OCR
RECOGNISING ACHIEVEMENT

Oxford, Cambridge and RSA Examinations

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS 2602/1

Pure Mathematics 2

Tuesday **5 JUNE 2001** Morning 1 hour 20 minutes

Additional materials:
Answer booklet
Graph paper
MEI Examination Formulae and Tables (MF12)

TIME 1 hour 20 minutes

INSTRUCTIONS TO CANDIDATES

Write your Name, Centre Number and Candidate Number in the spaces provided on the answer booklet.

Answer **all** questions.

You are permitted to use only a scientific calculator for this paper.

INFORMATION FOR CANDIDATES

The approximate allocation of marks is given in brackets [ ] at the end of each question or part question.

You are advised that an answer may receive no marks unless you show sufficient detail of the working to indicate that a correct method is being used.

Final answers should be given to a degree of accuracy appropriate to the context.

The total number of marks for this paper is 60.

This question paper consists of 5 printed pages and 3 blank pages.

1 A forest contains oaks, beeches and pine trees.

(i) The number of oak trees at the end of n years is modelled by u_n, where

$$u_{n+1} = 0.8\,u_n + 25,$$

and u_0 represents the initial number of oak trees.

Calculate u_1, u_2 and u_3 in the cases when

(*A*) $u_0 = 250$,

(*B*) $u_0 = 125$.

Comment briefly on your results. [5]

(ii) The number of beeches at the end of n years is modelled by v_n, where

$$v_{n+1} = rv_n$$

and r is a constant. The initial number of beeches is v_0, where $v_0 = 1000$.

Show that the numbers of beeches at the end of 1, 2, 3, ... years form a geometric progression. If the number of beeches halves after 10 years, find the value of r, giving your answer correct to 2 decimal places. [4]

(iii) The number of pines at the end of n years is modelled by w_n, where

$$w_{n+1} = w_n + 10(n+1)$$

and initially there are no pine trees, so $w_0 = 0$.

Using this model,

$$w_1 = 0 + 10 \times 1 = 10,$$
$$w_2 = 10 + 10 \times 2 = 10 + 20,$$
$$w_3 = 10 + 20 + 10 \times 3 = 10 + 20 + 30.$$

Write down a similar expression for w_4. Show that

$$w_n = 5n(n+1).$$

At the end of Y years, the number of pines first exceeds 1000. Find Y. [5]

[Total: 14]

2 Fig. 2 shows a sketch of the graph of $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = \frac{\ln x}{x} \qquad (x > 0).$$

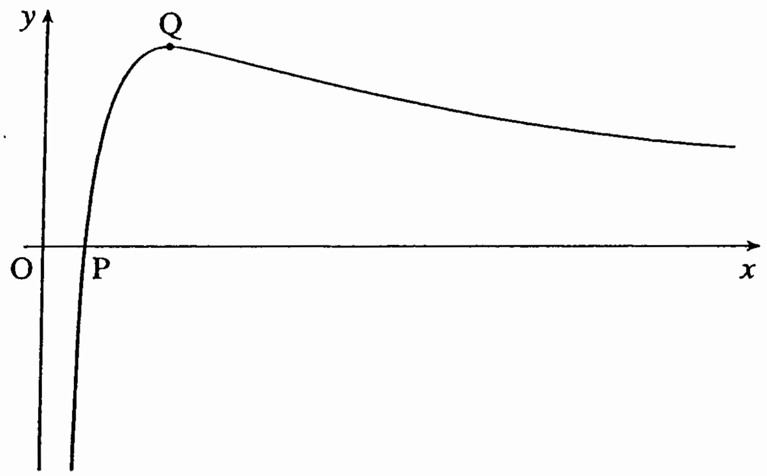


Fig. 2

The graph crosses the x-axis at the point P and has a turning point at Q.

(i) Write down the x-coordinate of P. [1]

(ii) Find the first and second derivatives $\mathrm{f}'(x)$ and $\mathrm{f}''(x)$, simplifying your answers as far as possible. [5]

(iii) Hence show that the x-coordinate of Q is e. Find the y-coordinate of Q in terms of e.

Find $\mathrm{f}''(\mathrm{e})$, and use this result to verify that Q is a maximum point. [5]

(iv) Find the exact area of the finite region between the graph $y = \mathrm{f}(x)$, the x-axis, and the line $x = 2$. [4]

[Total: 15]

3 A new car is tested for the amount of petrol it uses. Suppose the rate of consumption at v miles per hour (mph) is p miles per gallon.

