Paper Reference(s)

**6667**

# Edexcel GCE

## Further Pure Mathematics FP1

## Advanced Level

## Specimen Paper

## Time: 1 hour 30 minutes

| Materials required for examination | Items included with question papers |
|---|---|
| Answer Book (AB16) | Nil |
| Graph Paper (ASG2) | |
| Mathematical Formulae (Lilac) | |

**Candidates may use any calculator EXCEPT those with the facility for symbolic algebra, differentiation and/or integration. Thus candidates may NOT use calculators such as the Texas Instruments TI-89, TI-92, Casio CFX-9970G, Hewlett Packard HP 48G.**

### Instructions to Candidates

In the boxes on the answer book, write the name of the examining body (Edexcel), your centre number, candidate number, the unit title (Further Pure Mathematics FP1), the paper reference (6667), your surname, initials and signature.
When a calculator is used, the answer should be given to an appropriate degree of accuracy.

### Information for Candidates

A booklet 'Mathematical Formulae and Statistical Tables' is provided.
Full marks may be obtained for answers to ALL questions.
This paper has eight questions.

### Advice to Candidates

You must ensure that your answers to parts of questions are clearly labelled.
You must show sufficient working to make your methods clear to the Examiner. Answers without working may gain no credit.

**1.** Prove that $\sum_{r=1}^{n}\left(r^2 - r - 1\right) = \frac{1}{3}(n-2)n(n+2)$.

**(5)**

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**2.**

$$f(x) = \ln x - 1 - \frac{1}{x}.$$

(*a*) Show that the root $\alpha$ of the equation f($x$) = 0 lies in the interval $3 < \alpha < 4$.

**(2)**

(*b*) Taking 3.6 as your starting value, apply the Newton-Raphson procedure once to f($x$) to obtain a second approximation to $\alpha$. Give your answer to 4 decimal places.

**(5)**

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**3.** Find the set of values of $x$ for which

$$\frac{x}{x-3} > \frac{1}{x-2}.$$

**(7)**

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**4.**

$$f(x) \equiv 2x^3 - 5x^2 + px - 5, \quad p \in \mathbb{R}.$$

The equation f ($x$) = 0 has (1 – 2i) as a root.

Solve the equation and determine the value of $p$.

**(7)**

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**5.** (*a*) Obtain the general solution of the differential equation

$$\frac{dS}{dt} - 0.1S = t.$$

**(6)**

(*b*) The differential equation in part (*a*) is used to model the assets, £$S$ million, of a bank $t$ years after it was set up. Given that the initial assets of the bank were £200 million, use your answer to part (*a*) to estimate, to the nearest £ million, the assets of the bank 10 years after it was set up.

**(4)**

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6. The curve $C$ has polar equation

$$r^2 = a^2 \cos 2\theta, \qquad \frac{-\pi}{4} \leq \theta \leq \frac{\pi}{4}.$$

(*a*) Sketch the curve $C$.

**(2)**

(*b*) Find the polar coordinates of the points where tangents to $C$ are parallel to the initial line.

**(6)**

(*c*) Find the area of the region bounded by $C$.

**(4)**

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7. Given that $z = -3 + 4\text{i}$ and $zw = -14 + 2\text{i}$, find

(*a*) $w$ in the form $p + \text{i}q$ where $p$ and $q$ are real,

**(4)**

(*b*) the modulus of $z$ and the argument of $z$ in radians to 2 decimal places

**(4)**

(*c*) the values of the real constants $m$ and $n$ such that

$$mz + nzw = -10 - 20\text{i}.$$

**(5)**

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**8.** (*a*) Given that $x = e^t$, show that

(i) $$\frac{dy}{dx} = e^{-t}\frac{dy}{dt},$$

(ii) $$\frac{d^2y}{dx^2} = e^{-2t}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right).$$

**(5)**

(*b*) Use you answers to part (*a*) to show that the substitution $x = e^t$ transforms the differential equation

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = x^3$$

into

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = e^{3t}.$$

**(3)**

(*c*) Hence find the general solution of

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = x^3.$$

**(6)**

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**END**

Paper Reference(s)

# 6668

# Edexcel GCE

## Further Pure Mathematics FP2

## Advanced Level

## Specimen Paper

## Time: 1 hour 30 minutes

| Materials required for examination | Items included with question papers |
| --- | --- |
| Answer Book (AB16) | Nil |
| Graph Paper (ASG2) | |
| Mathematical Formulae (Lilac) | |

**Candidates may use any calculator EXCEPT those with the facility for symbolic algebra, differentiation and/or integration. Thus candidates may NOT use calculators such as the Texas Instruments TI 89, TI 92, Casio CFX-9970G, Hewlett Packard HP 48G.**

### Instructions to Candidates

In the boxes on the answer book, write the name of the examining body (Edexcel), your centre number, candidate number, the unit title (Further Pure Mathematics FP2), the paper reference (6668), your surname, initials and signature.
When a calculator is used, the answer should be given to an appropriate degree of accuracy.

