OCR
RECOGNISING ACHIEVEMENT

General Certificate of Education
Advanced Supplementary (AS) and Advanced Level

former Oxford and Cambridge modular syllabus

MEI STRUCTURED MATHEMATICS 5511

Mechanics 5

Thursday **7 JUNE 2001** Afternoon 1 hour 20 minutes

Additional materials:
Answer paper
Graph paper
Students' Handbook

TIME 1 hour 20 minutes

INSTRUCTIONS TO CANDIDATES

Write your name, Centre number and candidate number in the spaces provided on the answer paper/answer booklet.

Answer any **three** questions.

Write your answers on the separate answer paper provided.

If you use more than one sheet of paper, fasten the sheets together.

INFORMATION FOR CANDIDATES

The approximate allocation of marks is given in brackets [ ] at the end of each question or part question.

You are advised that an answer may receive no marks unless sufficient detail of the working is shown on the answer paper to indicate that a correct method is being used.

Take $g = 9.8\ \text{m s}^{-2}$ unless otherwise instructed.

This question paper consists of 5 printed pages and 3 blank pages.

HN/3

[Turn over

Option 1: Variable forces

1 In tests of a new engine for a sports car, the car is driven along a level road. The mass of the car is m and v is the speed of the car at time t. The driving force is equal to mav and the resistance is equal to mbv^2 where a and b are constants. After a time t the car has travelled a distance x.

(i) Write down a differential equation for v that describes the motion of the car as a function of t. Hence write down a differential equation for $\frac{dv}{dx}$. [5]

(ii) Show that $v = \frac{a}{b} + Be^{-bx}$, where B is a constant. Give interpretations of the quantities $\frac{a}{b}$ and $\frac{a}{b} + B$ in terms of the speed of the car. [7]

At the start of a particular trial, the speed of the car increases from V to $2V$ as x increases from 0 to L, where $V < \frac{a}{2b}$.

(iii) Find B in terms of V, a and b. Show that $L = \frac{1}{b}\ln\left(\frac{\frac{a}{b} - V}{\frac{a}{b} - 2V}\right)$.

Show further that the car has its maximum acceleration when $v = \frac{a}{2b}$ and that this occurs when it has moved a distance $\frac{1}{b}\ln\left(2 - \frac{2bV}{a}\right)$. [8]

Option 2: Relative motion

2 A cotton thread is unwound from around the circumference of a **fixed** circular reel of radius a and centre O, as shown in Fig. 2. At time t, a length BC of the thread has been unwound and the angle between OB and the fixed direction OA is θ. As the point B moves round the circumference, $\dot{\theta}$ remains constant and the free length BC is kept straight. The end C was originally at A, where $\theta = 0$.

Unit vectors **i** and **j** are chosen along and perpendicular to OA, with O as the origin.

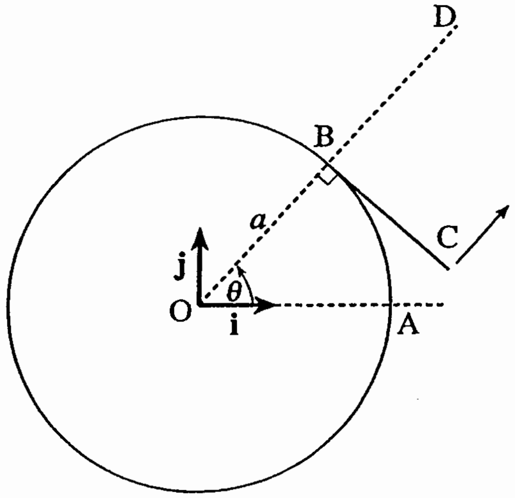

Fig. 2

(i) Explain why BC = $a\theta$. Hence write down the position vector of C relative to B and deduce the position vector of C relative to O. [4]

(ii) Find the velocity of C relative to B. Show that the magnitude of the component of this velocity in the direction $\overrightarrow{OB}$ is given by $a\theta\dot{\theta}$. Find the magnitude of the component in the direction $\overrightarrow{BC}$. Hence show that the velocity of C relative to B is in a direction $\arctan\theta$ to $\overrightarrow{BC}$. [5]

