OCR
RECOGNISING ACHIEVEMENT

OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS 2605

Pure Mathematics 5

Monday **13 JANUARY 2003** Afternoon 1 hour 20 minutes

Additional materials:
Answer booklet
Graph paper
MEI Examination Formulae and Tables (MF12)

TIME 1 hour 20 minutes

INSTRUCTIONS TO CANDIDATES

- Write your Name, Centre Number and Candidate Number in the spaces provided on the answer booklet.
- Answer any **three** questions.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The allocation of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive no marks unless you show sufficient detail of the working to indicate that a correct method is being used.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The total number of marks for this paper is 60.

This question paper consists of 3 printed pages and 1 blank page.

[Turn over

1 **(i)** Starting from the definition $\cosh x = \frac{1}{2}(e^x + e^{-x})$, prove that

$$2\cosh^2 x - 1 = \cosh 2x.$$ [3]

(ii) Solve the equation

$$\cosh 2x - 10\cosh x + 13 = 0,$$

giving the answers in logarithmic form. [7]

(iii) Show that the curve $y = \cosh 2x - 10\cosh x + 13$ has three stationary points, and find the y-coordinates of the stationary points. [8]

(iv) Sketch the curve $y = \cosh 2x - 10\cosh x + 13$. [2]

2 **(a)** Using de Moivre's theorem, express $\sin 5\theta$ in terms of $\sin\theta$. [6]

(b) **(i)** By considering $\left(z + \frac{1}{z}\right)^6$, where $z = \cos\theta + \mathrm{j}\sin\theta$, show that

$$\cos^6\theta = \tfrac{1}{32}\cos 6\theta + \tfrac{3}{16}\cos 4\theta + \tfrac{15}{32}\cos 2\theta + \tfrac{5}{16}.$$ [7]

(ii) Hence find $\int_0^{\frac{1}{2}} (1 - x^2)^{\frac{5}{2}}\,dx$, giving your answer in an exact form. [7]

3 **(a)** The equation $x^4 - x^3 + 2x^2 + 5x - 1 = 0$ has roots α, β, γ and δ.

Find a quartic equation with integer coefficients which has roots α^2, β^2, γ^2 and δ^2. [7]

(b) When the polynomial $f(x) = 4x^6 + kx^5 + mx^2$ is divided by $(x + 2)(2x - 1)$, the quotient is $g(x)$ and the remainder is $-43x + 42$, so that

$$f(x) = (x + 2)(2x - 1)g(x) - 43x + 42.$$

(i) Find the integers k and m, and show that $f'(-2) = 32$. [7]

(ii) Find the remainder when $f(x)$ is divided by $(x + 2)$. [1]

(iii) Find the remainder when $f(x)$ is divided by $(x + 2)^2$. [5]

4 **(a)** A parabola has parametric equations $x = at^2$, $y = 2at$.

(i) Show that the chord joining the points $P_1(at_1^2, 2at_1)$ and $P_2(at_2^2, 2at_2)$ on the parabola has equation

$$(t_1 + t_2)y = 2x + 2at_1t_2.$$ [4]

(ii) Hence or otherwise find the equation of the tangent to the parabola at a general point $(at^2, 2at)$. [3]

The tangents to the parabola at P_1 and P_2 meet at the point (p, q).

(iii) Show that $t_1t_2 = \frac{p}{a}$, and find an expression for $t_1 + t_2$. [4]

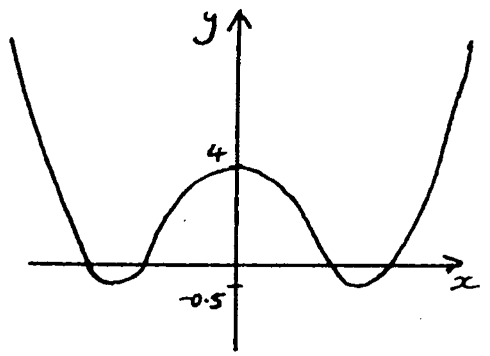
(iv) Show that P_1P_2 crosses the x-axis at the point $(-p, 0)$. [3]

(b) A conic has polar equation $\frac{5}{r} = 2 + 3\cos\theta$.