### Information for Candidates

A booklet 'Mathematical Formulae and Statistical Tables' is provided.
Full marks may be obtained for answers to ALL questions.
This paper has eight questions.

### Advice to Candidates

You must ensure that your answers to parts of questions are clearly labelled.
You must show sufficient working to make your methods clear to the Examiner. Answers without working may gain no credit.

**1.** The displacement $x$ of a particle from a fixed point $O$ at time $t$ is given by $x = \sinh t$.
At time $T$ the displacement $x = \dfrac{4}{3}$.

($a$) Find $\cosh T$.

**(2)**

($b$) Hence find $\mathrm{e}^T$ and $T$.

**(3)**

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**2.** Given that $y = \arcsin x$ prove that

($a$) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{\sqrt{(1 - x^2)}}$,

**(3)**

($b$) $(1 - x^2)\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} - x\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$.

**(4)**

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**Figure 1**

**3.**

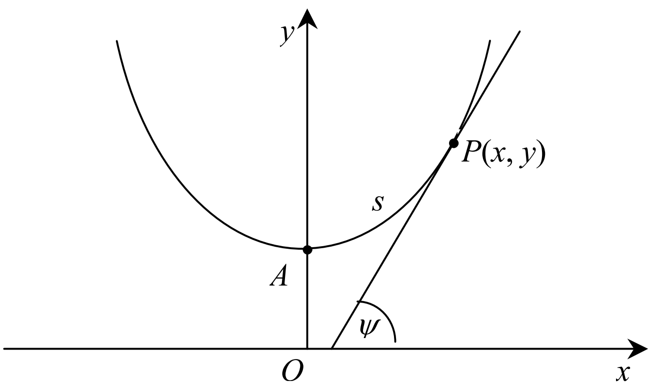


Figure 1 shows the curve $C$ with equation $y = \cosh x$. The tangent at $P$ makes an angle $\psi$ with the $x$-axis and the arc length from $A(0, 1)$ to $P(x, y)$ is $s$.

($a$) Show that $s = \sinh x$.

**(3)**

($a$) By considering the gradient of the tangent at $P$ show that the intrinsic equation of $C$ is $s = \tan \psi$.

**(2)**

($c$) Find the radius of curvature $\rho$ at the point where $\psi = \dfrac{\pi}{4}$.

**(3)**

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**4.** $$I_n = \int_0^{\frac{\pi}{2}} x^n \sin x \quad dx.$$

(a) Show that for $n \geq 2$

$$I_n = n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)I_{n-2}.$$

**(4)**

(*b*) Hence obtain $I_3$, giving your answers in terms of $\pi$.

**(4)**

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**5.** (*a*) Find $\int\sqrt{(x^2 + 4)}\ dx$.

**(7)**

The curve $C$ has equation $y^2 - x^2 = 4$.

(*b*) Use your answer to part (*a*) to find the area of the finite region bounded by $C$, the positive $x$-axis, the positive $y$-axis and the line $x = 2$, giving your answer in the form $p + \ln q$ where $p$ and $q$ are constants to be found.

**(4)**

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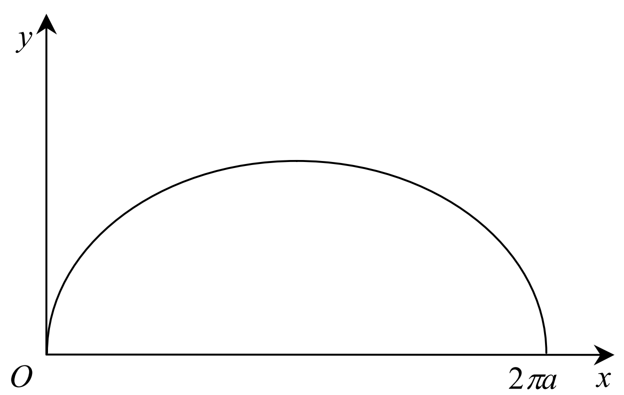
**Figure 2**

**6.**




The parametric equations of the curve $C$ shown in Fig. 2 are

$$x = a(t - \sin t), \quad y = a(1 - \cos t), \quad 0 \leq t \leq 2\pi.$$

(*a*) Find, by using integration, the length of $C$.