At a steady 49 mph, its rate of consumption is 45 miles per gallon. At a steady 81 mph, its rate of consumption is 35 miles per gallon.

One model for the petrol consumption is

$$p = ab^v,$$

where a and b are positive constants.

(i) Show that plotting $\ln p$ against v gives a straight line graph if this model is appropriate. [2]

(ii) Fig. 3 shows a straight line drawn through the points $(49, \ln 45)$ and $(81, \ln 35)$. Use the graph to find the petrol consumption of the car at 25 mph and at 64 mph according to this model. [3]

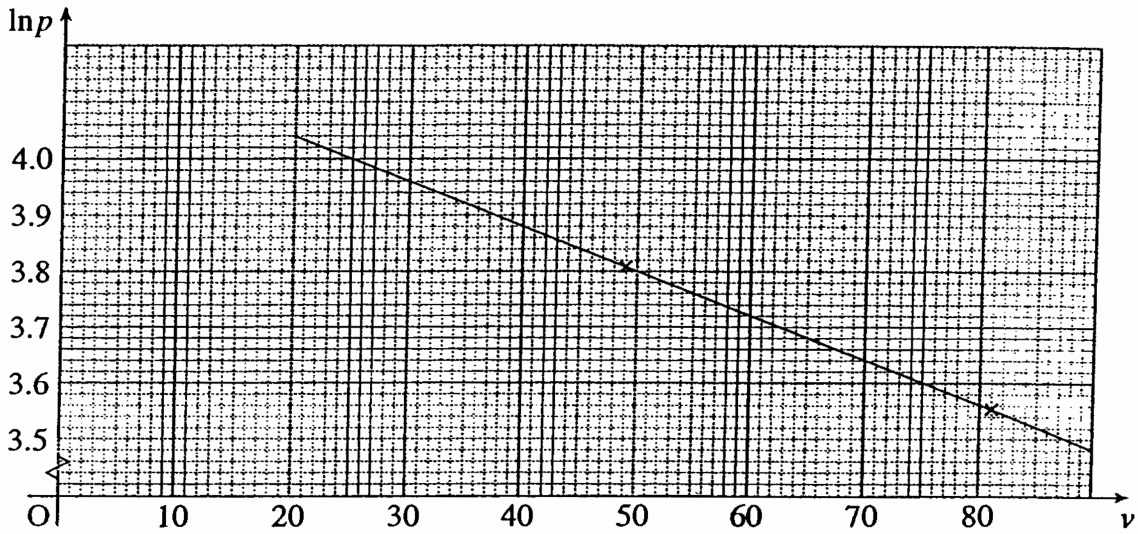


Fig. 3

An alternative model for the petrol consumption is

$$p = cv^{-d},$$

where c and d are positive constants.

(iii) Show that, using this model,

$$d = \frac{\ln 45 - \ln 35}{\ln 81 - \ln 49}.$$

Use the laws of logarithms to simplify this expression, and hence show that $d = \frac{1}{2}$. Show also that $c = 315$.

Find the petrol consumption of the car at 25 mph and at 64 mph according to this model. [9]

(iv) Further testing of the car yields the results $v = 25, p = 55$ and $v = 64, p = 40$. Comment on the suitability of the two models. [2]

[Total: 16]

Mark Scheme

4 Fig. 4.1 shows a sketch of the graph of $y = f(x)$, for $0 \leqslant x \leqslant 4$, where

$$f(x) = \sqrt{4 - x}.$$

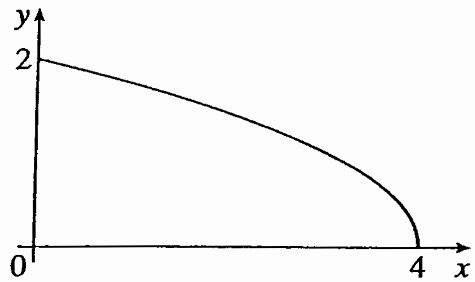


Fig. 4.1

(i) Find the gradient of the graph where $x = 0$. [3]

(ii) Find the inverse function $f^{-1}(x)$. Copy Fig. 4.1, and draw the graph of $y = f^{-1}(x)$ on the same diagram. What is the connection between the graph of $y = f(x)$ and the graph of $y = f^{-1}(x)$? [5]

(iii) Figs. 4.2, 4.3 and 4.4 below show the graph of $y = f(x)$, together with the graphs of $y = f_1(x)$, $y = f_2(x)$ and $y = f_3(x)$ respectively, each of which is a simple transformation of the graph of $y = f(x)$.