(i) Find the eccentricity, and state which type of conic the equation represents. [2]

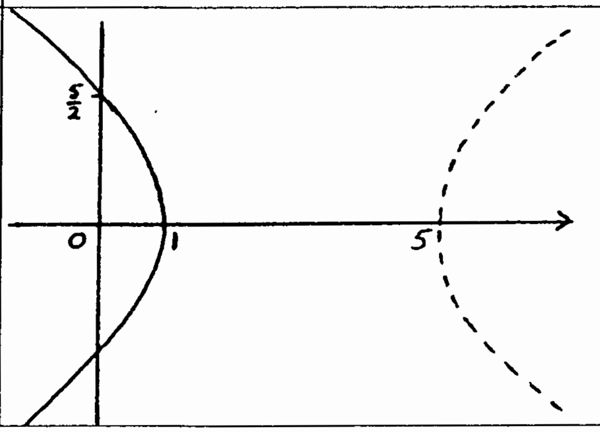
(ii) Sketch the conic, using a continuous line for sections where $r > 0$ and a broken line for sections where $r < 0$. [4]

Mark Scheme

1 (i)	LHS $= 2\left(\frac{1}{2}(e^x + e^{-x})\right)^2 - 1$ $= \frac{1}{2}(e^{2x} + 2 + e^{-2x}) - 1$ $= \frac{1}{2}(e^{2x} + e^{-2x}) = \cosh 2x$	B1 M1 A1 3	For $(e^{2x} + 2 + e^{-2x})$ Using $\cosh 2x = \frac{1}{2}(e^{2x} + e^{-2x})$ Completion
(ii)	$2\cosh^2 x - 1 - 10\cosh x + 13 = 0$ $\cosh^2 x - 5\cosh x + 6 = 0$ $\cosh x = 2,\ 3$ $x = \pm \operatorname{arcosh} 2,\ \pm \operatorname{arcosh} 3$ $= \pm \ln(2 + \sqrt{3}),\ \pm \ln(3 + \sqrt{8})$	M1 M1 A1A1 A1 ft M1 A1 7	Obtaining quadratic in $\cosh x$ Solving to obtain a value of $\cosh x$ For $\pm$ Conversion to logarithmic form
(iii)	$\frac{dy}{dx} = 2\sinh 2x - 10\sinh x$ $= 4\sinh x\cosh x - 10\sinh x$ $= 2\sinh x(2\cosh x - 5)$ $= 0$ when $x = 0$ or $x = \pm \operatorname{arcosh}\frac{5}{2}$ When $x = 0$, $y = 1 - 10 + 13 = 4$ When $\cosh x = \frac{5}{2}$, $y = 2\left(\frac{5}{2}\right)^2 - 1 - 10 \times \frac{5}{2} + 13 = -0.5$	B1B1 M1 B1 A1 B1 M1A1 8	Using $\sinh 2x = 2\sinh x\cosh x$ $\pm$ required
(iv)		B2 cao 2	Fully correct shape and correctly positioned *(No need for any numbers on axes)* Give B1 ft for a curve consistent with results obtained in (ii) and (iii)

2 (a)	$\cos 5\theta + \mathrm{j}\sin 5\theta = (\cos\theta + \mathrm{j}\sin\theta)^5$		
	$= \cos^5\theta + 5\cos^4\theta(\mathrm{j}\sin\theta) + 10\cos^3\theta(\mathrm{j}\sin\theta)^2 + \dots$	M1	Expanding $(\cos\theta + \mathrm{j}\sin\theta)^5$
		M1	$\sin 5\theta = \mathrm{Im}(\cos\theta + \mathrm{j}\sin\theta)^5$
	$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$	A2	Give A1 for 3 terms correct
	$= 5(1-s^2)^2 s - 10(1-s^2)s^3 + s^5$	M1	
	$= 5\sin\theta - 20\sin^3\theta + 16\sin^5\theta$	A1 6	
(b)(i)	$z + \frac{1}{z} = 2\cos\theta$, $\left(z + \frac{1}{z}\right)^6 = 64\cos^6\theta$	B1B1	
	$\left(z + \frac{1}{z}\right)^6 = z^6 + 6z^4 + 15z^2 + 20 + \frac{15}{z^2} + \frac{6}{z^4} + \frac{1}{z^6}$	M1A1	
	$= 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20$	M2	Give M1 for one cos term
	$\cos^6\theta = \frac{1}{32}\cos 6\theta + \frac{3}{16}\cos 4\theta + \frac{15}{32}\cos 2\theta + \frac{5}{16}$	A1 (ag) 7	
(ii)	Putting $x = \sin\theta$, the integral becomes	M1	
	$\int_0^{\frac{1}{6}\pi} (1 - \sin^2\theta)^{\frac{5}{2}} \cos\theta \, \mathrm{d}\theta$	A1A1	
	$= \int_0^{\frac{1}{6}\pi} \cos^6\theta \, \mathrm{d}\theta$	A1	Limits required (may be implied by later work)
	$= \left[\frac{1}{192}\sin 6\theta + \frac{3}{64}\sin 4\theta + \frac{15}{64}\sin 2\theta + \frac{5}{16}\theta\right]_0^{\frac{1}{6}\pi}$	B1	
	$= 0 + \frac{3}{64}\frac{\sqrt{3}}{2} + \frac{15}{64}\frac{\sqrt{3}}{2} + \frac{5}{16}\frac{\pi}{6}$	M1	Exact substitution
	$= \frac{9\sqrt{3}}{64} + \frac{5\pi}{96}$	A1 7	