(iii) Find the velocity and acceleration of C relative to O. [5]

(iv) D is the point on OB produced so that OD = $2a$. Find the position vector of D relative to O. Hence show that the acceleration of the end C of the cotton thread is always directed towards D. Find the magnitude of this acceleration. [6]

Option 3: Motion described in Polar Coordinates

3 A communication satellite moves in an elliptical orbit round the Earth. The satellite is modelled as a particle of mass m moving under a central force of attraction $\frac{\mu m}{r^2}$ towards the centre O of the Earth, where r is the distance of the satellite from O at time t and μ is a constant.

(i) Write down the equations of motion for the satellite in terms of polar coordinates r and θ. Hence show that $r^2\dot{\theta} = h$, where h is a constant. [4]

(ii) By using the substitution $u = \frac{1}{r}$, prove that the equation of the path of the particle can be written as $\frac{l}{r} = 1 + e\cos\theta$, where l and e are constants. [8]

(iii) Show that the radial component of the velocity is $\frac{he\sin\theta}{l}$, and find an expression for the transverse component in terms of the same quantities. Deduce that the maximum speed in the orbit is $\frac{h(1+e)}{l}$. [5]

(iv) Find the time the satellite takes to make one complete orbit of the Earth in terms of h, l and e.

(You may assume that $\int_0^{\pi} \frac{1}{(1+e\cos\theta)^2}\,d\theta = \frac{\pi}{(1-e^2)^{\frac{3}{2}}}$, $e < 1$.) [3]

Option 4: Rotation of a rigid body

4 A pulley is modelled as a circular disc of radius r whose plane is vertical. It can turn freely about a fixed horizontal axis through its centre and the moment of inertia about this axis is I. Particles of mass m_1 and m_2, where $m_2 > m_1$, are attached to the ends of a light rough string which hangs vertically over the pulley, as shown in Fig. 4. T_1 and T_2 are the tensions in the hanging parts of the string and the string is inextensible. During the motion the string does not slip on the pulley.

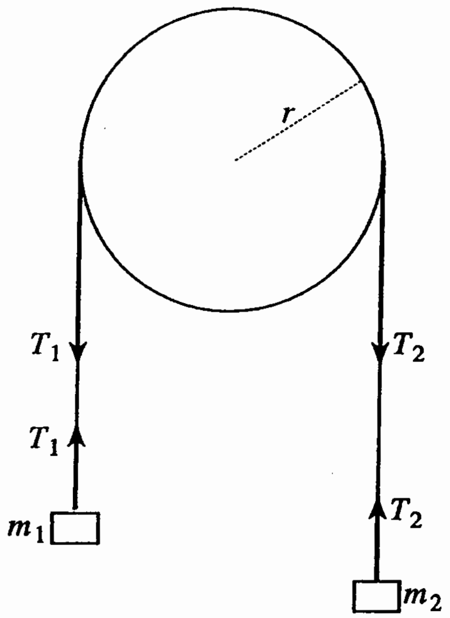

Fig. 4

(i) Write down the equations of motion of the two masses and of the pulley. [4]

(ii) Find the acceleration of the masses and the tensions T_1 and T_2. [4]

A mass m is now attached to the lowest point of the pulley and the system is released from rest.

(iii) Write down the energy equation for the system in terms of θ, the angle through which the pulley has turned. Hence show that the angular acceleration $\ddot{\theta}$ becomes zero provided that

$$\frac{m_2 - m_1}{m} \leqslant 1.$$ [7]

(iv) By considering the energy equation for the system, or otherwise, show that the pulley comes instantaneously to rest before $\theta = \pi$ provided that

$$\frac{m_2 - m_1}{m} < \frac{2}{\pi}.$$ [5]