**(6)**

The curve $C$ is rotated through $2\pi$ about $Ox$.

(*b*) Find the surface area of the solid generated.

**(5)**

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**7.** (*a*) Using the definitions of $\sinh x$ and $\cosh x$ in terms of exponential functions, express $\tanh x$ in terms of $\mathrm{e}^{x}$ and $\mathrm{e}^{-x}$.

**(1)**

(*b*) Sketch the graph of $y = \tanh x$.

**(2)**

(*c*) Prove that $\operatorname{artanh} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$.

**(4)**

(*d*) Hence obtain $\frac{\mathrm{d}}{\mathrm{d}x}(\operatorname{artanh} x)$ and use integration by parts to show that

$$\int \operatorname{artanh} x \;\mathrm{d}x = x \operatorname{artanh} x + \frac{1}{2}\ln\left(1 - x^{2}\right) + \text{constant.}$$

**(5)**

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**8.** The hyperbola $C$ has equation $\frac{x^{2}}{a^{2}} - \frac{y^{2}}{b^{2}} = 1$.

(*a*) Show that an equation of the normal to $C$ at $P(a\sec\theta, b\tan\theta)$ is

$$by + ax\sin\theta = \left(a^{2} + b^{2}\right)\tan\theta.$$

**(6)**

The normal at $P$ cuts the coordinate axes at $A$ and $B$. The mid-point of $AB$ is $M$.

(*b*) Find, in cartesian form, an equation of the locus of $M$ as $\theta$ varies.

**(7)**

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**END**

Paper Reference(s)

# 6669

# Edexcel GCE

## Further Pure Mathematics FP3

## Advanced Level

## Specimen Paper

## Time: 1 hour 30 minutes

| Materials required for examination | Items included with question papers |
| --- | --- |
| Answer Book (AB16) | Nil |
| Graph Paper (ASG2) | |
| Mathematical Formulae (Lilac) | |

**Candidates may use any calculator EXCEPT those with the facility for symbolic algebra, differentiation and/or integration. Thus candidates may NOT use calculators such as the Texas Instruments TI 89, TI 92, Casio CFX 9970G, Hewlett Packard HP 48G.**

### Instructions to Candidates

In the boxes on the answer book, write the name of the examining body (Edexcel), your centre number, candidate number, the unit title (Further Pure Mathematics FP3), the paper reference (6669), your surname, initials and signature.
When a calculator is used, the answer should be given to an appropriate degree of accuracy.

### Information for Candidates

A booklet 'Mathematical Formulae and Statistical Tables' is provided.
Full marks may be obtained for answers to ALL questions.
This paper has eight questions.

### Advice to Candidates

You must ensure that your answers to parts of questions are clearly labelled.
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**1.** $$\frac{dy}{dx} = x^2 - y, \quad y = 1 \text{ at } x = 0.$$

Use the approximation $\left(\frac{dy}{dx}\right)_0 \approx \frac{y_1 - y_0}{h}$ with a step length of 0.1 to estimate the values of $y$ at $x = 0.1$ and $x = 0.2$, giving your answers to 2 significant figures.

**(6)**

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**2.** (*a*) Show that the transformation

$$w = \frac{z - \mathrm{i}}{z + 1}$$

maps the circle $|z| = 1$ in the $z$-plane to the line $|w - 1| = |w + \mathrm{i}|$ in the $w$-plane.

**(4)**

The region $|z| \leq 1$ in the $z$-plane is mapped to the region $R$ in the $w$-plane.

(*b*) Shade the region $R$ on an Argand diagram.

**(2)**

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**3.** Prove by induction that, all integers $n$, $n \geq 1$,

$$\sum_{r=1}^{n} r > \frac{1}{2}n^2 .$$

**(7)**

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**4.** $$\frac{d^2y}{dx^2} + y\frac{dy}{dx} = x, \quad y = 0, \ \frac{dy}{dx} = 2 \text{ at } x = 1.$$

Find a series solution of the differential equation in ascending powers of $(x - 1)$ up to and including the term in $(x - 1)^3$.

**(7)**

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**5.** $$\mathbf{A} = \begin{pmatrix} 7 & 6 \\ 6 & 2 \end{pmatrix}.$$

(a) Find the eigenvalues of **A**.

