frac{\mathrm{d}}{\mathrm{d}t}(mv)$ and then used an integrating factor, rather than the simpler approach of integrating it directly. Many attempts in part (ii) were hampered by difficulties with the integration, with candidates not realising that they needed to divide (as polynomials) to find a suitable expression for the integrand. Part (iii) was usually more successful, although algebraic slips were common.