Find expressions in terms of x for each of the functions $f_1(x)$, $f_2(x)$ and $f_3(x)$. [3]

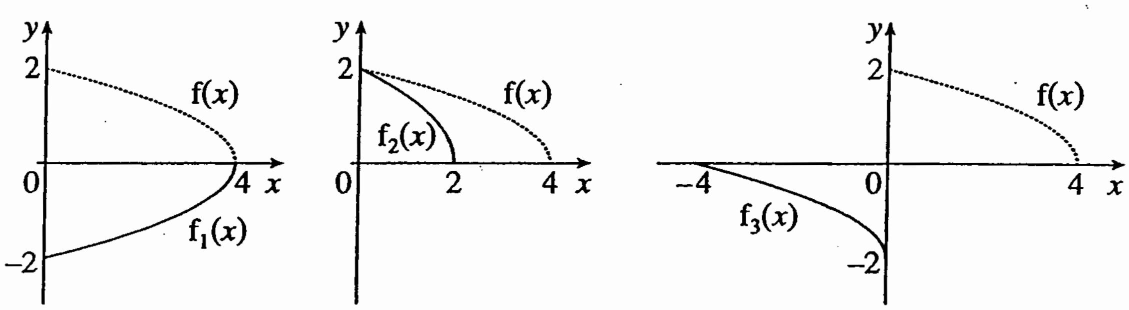


Fig. 4.2 **Fig. 4.3** **Fig. 4.4**

(iv) The function $g(x)$ is defined in such a way that the composite function $gf(x)$ is given by

$$gf(x) = x - 4.$$

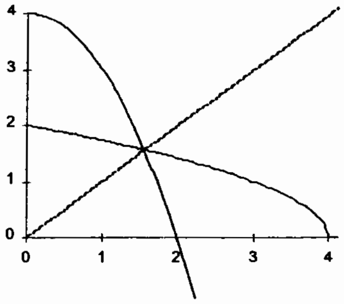
Find the functions $g(x)$ and $fg(x)$. [4]

[Total: 15]

1(i) $(A)\ u_0 = 250$ $u_1 = 0.8 \times 250 + 25 = 225$ $u_2 = 0.8 \times 225 + 25 = 205$ $u_3 = 0.8 \times 205 + 25 = 189$ $(B)\ u_0 = 125$ $u_1 = 0.8 \times 125 + 25 = 125$ $u_2 = 0.8 \times 125 + 25 = 125$ $u_3 = 0.8 \times 125 + 25 = 125$ Number of trees is declining. Number of trees is constant.	B1 B1 B1 B1 B1 [5]	225 205 and 189 cao 125, 125, 125 'Converging to 125' B1 B1 No ft on their u_1, u_2, u_3
(ii) $v_0 = 1000$ $v_1 = 1000\,r$ $v_2 = 1000\,r^2 \ldots$ So GP with common ratio r. $v_{10} = 1000\,r^{10} = 500$ $\Rightarrow\ r^{10} = 0.5$ $\Rightarrow\ r = 0.93(\ldots)$	B1 M1 M1 A1 [4]	1000, $1000r$, $1000r^2$, … or 'common ratio between terms' Accept 'common ratio is r' $1000r^{10} = 500$ condone $1000r^9 = 500$ solves using $0.5^{0.1}$ or logs or trial and improvement (soi) $R = 0.93$ www Unsupported (viz no equation quoted) $R = 0.93$ SCB1
(iii) $w_4 = 10 + 20 + 30 + 10 \times 4$ $= 10 + 20 + 30 + 40$ $w_n = 10 + 20 + 30 + \ldots + 10n$ A.P. with $a = 10$, $d = 10$ $w_n = \frac{n}{2}[20 + (n-1)10]$ $= \frac{n}{2}[10 + 10n]$ $= 5n(n+1)$ * $5Y(Y+1) > 1000$ $\Rightarrow\ Y(Y+1) > 200$ $\Rightarrow\ Y = 14$	B1 M1 E1 M1 A1 [5]	10+20+30+10×4 or 10+20+30+40 or $\frac{n+1}{2}[2 \times 0 + n \times 10]$ $= 5n(n+1)$ * www Allow equality. Trial and improvement (soi) or quadratic method, but must be solving correct equation, with correct discriminant. $Y = 14$ cao www.

