1. A sequence $\{u_n\}$ is defined by $u_0 = 0,\ \ u_{n+1} = 1 + 2u_n$ for $n = 1, 2, \ldots$
Prove by induction that, for $n \geq 1$, $u_n = 2^n - 1$. **(4 marks)**

2. Find the eigenvalues of the matrix $\mathbf{A} = \begin{pmatrix} 2 & 5 \\ 2 & -1 \end{pmatrix}$. **(5 marks)**

3. Obtain the Maclaurin expansion of $e^x \cos \pi x$ as far as the term in x^2. **(5 marks)**

   Use your expansion to find an approximate value of $e^{1/10} \cos \frac{\pi}{10}$, to 3 significant figures. **(2 marks)**

4. Given that $z^4 - 32i = 0$,
   (a) find the two possible values of z^2 in the form $a + ib$. **(3 marks)**
   (b) Hence find all possible values of z in the form $r(\cos \theta + i \sin \theta)$ where $r > 0, -\pi < \theta \leq \pi$. **(6 marks)**

5. The linear transformation T from $\mathbb{R}^3$ to $\mathbb{R}^3$ maps the points (1, 0, 0), (1, 1, 0) and (1, 1, 1) to the points (–2, 1, 3), (–1, 1, –1) and (0, 0, 1) respectively.
   (a) Find the 3 x 3 matrix which represents T. **(3 marks)**
   (b) Find the matrix representing the inverse transformation to T. **(5 marks)**
   (c) Find the point which is mapped to (2, –1, 0) by T. **(2 marks)**

6. A transformation of the complex plane is defined by $w = z^2 + 1$, where $z = x + iy,\ \ w = u + iv$.
   (a) Find, in the form $re^{i\theta}$, the points which remain unchanged under this transformation. **(6 marks)**
   (b) Find an equation in v and u for the image of the line $\text{Re}(z) = 1$ under the transformation and state what type of curve this image is. **(6 marks)**

7. The differential equation $\frac{dy}{dx} - 4xy = 2x$, with $y = -1$ when $x = 1$, is to be solved numerically using step-by-step methods.

(a) Show that the approximation $\left(\frac{dy}{dx}\right)_0 \approx \frac{y_1 - y_0}{h}$ with step length 0·1 and $x_0 = 1$,

leads to the equation $y_1 = y_{-1} - 0{\cdot}4$. **(4 marks)**

(b) Use the approximation $\left(\frac{dy}{dx}\right)_0 \approx \frac{y_0 - y_{-1}}{h}$ to estimate the value of y when $x = 0{\cdot}9$. **(3 marks)**

(c) Hence find approximates value for y when $x = 1{\cdot}1$ and when $x = 1{\cdot}2$. **(6 marks)**

8. The equation of a plane is $\mathbf{r} = (1 + 3\lambda - \mu)\,\mathbf{i} + (2 - \lambda + \mu)\,\mathbf{j} + (\lambda - 1)\,\mathbf{k}$, where λ and μ are real parameters.

(a) By converting this equation to scalar product form, or otherwise, find a unit vector normal to the plane. **(7 marks)**

(b) Find the perpendicular distance from the origin to the plane. **(1 mark)**

A second plane has cartesian equation $4x + 3y - 5z = 25$.

(c) Show that the points (8, 1, 2) and (5, 10, 5) lie in both planes. **(2 marks)**

(d) Hence or otherwise find a vector equation for the line of intersection of the planes. **(2 marks)**

(e) Find, to the nearest degree, the angle between the two planes. **(3 marks)**