3 (a)	Let $y = x^2$, $x = \sqrt{y}$	M1	
	$y^2 - y^{\frac{3}{2}} + 2y + 5y^{\frac{1}{2}} - 1 = 0$	A2	Give A1 for 3 terms correct
	$(y^2 + 2y - 1)^2 = \left(y^{\frac{3}{2}} - 5y^{\frac{1}{2}} \right)^2$	M1	Eliminating fractional powers
	$y^4 + 4y^2 + 1 + 4y^3 - 2y^2 - 4y = y^3 - 10y^2 + 25y$	A1A1	
	$y^4 + 3y^3 + 12y^2 - 29y + 1 = 0$	A1 7	
	OR $\sum\alpha = 1$, $\sum\alpha\beta = 2$, $\sum\alpha\beta\gamma = -5$, $\alpha\beta\gamma\delta = -1$ $\sum\alpha^2 = (\sum\alpha)^2 - 2\sum\alpha\beta = -3$ B1 $\sum\alpha^2\beta^2 = (\sum\alpha\beta)^2 - 2\sum\alpha^2\beta\gamma - 6\alpha\beta\gamma\delta$ $= (\sum\alpha\beta)^2 - 2\sum\alpha\sum\alpha\beta\gamma + 2\alpha\beta\gamma\delta$ $= 12$ M1A1 $\sum\alpha^2\beta^2\gamma^2 = (\sum\alpha\beta\gamma)^2 - 2\alpha\beta\gamma\delta\sum\alpha\beta$ $= 29$ M1A1 $\alpha^2\beta^2\gamma^2\delta^2 = 1$ Equation is $y^4 + 3y^3 + 12y^2 - 29y + 1 = 0$ M1A1		
(b)(i)	Putting $x = -2$, $f(-2) = 86 + 42 = 128$	M1	Substituting $x = -2$ *or* $x = \frac{1}{2}$
	$256 - 32k + 4m = 128$	A1	
	$-8k + m = -32$		
	Putting $x = \frac{1}{2}$, $f(\frac{1}{2}) = -\frac{43}{2} + 42 = \frac{41}{2}$		
	$\frac{4}{64} + \frac{k}{32} + \frac{m}{4} = \frac{41}{2}$	A1	
	$k + 8m = 654$		
	Solving, $k = 14$, $m = 80$	M1A1	
	$f(x) = 4x^6 + 14x^5 + 80x^2$		
	$f'(x) = 24x^5 + 70x^4 + 160x$	M1	Substituting *k*, *m* and differentiating
	$f'(-2) = -768 + 1120 - 320 = 32$	A1 (ag) 7	
(ii)	Remainder is $f(-2) = 128$	B1 1	
(iii)	Let $f(x) = (x+2)^2 h(x) + ax + b$	B1	
	Putting $x = -2$, $128 = -2a + b$	B1	
	$f'(x) = (x+2)^2 h'(x) + 2(x+2)h(x) + a$	M1	Use of product rule
	Putting $x = -2$, $32 = a$	A1	*Must be correctly obtained*
	Hence $b = 192$, and remainder is $32x + 192$	B1 5	
	OR By long division, quotient is M2		Fully correct method leading to a linear remainder
	$4x^4 - 2x^3 - 8x^2 + 40x - 48$ A2		Give A1 ft for first three terms $4x^4 + (k-16)x^3 + (48-4k)x^2 + \ldots$
	remainder is $32x + 192$ B1		