2(i) $\ln x = 0 \Rightarrow x = 1$	B1 [1]	$x = 1$.
(ii) $f'(x) = \dfrac{x.\frac{1}{x} - \ln x.1}{x^2} = \dfrac{1-\ln x}{x^2}$ *or* $x^{-1}.\frac{1}{x} - x^{-2}\ln x = x^{-2} - x^{-2}\ln x$ $f''(x) = \dfrac{x^2.(-\frac{1}{x}) - (1-\ln x).2x}{x^4}$ $= \dfrac{-x - 2x + 2x\ln x}{x^4}$ $= \dfrac{2\ln x - 3}{x^3}$	M1 A1 A1 M1 A1 A1 M1 A1 [5]	Quotient rule (consistent with their derivatives) correct numerator $\dfrac{1-\ln x}{x^2}$ cao Product rule correct expression simplified correctly (allow –ve indices) Any expression for $f''(x)$ consistent with their $f'(x)$. Condone missing brackets $\dfrac{2\ln x - 3}{x^3}$ or $\dfrac{2x\ln x - 3x}{x^4}$ or $2x^{-3}\ln x - 3x^{-3}$
(iii) $f'(x) = 0 \Rightarrow 1 - \ln x = 0$ $\Rightarrow x = \text{e}$ * *or* $f'(\text{e}) = \dfrac{1-\ln \text{e}}{\text{e}^2} = 0$ when $x = \text{e}, y = \dfrac{\ln \text{e}}{\text{e}} = \dfrac{1}{\text{e}}$ $f''(\text{e}) = \dfrac{2\ln \text{e} - 3}{\text{e}^3} = -\dfrac{1}{\text{e}^3} < 0$ $f''(\text{e}) < 0 \Rightarrow$ Q is a maximum point	M1 E1 B1 M1 A1 cao [5]	Their $f'(x) = 0$ soi or calculates $f'(\text{e})$ $\Rightarrow x = \text{e}$ or $1 - \ln \text{e} = 0$ www $y = \dfrac{1}{\text{e}}$ Substituting e into their $f''(x)$ $f''(x) = -\dfrac{1}{\text{e}^3}$ or $-0.0498\ldots < 0$ $\Rightarrow$ Q is a maximum point [must evaluate $f''(\text{e})$]
(iv) $A = \int_1^2 \dfrac{\ln x}{x} dx$ let $u = \ln x \Rightarrow du$ $= \dfrac{1}{x}\text{d}x$ $= \int_{\ln 1}^{\ln 2} u\,\text{d}u$ $= \left[\frac{1}{2}u^2\right]_0^{\ln 2}$ $= \frac{1}{2}(\ln 2)^2$	M1 M1 A1 A1cao [4]	Correct integral and limits. Using substitution $u = \ln x$ to get $\int u\,\text{d}u$ $\left[\frac{1}{2}u^2\right]$ $\left[\frac{1}{2}(\ln x)^2\right]$ by inspection SC B2 www

3 (i) $p = a\,b^v \Rightarrow \ln p = \ln a + v \ln b$ this is of form $y = m\,x + c$	M1 A1 [2]	$\ln p = \ln a + v \ln b$ compares with straight line equation
(ii) $v = 25, \Rightarrow \ln p = 4.0$ $\Rightarrow p = e^4 = 54.6$ mpg $v = 64, \Rightarrow \ln p = 3.69$ $\Rightarrow p = e^{3.69} = 40.0$ mpg	M1 A1 A1 [3]	Reading off one value of $\ln p$ from graph (4.0, 3.69) 54.59… or answers rounded from this. 39.5 to 40.5
(iii) $p = c\,v^{-d}$ $\Rightarrow \ln p = \ln c - d \ln v$ $\Rightarrow \ln 45 = \ln c - d \ln 49$ $\underline{\ln 35 = \ln c - d \ln 81}$ $\Rightarrow \ln 45 - \ln 35 = d(\ln 81 - \ln 49)$ $\Rightarrow d = \dfrac{\ln 45 - \ln 35}{\ln 81 - \ln 49}$ * $= \dfrac{\ln \frac{45}{35}}{\ln \frac{81}{49}} = \dfrac{\ln \frac{9}{7}}{2 \ln \frac{9}{7}} = \dfrac{1}{2}$ $\Rightarrow \ln c = \ln 45 + \dfrac{1}{2} \ln 49$ $= \ln 49 + \ln 7$ $= \ln 315$ *or* $45 = c.49^{-1/2}$ or $35 = c.81^{-1/2}$ $\Rightarrow c = 315$ $v = 25 \Rightarrow p = \dfrac{315}{5} = 63$ mpg $v = 64 \Rightarrow p = \dfrac{315}{8} = 39.375$ mpg	M1 M1 E1 M1 A1 M1 B1 B1 B1 [9]	$\ln p = \ln c - d \ln v$ substituting (45,49) or (35,81) or gradient $= \dfrac{\ln 35 - \ln 45}{\ln 81 - \ln 49}$ or equiv. $\Rightarrow d = \dfrac{\ln 45 - \ln 35}{\ln 81 - \ln 49}$ * www use of $\ln a - \ln b = \ln \dfrac{a}{b}$ *or* ln 45 = ln 5 + ln 9, etc,… $\Rightarrow d = \dfrac{1}{2}$ (must be exact method) substituting for d, p and v in ln equation or power equation Deriving $c = 315$ (exactly) 63 cao 39.375 or answers rounded from this
(iv) Both agree well for $v = 64$ First model better for $v = 25$	B1 B1 [2]	Agree for $v = 64$ First model better for $v = 25$ [or B1 1st model better, B1 2nd model improves at higher speeds, or equivalent.]