Mark Scheme

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Question 1

(i)	$m\frac{\mathrm{d}v}{\mathrm{d}t} = mav - mbv^2$	M1A1 (2terms), A1 (third term)	
	$\frac{\mathrm{d}v}{\mathrm{d}x} = a - bv$	M1A1	[5]
(ii)	$\int \frac{\mathrm{d}v}{a - bv} = \int \mathrm{d}x$	M1 separation	
	$\frac{-1}{b}\ln(a - bv) = x + \text{constant}$	M1 integration, A1	
	$v = A + Be^{-bx}$	M1 solving, A1	
	$A = \frac{a}{b}$ is speed at infinity,	B1	
	$\frac{a}{b} + B$ is speed at $x = 0$.	B1	[7]
(iii)	$A + B = V$	M1 either boundary condition	
	$A + Be^{-bL} = 2V$	A1 both boundary conditions	
	$B = V - \frac{a}{b}$	B1	
	$\frac{a}{b} + \left(V - \frac{a}{b}\right)e^{-bL} = 2V$	M1 elim B	
	$L = \frac{1}{b}\ln\left(\frac{\frac{a}{b} - V}{\frac{a}{b} - 2V}\right)$	A1 ag	
	$\ddot{v} = 0$ when $v = \frac{a}{2b}$	B1	
	$\frac{a}{2b} = \frac{a}{b} + \left(V - \frac{a}{b}\right)e^{-bX}$	M1, substitution	
	distance $= \frac{1}{b}\ln\left(2 - \frac{2bV}{a}\right)$	A1,ag for solving	[8]

Question 2

		Mark	
(i)	$BC = a\theta$	B1	
	${}_{C}\mathbf{r}_{B} = a\theta\sin\theta\,\mathbf{i} - a\theta\cos\theta\,\mathbf{j}$	B1	
	${}_{C}\mathbf{r}_{O} = (a\cos\theta + a\theta\sin\theta)\mathbf{i} + (a\sin\theta - a\theta\cos\theta)\mathbf{j}$	M1F1	[4]
(ii)	${}_{C}\dot{\mathbf{r}}_{B} = \left.\begin{matrix}(a\dot{\theta}\sin\theta + a\theta\dot{\theta}\cos\theta)\mathbf{i} \\ + (a\theta\dot{\theta}\sin\theta - a\dot{\theta}\cos\theta)\mathbf{j}\end{matrix}\right\}$	M1A1	
	$= a\theta\dot{\theta}\dfrac{\overrightarrow{OB}}{\text{mod}\left(\overrightarrow{OB}\right)} + a\dot{\theta}\dfrac{\overrightarrow{BC}}{\text{mod}\left(\overrightarrow{BC}\right)}$	M1A1	
	angle is arctan θ	B1	[5]
(iii)	${}_{C}\dot{\mathbf{r}}_{O} = (a\theta\dot{\theta}\cos\theta)\mathbf{i} + (a\theta\dot{\theta}\sin\theta)\mathbf{j}$	M1A1	
	$\ddot{\theta} = 0$	B1	
	${}_{C}\ddot{\mathbf{r}}_{O} = \left.\begin{matrix}\dot{\theta}^2(a\cos\theta - a\theta\sin\theta)\mathbf{i} \\ + \dot{\theta}^2(a\sin\theta + a\theta\cos\theta)\mathbf{j}\end{matrix}\right\}$	M1A1	
			[5]
(iv)	$\overrightarrow{OD} = 2a\cos\theta\,\mathbf{i} + 2a\sin\theta\,\mathbf{j}$	B1	
	$\overrightarrow{CD} = (a\cos\theta - a\theta\sin\theta)\mathbf{i} + (a\sin\theta + a\theta\cos\theta)\mathbf{j}$	M1A1	
	${}_{C}\ddot{\mathbf{r}}_{O} = \dot{\theta}^2\,{}_{D}\mathbf{r}_{C}$	A1, as required	
	$\lvert{}_{C}\ddot{\mathbf{r}}_{O}\rvert = a\dot{\theta}^2\sqrt{1+\theta^2}$	M1A1	[6]

