PURE MATHS 5 (A) TEST PAPER 7 : ANSWERS AND MARK SCHEME

1. $1 + 2\sinh^2 x = 1 + 2\left(\frac{e^x - e^{-x}}{2}\right)^2 = 1 + \frac{e^{2x} - 2 + e^{-2x}}{2} = \frac{e^{2x} + e^{-2x}}{2}$ M1 A1 A1

$= \cosh 2x$ A1 4

2. (a) $\int \frac{1}{(x+1)^2 + 7^2}\,dx = \frac{1}{7}\arctan\left(\frac{x+1}{7}\right) + c$ M1 A1 A1

(b) Area $= \frac{1}{7}(\arctan 1 - \arctan 0) = \frac{\pi}{28}$ M1 A1 5

3. $\frac{dy}{dx} = \frac{6\cosh t \sinh t}{6\sinh^2 t \cosh t} = \operatorname{cosech} t$ $\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{1/2} = \coth t$ M1 A1 A1

Arc length $= \int_0^{\ln 3} \coth t . 6\sinh^2 t \cosh t\,dt = \left[2\cosh^3 t\right]_0^{\ln 3}$ M1 A1 A1

$= 2\left[\left(\frac{3 + 1/3}{2}\right)^3 + 1\right] = \frac{196}{27}$ M1 A1 8

4. (a) $ae = 2,\ \frac{a}{e} = 4$ $a^2 = 8$ $a = 2\sqrt{2},\ e = \frac{\sqrt{2}}{2}$ B1 M1 A1

$b^2 = a^2(1 - e^2) = 4$ Equation is $\frac{x^2}{8} + \frac{y^2}{4} = 3$ $x^2 + 2y^2 = 24$ M1 A1

(b) Differentiate implicitly : $2x + 4y\frac{dy}{dx} = 0$ $\frac{dy}{dx} = \frac{-x}{2y} = -1$ at (4, 2) M1 A1

$2 + 4y\frac{d^2y}{dx^2} + 4\frac{dy}{dx} = 0$ $\frac{d^2y}{dx^2} = \frac{1}{4}$ $\rho = \frac{2^{3/2}}{1/4} = 8\sqrt{2}$ M1 A1 A1 10

5. (a) $I_n(x) = \int_0^x \sec^{n-2} t \sec^2 t\,dt$ B1

Let $u = \sec^{n-2} t$, $dv = \sec^2 t\,dt$ $v = \tan t$ B1

$du = (n-2)\sec^{n-3} t \sec t \tan t\,dt = (n-2)\sec^{n-2} t \tan t\,dt$ M1 A1

$I_n(x) = [\sec^{n-2} t \tan t]_0^x - \int_0^x (n-2)\sec^{n-2} t(\sec^2 t - 1)\,dt$ M1 A1

$= \sec^{n-2} x \tan x - (n-2)(I_n(x) - I_{n-2}(x))$ A1

Hence $(n-1)I_n(x) = \sec^{n-2} x \tan x + (n-2)I_{n-2}(x)$ A1

(b) $I_2\left(\frac{\pi}{4}\right) = \int_0^{\pi/4} \sec^2 t\,dt = [\tan t]_0^{\pi/4} = 1$ M1 A1

$3I_4\left(\frac{\pi}{4}\right) = \sec^2\frac{\pi}{4}\tan\frac{\pi}{4} + 2(1) = 4$ $I_4\left(\frac{\pi}{4}\right) = \frac{4}{3}$ M1 A1 A1 13

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6. (a) Curve sketched — B3

(b) Area $= \int_{-1/2}^{1/2} x + \frac{1}{\sqrt{1-x^2}}\,dx = \left[\frac{x^2}{2} + \arcsin x\right]_{-1/2}^{1/2}$ — M1 A1 A1

$= \left[\frac{1}{8} + \frac{\pi}{6}\right] - \left[\frac{1}{8} - \frac{\pi}{6}\right] = \frac{\pi}{3}$ — M1 A1

(c) $\frac{d}{dx}\left(\left(1-x^2\right)^{1/2}\right) = \frac{1}{2}(1-x^2)^{-1/2}.2x = \frac{-x}{\sqrt{1-x^2}}$ — M1 A1

(d) Volume $= \pi\int_{-1/2}^{1/2} x^2 + \frac{2x}{\sqrt{1-x^2}} + \frac{1}{1-x^2}\,dx$ — M1 A1

$= \pi\left[\frac{x^3}{3} - 2\sqrt{1-x^2} + \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)\right]_{-1/2}^{1/2}$ — M1 A1 A1

$= \pi\left\{\left[\frac{1}{24} - \sqrt{3} + \frac{1}{2}\ln 3\right] - \left[-\frac{1}{24} - \sqrt{3} + \frac{1}{2}\ln\frac{1}{3}\right]\right\} = \pi\left(\frac{1}{12} + \ln 3\right)$ — M1 A1 — 17

7. (a) $\frac{dy}{dx} = \frac{2a}{2ap} = \frac{1}{p}$ Normal at P is $y - 2ap = -p(x - ap^2)$ — M1 A1 M1

$y - 2ap = ap^3 - px$ $\quad px + y = ap(p^2 + 2)$ — M1 A1

(b) At Q, $2aq + apq^2 = ap^3 + 2ap$ $\quad pq^2 + 2q - p^2 - 2p = 0$ — M1 A1

$q = \frac{-2 \pm \sqrt{4 + 4p(p^3 + 2p)}}{2p} = \frac{-1 \pm \sqrt{p^4 + 4p^2 + 1}}{p} = \frac{-1 \pm (p^2 + 1)}{P}$ — M1 A1

$= p$ or $-p - \frac{2}{p}$ $\quad$ Hence $q = -p - \frac{2}{p}$ — M1 A1

(c) Mid-point is $\left(\frac{a(p^2+q^2)}{2}, a(p+q)\right) = \left(a\left(p^2 + \frac{2}{p^2} + 2\right), -\frac{2a}{p}\right)$ — M1 A1 A1

$y = \frac{-2a}{p}$ $\quad p = \frac{-2a}{y}$ $\quad x = a\left(\frac{4a^2}{y^2} + \frac{2y^2}{4a^2} + 2\right)$ — A1 M1 A1

$2ay^2x = 8a^4 + y^4 + 4a^2y^2$ — A1 — 18