4 (i) $f'(x) = \frac{1}{2}(4-x)^{-\frac{1}{2}}.(-1)$ $= -\frac{1}{2\sqrt{4-x}}$ $\Rightarrow f'(0) = -\frac{1}{4}$	B1 B1 B1ft [3]	$\frac{1}{2}(4-x)^{-\frac{1}{2}}$ $\times(-1)$ correct expression for $f'(x)$ $f'(0) = -\frac{1}{4}$
(ii) $y = \sqrt{(4-x)}$ $\Rightarrow y^2 = 4 - x$ $\Rightarrow x = 4 - y^2$ $\Rightarrow f^{-1}(x) = 4 - x^2$  f^{-1} is f reflected in $y = x$	M1 A1 A1 B1 B1 [5]	Solving for y or reversing flowchart method $x = 4 - y^2$ or $\to x^2 \to 4-x$ $f^{-1}(x) = 4 - x^2$ [or $f^{-1}(y) = 4 - y^2$] f^{-1} correct shape through (0,4) and (2,0) reflection in $y = x$
(iii) $f_1(x) = -f(x) = -\sqrt{(4-x)}$ $f_2(x) = f(2x) = -\sqrt{(4-2x)}$ $f_3(x) = f(x+4) - 2 = \sqrt{(-x)} - 2$	B1 B1 B1 [3]	$-\sqrt{(4-x)}$ $-\sqrt{(4-2x)}$ $\sqrt{(-x)} - 2$ If no substitution for f: SCB2 $-f(x)$, $f(2x)$ and $f(x+4) - 2$ SCB1 2 correct from $-f(x)$, $f(2x)$ and $f(x+4) - 2$
(iv) $g\,f(x) = g(\sqrt{(4-x)}) = x - 4$ $\Rightarrow g(x) = -x^2$ and so $f\,g(x) = f(-x^2) = \sqrt{[4-(-x^2)]}$ $= \sqrt{(4+x^2)}$	B2 M1 A1 [4]	$g(x) = -x^2$ [$g(x) = x^2$ SC1] $f\,g(x) = f(\text{their } -x^2)$ $\sqrt{(4+x^2)}$ cao

Examiner’s Report

candidates tried integration by parts, and credit was given for progress using this method. The meaning of 'exact' in this context does not seem to be well understood, and even good candidates often approximated – this was condoned provided they wrote $\frac{1}{2}(\ln 2)^2$ first.

$$x = 1\ ;\ \frac{1-\ln x}{x^2}\ ;\ \frac{2\ln x - 3}{x^3}\ ;\ -1/e^3\ ;\ \frac{1}{2}(\ln 2)^2.$$

Question 3 (Logarithms)

(i) Generally reduction to linear form is well known. However, students need to take care to identify 'm' and 'c' correctly. A small minority here identified 'm' with v instead of ln b.

(ii) Most candidates could read off the values for ln p from the graph, and correctly find p.

(iii) This was less well done. Many candidates wrote ln p = ln c – d ln v, but getting from here to the expression for d was too demanding for all but A grade candidates. The main error was equating the gradient with d instead of $-d$, and then 'fiddling'. Thereafter, many candidates scored 1 for using ln (a/b) = ln a – ln b, but very few derived $d = ½$ exactly, failing to spot that ln (81/49) = ln $(9/7)^2$ = 2 ln (9/7). Quite a few then managed to find c exactly, but approximating for 1/7 here lost them the 'E' mark. The final two 'B' marks were well answered, however.