Question 3

(i)	$m\left(r\ddot{\theta}+2\dot{r}\dot{\theta}\right)=0$ $m\left(\ddot{r}-r\dot{\theta}^2\right)=\dfrac{-\mu m}{r^2}$	M1A1A1	
	$r^2\dot{\theta}=h$	B1, requires showing	[4]
(ii)	$\dot{r}=-hu'$	M1A1	
	$\ddot{r}=-h^2u^2u''$	M1A1	
	$u''+u=\dfrac{\mu}{h^2}$	M1, for substitution	
	$u=\dfrac{\mu}{h^2}+A\cos(\theta+B)$	M1A1 solving	
	$\dfrac{l}{r}=1+e\cos\theta,\quad l=\dfrac{h^2}{\mu},\quad A=\dfrac{\mu e}{h^2}$	A1, ag	[8]
(iii)	$\dot{r}=\dfrac{hle\sin\theta}{r^2(1+e\cos\theta)^2}=\dfrac{he\sin\theta}{l}$	M1A1	
	$r\dot{\theta}=\dfrac{h}{r}=\dfrac{h(1+e\cos\theta)}{l}$	B1	
	speed = $\left(\dot{r}^2+r^2\dot{\theta}^2\right)^{1/2}$		
	$=\dfrac{h}{l}\left(1+e^2+2e\cos\theta\right)^{1/2}$	M1	
	max. speed = $\dfrac{h}{l}(1+e)$	A1, ag	[5]
(iv)	use of $r^2\dot{\theta}=h$	M1	
	$\int \mathrm{d}t=\dfrac{2l^2}{h}\displaystyle\int_0^{\pi}\dfrac{\mathrm{d}\theta}{(1+e\cos\theta)^2}$	M1	
	$=\dfrac{2\pi l^2}{h}\left(1-e^2\right)^{-3/2}$	A1	[3]

Question 4

(i)	$T_1 - m_1 g = m_1 a$		
	$m_2 g - T_2 = m_2 a$	M1A1	
	$T_2 - T_1 = \dfrac{Ia}{r^2}$	M1A1	[4]
(ii)	$a = \dfrac{(m_2 - m_1)g}{m_1 + m_2 + \frac{I}{r^2}}$	M1A1	
	$T_1 = \dfrac{m_1 g\left(2m_2 + \frac{I}{r^2}\right)}{m_1 + m_2 + \frac{I}{r^2}}$	M1A1 for tensions	
	$T_2 = \dfrac{m_2 g\left(2m_1 + \frac{I}{r^2}\right)}{m_1 + m_2 + \frac{I}{r^2}}$		[4]
(iii)	$\left(\frac{1}{2}m_1 r^2\dot\theta^2 + \frac{1}{2}m_2 r^2\dot\theta^2 + \frac{1}{2}mr^2\dot\theta^2\right) + \frac{1}{2}I\dot\theta^2$ $+ m_1 gr\theta - m_2 gr\theta - mgr\cos\theta = -mgr$	M1A1A1	
	$(m_1 + m_2 + m)r^2\ddot\theta + I\ddot\theta + (m_1 - m_2 + m\sin\theta)gr = 0$	M1F1	
	Accn = 0 when $m_1 - m_2 + m\sin\theta = 0$	M1	
	θ is real if $\dfrac{m_2 - m_1}{m} \le 1$	A1, ag	[7]
(iv)	Put $\theta = \pi$ in energy equation	M1	
	$\frac{1}{2}r^2\dot\theta^2\left(m_1 + m_2 + m + \frac{I}{r^2}\right) = gr\pi(m_2 - m_1) - 2mgr$	A1	
	Since $\dot\theta^2 \ge 0$ if pulley reaches $\theta = \pi$,	M1	
	$gr\pi(m_2 - m_1) \ge 2mgr \Rightarrow \dfrac{m_2 - m_1}{m} \ge \dfrac{2}{\pi}$	A1	
	Hence if $\dfrac{m_2 - m_1}{m} < \dfrac{2}{\pi}$, pulley stops for $\theta < \pi$	A1	[5]

Examiner’s Report

Mechanics 5 (5511)

General Comments

The number of candidates for this paper was up on the previous year at 87 compared to 76. The candidates found the paper rather harder than recent ones and the average mark was down as a result. The main problem I think was that the candidates found the final parts of the questions too hard this year.