4(a)(i)	Gradient is $\frac{2at_2 - 2at_1}{at_2^2 - at_1^2} = \frac{2}{t_1 + t_2}$ Equation is $y - 2at_1 = \frac{2}{t_1 + t_2}(x - at_1^2)$ $(t_1 + t_2)y - 2at_1^2 - 2at_1t_2 = 2x - 2at_1^2$ $(t_1 + t_2)y = 2x + 2at_1t_2$	M1A1 M1 A1 (ag) 4	
(ii)	Let $t_1 \to t$, $t_2 \to t$ in equation of chord, Tangent is $2ty = 2x + 2at^2$ $ty = x + at^2$	M2 A1 3	
	OR Gradient $\frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t}$ M1 Equation is $y - 2at = \frac{1}{t}(x - at^2)$ M1A1		
(iii)	Tangent passes through (p, q) if $tq = p + at^2$ $at^2 - qt + p = 0$ t_1, t_2 are the roots of this equation, so $t_1t_2 = \frac{p}{a}$ $t_1 + t_2 = \frac{q}{a}$	B1 ft M1 A1 (ag) A1 4	Sum *or* product of roots
	OR Tangents $t_1y = x + at_1^2$, $t_2y = x + at_2^2$ meet at $(at_1t_2,\ a(t_1 + t_2))$ M1A1 $at_1t_2 = p$, so $t_1t_2 = \frac{p}{a}$ A1 $a(t_1 + t_2) = q$, so $t_1 + t_2 = \frac{q}{a}$ A1		
(iv)	Equation of P_1P_2 is $\frac{q}{a}y = 2x + 2a\frac{p}{a}$ When $y = 0$, $0 = 2x + 2p$ $x = -p$	M1 M1 A1 (ag) 3	
(b)(i)	$e = \frac{3}{2}$; hyperbola	B1B1 2	
(ii)		B1 B1 B1 B1 4	LH branch: correct shape RH branch: correct shape Intercepts marked Correct use of continuous and broken lines

Examiner’s Report

2605 Pure Mathematics 5

General Comments

The candidates for this paper covered the full range of ability. There were some excellent scripts, with 30% of candidates scoring 50 marks or more (out of 60), but about a third scored less than 30 marks. Question 4 (which was attempted by only 30% of candidates) was much less popular than the other three questions.

Comments on Individual Questions

Question 1 (Hyperbolic functions)

The proof in part (i) was very well done; just a few candidates ignored the given starting point and assumed results such as $\cosh 2x = \cosh^2 x + \sinh^2 x$. In part (ii) most candidates knew how to solve the equation, although the negative roots were often omitted. Some wrote the equation in terms of exponentials, without using the result from part (i), but this method was very rarely successful. Part (iii) caused some difficulty, as many candidates were unable to solve $2\sinh 2x - 10\sinh x = 0$. Sketching the graph in part (iv) was generally well done.

(ii) $x = \pm\ln(2+\sqrt{3}),\ \pm\ln(3+\sqrt{8})$; (iii) $4,\ -0.5,\ -0.5$.

Question 2 (Complex numbers and Integration)

This was the best answered question, with half the attempts scoring 16 marks or more (out of 20), and a quarter scoring full marks. The applications of de Moivre's theorem to trigonometry in parts (a) and (b)(i) were well understood. The work in part (b)(ii) was most impressive, and the correct value of the definite integral was frequently obtained.

(a) $5\sin\theta - 20\sin^3\theta + 16\sin^5\theta$; (b)(ii) $\dfrac{9\sqrt{3}}{64} + \dfrac{5\pi}{96}$.

Question 3 (Algebra)

This was the worst answered question, with an average mark of about 12. In part (a) most candidates used the substitution $y = x^2$ but very many were unable to eliminate the fractional powers correctly. Those who used sums of products of the roots were very rarely successful. Parts (b)(i) and (ii) were generally well answered, although arithmetic slips were very common. The final part (b)(iii) was sometimes omitted, but most proceeded by differentiating an appropriate identity, and a few preferred long division; a fair proportion obtained the correct remainder.

(a) $y^4 + 3y^3 + 12y^2 - 29y + 1 = 0$; (b)(i) $k = 14,\ m = 80$; (ii) 128; (iii) $32x + 192$.

Question 4 (Conics)

There were just a few serious attempts at this question, and some fragments in which just one or two parts were attempted. The final part (b)(ii) was found difficult and the hyperbola was rarely shown in the correct position. All the other parts were well understood in the sense that whenever they were attempted they were usually answered correctly.

(a)(ii) $ty = x + at^2$; (iii) $t_1 + t_2 = \dfrac{q}{a}$; (b)(i) $e = \frac{3}{2}$, hyperbola.