(iv) Many candidates got the first mark here. However, to get the second, they needed to spot that the second model gave poor results at low speeds, and this proved to be more subtle.

p = 54.6, 40.0 ; c = 315 ; 63 mpg, 39.375 mpg.

Question 4 (Functions and transformations)

(i) The chain rule was reasonably well answered. Even if they failed this, there was generous follow through to be gained in finding the gradient at $x = 0$.

(ii) This question was generally answered either well or badly. If they knew what they were doing, they usually got full marks. The sketch of f^{-1} needed to show the intercepts with the axes, and most candidates did this well. Reflection in $y = x$ was also known quite well.

(iii) The transformations for f_1 and f_2 were quite well done, although some confused $f(-x)$ with $-f(x)$ and $f(2x)$ with $2f(x)$. The third function was more difficult and the preserve of better candidates. Quite a few failed to convert their transformed functions into functions of x, but only lost one mark out of the three here.

(iv) Again, this was often either all or nothing, although candidates gained B1 for $g(x) = x^2$ and were allowed some follow through on their $g(x)$.

$-1/4$; $f^{-1}(x) = 4 - x^2$; $-\sqrt{(4-x)}$, $-\sqrt{(4-2x)}$, $\sqrt{(-x)} - 2$, $g(x) = -x^2$, $fg(x) = -\sqrt{(4+x^2)}$.

Pure Mathematics 2 (2602)

General Comments

The paper attracted the full range of marks, with all but the very weakest candidates scoring over 10 marks. The majority of candidates appeared to be well prepared for the paper, and all the questions gained reasonable marks – question 4 being the poorest answered and question 1 gaining the highest mean score. The overall mark distribution was somewhat bimodal, with evidence of some very good scripts (probably mainly from Further Mathematics students), but also a tail of very low marks.

Poor algebra and notation continue to be a general concern. There was evidence that some candidates ran into time trouble, but most managed to get through the paper and nearly all attempted all four questions. However, there was some evidence of incomplete syllabus knowledge from AS candidates struggling to complete this as their third unit in the first year of their course.

It is worth emphasising the importance of learning the formulae that are now not given for the 'new' specification. In particular, for this paper, students needed to know the series formulae for question 1 and the quotient rule for question 2.

Comments on Individual Questions

Question 1 (Sequences and Series)

(i) The first five marks proved to be friendly, with all but the very weakest candidates scoring heavily, and many gaining full marks. The two comments required were 'declining' and 'constant', but answers which effectively combined these, e.g. 'converging to a limit of 125', gained full marks.

(ii) Most students recognised the geometric progression and showed enough working to gain the B1 mark. The main problem with this part proved to be starting the sequence at u_0 instead of u_1, which led to $r^9 = 0.5$ instead of $r^{10} = 0.5$. This lost one mark only. However, using the sum of the series formula lost all three marks. Most candidates scored at least 3 marks here.

(iii) The first mark was gained by virtually all candidates. However, deriving $w_n = 5n(n + 1)$ was disappointingly answered. Again, there was evidence of confusion about how many terms were being summed, leading to evidence of 'fudging' to achieve the result. Some candidates used difference methods, showing a constant second difference to show a quadratic expression: this was condoned if the work was convincing.

Candidates often recovered to get the final two marks, giving $Y = 14$, either using the quadratic formula or trial and error methods.

$u_1 = 225$, $u_2 = 205$, $u_3 = 189$, declining; $u_0 = u_2 = u_3 = 125$ constant; $r = 0.93$; $Y = 14$

Question 2 (Calculus)

(i) This proved to be an easy starter for 1 mark.

(ii) The quotient rule was generally well done, at least up to the first derivative. Poor algebra then often lost them the final A1 – the mark scheme did not penalise them heavily for this. Some candidates used the product rule instead, which in fact is no more difficult in this case.

(iii) Some candidates did not know the result ln e = 1, but verified this approximately using 2.718...; this lost them the second E1 mark. Ln e / e was not enough to earn the B1 mark for $y = 1/e$. Most substituted e into their f "(x) to gain the M1, but only fully correct solutions gained the final mark.

(iv) Most candidates expressed the area as an integral with the correct limits. However, the integration by substitution proved to be difficult with many candidates not knowing what to substitute. Some