PURE MATHS 5 (A) TEST PAPER 4 : ANSWERS AND MARK SCHEME

1. $\int_4^5 \frac{1}{\sqrt{(x+4)^2-16}}\,dx = [\text{arcosh}\left(\frac{x}{4}\right)+c]_4^5 = \text{arcosh}\left(\frac{5}{4}\right) - \text{arcosh } 1$ M1 A1 A1

$= \ln\left(\frac{5}{4}+\sqrt{\frac{25}{16}-1}\right) - 0 = \ln 2$ M1 A1 A1 6

2. $\frac{dy}{dx} + y\tanh x = \text{sech } x$ I.F. $= \exp(\int \tanh x\,dx) = \exp(\ln\cosh x) = \cosh x$ B1 M1 A1

$\cosh x\,\frac{dy}{dx} + y\sinh x = 1$ $\frac{d}{dx}(y\cosh x) = 1$ M1 A1

$y\cosh x = x + c$ $y = (x+c)\,\text{sech } x$ M1 A1 7

3. (a) Let $u = x^n$, $dv = e^{2x}\,dx$ $du = nx^{n-1}\,dx$, $v = \frac{1}{2}e^{2x}$ M1 A1

$I_n = \left[\frac{1}{2}x^n e^{2x}\right]_0^1 - \frac{n}{2}\int_0^1 x^{n-1}e^{2x}dx = \frac{e^2}{2} - \frac{n}{2}I_{n-1}$ M1 A1 A1

(b) $I_0 = \left[\frac{e^{2x}}{2}\right]_0^1 = \frac{e^2-1}{2}$ M1 A1

(c) $I_1 = \frac{1}{2}\left(e^2 - \frac{e^2-1}{2}\right) = \frac{e^2+1}{4}$ $I_2 = \frac{1}{2}\left(e^2 - 2I_1\right) = \frac{e^2-1}{4}$ M1 A1 A1 10

4. (a) $x = \frac{1}{2}\cosh y$ $\frac{dx}{dy} = \frac{1}{2}\sinh y = \frac{1}{2}\sqrt{4x^2-1}$ $\frac{dy}{dx} = \frac{2}{\sqrt{4x^2-1}}$ B1 M1 A1 A1

(b) Let $u = \text{arcosh } 2x$, $dv = dx$ $du = \frac{2}{\sqrt{4x^2-1}}\,dx$, $v = x$ M1 A1 A1

$\int \text{arcosh } x\,dx = x\,\text{arcosh } 2x - \int 2x(4x^2-1)^{-1/2}\,dx$ M1 A1

$= x\,\text{arcosh } 2x - \frac{1}{2}(4x^2-1)^{1/2} + c$ M1 A1 11

5. (a) $\int \text{sech } x\,dx = \int \frac{2}{e^x + e^{-x}}\,dx$ $u = e^x$, $du = e^x\,dx$ B1 M1 A1

$\int \frac{2}{u+u^{-1}}.\frac{1}{u}du = \int \frac{2}{u^2+1}du = 2\arctan u + c = 2\arctan(e^x) + c$ M1 A1 M1 A1

(Alternative answer : $\arctan(\sinh x) + c$)

(b) Region below graph of $y = \text{sech } x$ between $x = 1$ and $x = \ln 5$ B2

(c) Area = 2 arctan 5 – 2 arctan e = 0.310 (to 3 s.f.) M1 A1 A1 12

6. (a) $\frac{dx}{dt} = 2a\sin 2t$ $\quad \frac{dy}{dt} = 2a(1+\cos 2t)$ $\quad \frac{dy}{dx} = \frac{1+\cos 2t}{\sin 2t}$ B1 B1 B1

Arc length $= \int_0^{\pi/2}\left(1+\frac{(1+\cos 2t)^2}{\sin^2 2t}\right).2a\sin 2t\ dt$ M1 A1

$= 2a\int_0^{\pi/2}\left(\sin^2 2t+\cos^2 2t+2\cos 2t+1\right)^{1/2}\ dt = 2a\sqrt{2}\int_0^{\pi/2}(\cos 2t+1)^{1/2}dt$ M1 A1

$= 4a\int_0^{\pi/2}\cos t\ dt = 4a[\sin t]_0^{\pi/2} = 4a$ M1 A1

(b) $\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} = \frac{\sin 2t(-2\sin 2t)-(1+\cos 2t)(2\cos 2t)}{(\sin^2 2t)(2a\sin 2t)} = \frac{-(\cos 2t+1)}{a\sin^3 2t}$ M1 A1 A1

When $t = \frac{\pi}{4}$, $\frac{dy}{dx}$=1 and $\frac{d^2y}{dx^2} = -\frac{1}{a}$ $\quad$ Hence $\rho = -2a\sqrt{2}$ M1 A1 14

7. (a) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ $\quad$ (b) $y = \pm\frac{b}{a}x$ B2 B2

(c) Gradient at $P = \frac{dy}{dp}/\frac{dx}{dp} = \frac{b\cosh p}{a\sinh p}$ M1 A1

Tangent is $y - b\sinh p = \frac{b\cosh p}{a\sinh p}(x - a\cosh p)$ A1

$ay\sinh p - bx\cosh p = ab\sinh^2 p - ab\cosh^2 p$ A1

$bx\cosh p - ay\sinh p = ab$ A1

(d) At A and B, $y = \pm\frac{b}{a}x$ so $bx(\cosh p \pm \sinh p) = ab$ M1 A1

Hence $x = \frac{a}{\cosh p \pm \sinh p}$, so at mid-point M of AB, x-coordinate is A1

$\frac{a}{2}\left(\frac{1}{\cosh p+\sinh p}+\frac{1}{\cosh p-\sinh p}\right) = \frac{a}{2}\left(\frac{2\cosh p}{\cosh^2 p-\sinh^2 p}\right) = a\cosh p$ M1 A1

Since M lies on the tangent, M is P A1 15