PURE MATHS 5 (A) TEST PAPER 5 : ANSWERS AND MARK SCHEME

1. $\int \frac{2}{1+y^2} dy = \int dx$ $\quad$ $2 \arctan y = x + c$ $\quad$ $y = \tan\left(\frac{x}{2} + c\right)$ $\quad$ B1 M1 A1 A1

$1 = \tan(0 + c) : c = \frac{\pi}{4}$ $\quad$ $y = \tan\left(\frac{x}{2} + \frac{\pi}{4}\right)$ $\quad$ M1 A1 $\quad$ 6

2. (a) $\frac{d}{dx}(\arccos 2x) = \frac{-1}{\sqrt{1-(2x)^2}}(2) = \frac{-2}{\sqrt{1-4x^2}}$ $\quad$ M1 A1 A1

(b) $\int_0^{1/4} -\frac{3}{2} \cdot \frac{-2}{\sqrt{1-4x^2}} dx = \left[-\frac{3}{2}\arccos(2x)\right]_0^{1/4} = -\frac{3}{2}\left[\frac{\pi}{3} - \frac{\pi}{2}\right] = \frac{\pi}{4}$ $\quad$ M1 A1 A1 $\quad$ 6

3. Let $y = \text{arcosh}\, x$, so $x = \cosh y = \frac{1}{2}(e^y + e^{-y})$ $\quad$ $e^y + e^{-y} = 2x$ $\quad$ B1 M1

$e^y - 2x + e^{-y} = 0$ $\quad$ $e^{2y} - 2xe^y + 1 = 0$ $\quad$ A1

$e^y = (2x \pm \sqrt{4x^2 - 4})/2 = x \pm \sqrt{x^2 - 1}$ $\quad$ $y > 0$, so $y = \ln(x + \sqrt{x^2 - 1})$ $\quad$ M1 A1

Thus arcosh (13/12) = ln (13/12 + 5/12) = ln (3/2) or ln 3 – ln 2 $\quad$ M1 A1 $\quad$ 7

4. (a) $x = at^2,\ y = 2at$ $\quad$ $36 = 24a$ $\quad$ $a = 3/2$ $\quad$ M1

$x = \frac{3}{2}t^2,\ y = 3t$ $\quad$ A1 A1

(b) $\frac{dy}{dx} = \frac{3}{3t} = \frac{1}{t} = 1$ $\quad$ $\frac{d^2y}{dx^2} = -\frac{1}{t^2} \cdot \frac{1}{3t} = -\frac{1}{3t^3} = -\frac{1}{3}$ when $t = 1$ $\quad$ B1 M1 A1

$\rho = 2^{3/2}/(-1/3) = -6\sqrt{2}$ $\quad$ M1 A1 $\quad$ 8

5. Let $u = x^{n-1}$, $dv = x(1 + x^2)^5\, dx$ $\quad$ $du = (n-1)x^{n-2}\, dx$, $v = \frac{1}{12}(1 + x^2)^6$ $\quad$ B1 M1 A1

$I_n = \frac{1}{12}x^{n-1}(1+x^2)^6 - \frac{n-1}{12}\int (1+x^2)^6 x^{n-2} dx$ $\quad$ M1 A1

$= \frac{1}{12}x^{n-1}(1+x^2)^6 - \frac{n-1}{12}\int (1+x^2)^5 (x^{n-2} + x^n) dx$ $\quad$ A1

$= \frac{1}{12}x^{n-1}(1+x^2)^6 - \frac{n-1}{12}(I_{n-2} + I_n)$ $\quad$ M1 A1

$12 + (n-1)\, I_n = x^{n-1}(1+x^2)^6 - (n-1)I_{n-2}$ so $f(n) = 11 + n$ $\quad$ M1 A1 $\quad$ 10

6. (a) $\frac{dy}{dx} = x$ $\quad$ $s = \int_0^x \sqrt{1+t^2}\, dt$ $\quad$ Let $t = \sinh u$, so $dt = \cosh u\, du$ $\quad$ B1 B1 M1 A1

Then $s = \int_0^{ar\sinh x} \sqrt{1 + \sinh^2 u} \cosh u\, du = \int_0^{ar\sinh x} \cosh^2 u\, du$ $\quad$ M1 A1

Also $\frac{dy}{dx} = \tan\psi$, so $\tan\psi = x$ $\quad$ B1

(b) $\int_0^{ar\sinh x} \cosh^2 u\, du = \frac{1}{2}\int_0^{ar\sinh x} \cosh 2u + 1\, du = \frac{1}{2}\left[\frac{1}{2}\sinh 2u + u\right]_0^{ar\sinh x}$ $\quad$ M1 A1 A1

$= \frac{1}{2}[x\sqrt{1+x^2} + \text{arsinh}\, x] = \frac{1}{2}[\sec\psi \tan\psi + \text{arsinh}(\tan\psi)]$ $\quad$ M1 A1 $\quad$ 12

PURE MATHS 5 (A) TEST PAPER 5 : ANSWERS AND MARK SCHEME

7. (a) $\frac{dy}{dx} = x^{-1/2}$ $\left(1+\left(\frac{dy}{dx}\right)^2\right)^{1/2} = \sqrt{1+\frac{1}{x}}$ $s = \int_1^8 \sqrt{x+\frac{1}{x}}dx$ B1 M1 A1

This is approx. $\frac{1}{2}.1\left(\sqrt{2}+2\left(\sqrt{\frac{3}{2}}+\sqrt{\frac{4}{3}}+\sqrt{\frac{5}{4}}+\sqrt{\frac{6}{5}}+\sqrt{\frac{7}{6}}+\sqrt{\frac{8}{7}}\right)+\sqrt{\frac{9}{8}}\right)$ M1 A1

= 8.0 to 2 s.f. A1

(b) Area $= 2\pi \int_1^8 2\sqrt{x}.\left(1+\frac{1}{x}\right)^{1/2} dx = 4\pi\int_1^8 \sqrt{1+x}dx$ M1 A1 A1

$= \frac{8\pi}{3}\left[(x+1)^{3/2}\right]_1^8 = \frac{8\pi}{3}\left(27-2\sqrt{2}\right)$ M1 A1 11

8. (a) $\frac{x^2}{a^2}+\frac{y^2}{k^2a^2}=1$ B2

(b) (i) $a^2(1-e^2) = k^2a^2$ $e = \sqrt{(1-k^2)}$ M1 A1

(ii) Foci are $(\pm ae, 0)$, i.e. $(\pm a\sqrt{(1-k^2)}, 0)$ M1 A1

(c) If $y = mx + c$ is a tangent, $\frac{x^2}{a^2}+\frac{(mx+c)^2}{k^2a^2}=1$ has one real root for x M1

This equation is $k^2x^2+m^2x^2+2mcx+c^2-k^2a^2=0$ A1

For one real root, $4m^2c^2 = 4(k^2+m^2)(c^2-k^2a^2)$, which simplifies to M1 A1

$a^2(m^2+k^2) = c^2$ A1

(d) For the given ellipse, $a = 3$ and $k = \frac{1}{2}$ Also $c = 9$ B1 B1

Hence $9(m^2+\frac{1}{4}) = 81$ $m = \pm\frac{1}{2}\sqrt{35}$ M1 A1 15