It has been remarked that there is not enough use in this paper of numerical constants rather than letters to denote values. It has always been my understanding that papers at this level are meant to be more general than mechanics papers at earlier levels.

Comments on Individual Questions

Question 1 (Differential equation with a variable force)

Despite the comment above, this question produced many very good answers. There were many high scores and an unusual (for this paper) number of candidates gaining full marks.

The first two parts were well answered, with some candidates interpreting *a*/*b* as the maximum speed rather than the speed at infinity. Quite a few candidates got themselves into a bind by assuming in part (ii) that the car starts from rest. This caused some difficulty in establishing the displayed answer.

(i) $\frac{a}{b}$ is speed at infinity, $\frac{a}{b}+B$ is speed at $x=0$; (iii) $B = V - \frac{a}{b}$

Question 2 (Relative motion)

This was not a popular question and not well done by the few who attempted it. A good many of these could write down the relevant vectors but then didn't know what to do with them. Candidates would have benefited from spending a short time planning their solutions.

(i) ${}_{C}\mathbf{r}_{B} = a\theta\sin\theta\,\mathbf{i} - a\theta\cos\theta\,\mathbf{j}$, ${}_{C}\mathbf{r}_{O} = (a\cos\theta + a\theta\sin\theta)\mathbf{i} + (a\sin\theta - a\theta\cos\theta)\mathbf{j}$

(ii) ${}_{C}\dot{\mathbf{r}}_{B} = a\theta\dot{\theta}\,\frac{{}_{O}\mathbf{r}_{B}}{|{}_{O}\mathbf{r}_{B}|} + a\dot{\theta}\,\frac{{}_{B}\mathbf{r}_{C}}{|{}_{B}\mathbf{r}_{C}|}$

(iii) ${}_{C}\dot{\mathbf{r}}_{O} = (a\theta\dot{\theta}\cos\theta)\mathbf{i} + (a\theta\dot{\theta}\sin\theta)\mathbf{j}$, (iv) ${}_{C}\ddot{\mathbf{r}}_{O} = \dot{\theta}^2\,{}_{D}\mathbf{r}_{C}$, $|{}_{C}\ddot{\mathbf{r}}_{O}| = a\dot{\theta}^2\sqrt{1+\theta^2}$

Question 3 (Motion described in polar co-ordinates)

The question on polar co-ordinates poses an annual dilemma for candidates in that they usually know how to do it but the actual manipulation and choice from the many equations proves too much. This year the choice was easier in the sense that the requirements to establish the equations of motion and their solutions in parts (i) and (ii) should be relatively familiar. On the whole, this proved to be the case but the actual manipulation caused more difficulty than it should have. The rather less familiar part (iii) to find speeds caused more problems and the final part to find the time for one orbit was beyond all but the best candidates. The problem in the final part was that the candidates did not know which expression to use to derive the time.

(iv) time for 1 orbit $= \dfrac{2\pi l^2}{h}\left(1-e^2\right)^{-3/2}$

Question 4 (Rotation of a rigid body)

I was very surprised how difficult many candidates found this question on a pulley. The first two parts to establish the equations of motion and solve them were reasonably well done, provided that the equations were correctly written down in part (i). The real problems began in part (iii) where the candidates were required to write down the energy equation. Most candidates got most of the terms but few got all of them. This, of course, created problems in deriving the required conditions in the rest of the question. Of those who managed to get through successfully to part (iv), only a handful could actually derive the final result.

(ii) $a = \frac{(m_2 - m_1)g}{m_1 + m_2 + \frac{I}{r^2}}$;

(iii) energy equation is $\left(\frac{1}{2}m_1r^2 + \frac{1}{2}m_2r^2 + \frac{1}{2}mr^2 + \frac{1}{2}I\right)\dot{\theta}^2 + m_1gr\theta - m_2gr\theta - mgr\cos\theta = -mgr$.