**(4)**

(a) Obtain the corresponding normalised eigenvectors.

**(6)**

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**6.** The points $A$, $B$, $C$, and $D$ have position vectors

$$\mathbf{a} = 2\mathbf{i} + \mathbf{k}, \quad \mathbf{b} = \mathbf{i} + 3\mathbf{j}, \quad \mathbf{c} = \mathbf{i} + 3\mathbf{j} + 2\mathbf{k}, \quad \mathbf{d} = 4\mathbf{j} + \mathbf{k}$$

respectively.

(a) Find $\overrightarrow{AB} \times \overrightarrow{AC}$ and hence find the area of triangle $ABC$.

**(7)**

(b) Find the volume of the tetrahedron $ABCD$.

**(2)**

(c) Find the perpendicular distance of $D$ from the plane containing $A$, $B$ and $C$.

**(3)**

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**7.**

$$\mathbf{A}(x) = \begin{pmatrix} 1 & x & -1 \\ 3 & 0 & 2 \\ 1 & 1 & 0 \end{pmatrix}, \quad x \neq \frac{5}{2}.$$

(a) Calculate the inverse of $\mathbf{A}(x)$.

**(8)**

$$\mathbf{B} = \begin{pmatrix} 1 & 3 & -1 \\ 3 & 0 & 2 \\ 1 & 1 & 0 \end{pmatrix}.$$

The image of the vector $\begin{pmatrix} p \\ q \\ r \end{pmatrix}$ when transformed by $\mathbf{B}$ is $\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$.

(b) Find the values of $p$, $q$ and $r$.

**(4)**

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**8.** (*a*) Given that $z = \mathrm{e}^{i\theta}$, show that

$$z^p + \frac{1}{z^p} = 2\cos p\theta,$$

where $p$ is a positive integer.

**(2)**

(*b*) Given that

$$\cos^4\theta = A\cos 4\theta + B\cos 2\theta + C,$$

find the values of the constants $A$, $B$ and $C$.

**(7)**

The region $R$ bounded by the curve with equation $y = \cos^2 x$, $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, and the $x$-axis is rotated through $2\pi$ about the $x$-axis.

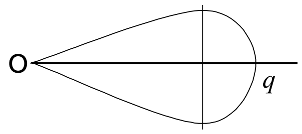
(*c*) Find the volume of the solid generated.

**(6)**

**END**

| Question number | Scheme | Marks |
|---|---|---|
| **1.** | $\sum_{r=1}^{n}\left(r^2-r-1\right)=\sum_{r=1}^{n}r^2-\sum_{r=1}^{n}r-\sum_{r=1}^{n}1$ | M1 |
| | $\left(\sum_{r=1}^{n}1=n\right)$ | B1 |
| | $=\frac{n}{6}(n+1)(2n+1)-\left(\frac{1}{2}\right)n(n+1)-n$ | |
| | $=\frac{n}{6}\left[2n^2-8\right]$ | M1 A1 |
| | $=\frac{1}{3}n(n-2)(n+2)$ | A1 (5) |
| | | **(5 marks)** |
| **2.** (*a*) | $\mathrm{f}(x)=\ln x-1-\frac{1}{x}$ | |
| | $\mathrm{f}(3)=\ln 3-1-\frac{1}{3}=-0.2347$ | |
| | $\mathrm{f}(4)=\ln 4-1-\frac{1}{4}=0.1363$ | |
| | $\mathrm{f}(3)$ and $\mathrm{f}(4)$ are of opposite sign and so $\mathrm{f}(x)$ has root in (3, 4) | M1 A1 (2) |
| (*b*) | $x_0=3.6$ | |
| | $\mathrm{f}'(x)=\frac{1}{x}+\frac{1}{x^2}$ | M1 |
| | $\mathrm{f}'(3.6)=0.354\ 381$ | A1 |
| | $\mathrm{f}(3.6)=0.003\ 156\ 04$ | |
| | Root $\approx 3.6-\frac{\mathrm{f}(3.6)}{\mathrm{f}'(3.6)}$ | M1 A1 ft |
| | $\approx 3.5911$ | A1 (5) |
| | | **(7 marks)** |

| Question number | Scheme | Marks |
|---|---|---|
| **3.** | $\dfrac{x}{x-3} > \dfrac{1}{x-2} \Rightarrow \dfrac{x}{x-3} - \dfrac{1}{x-2} > 0 \Rightarrow \dfrac{x^2-3x+3}{(x-3)(x-2)} > 0$ | M1 A1 |
| | Numerator always positive | B1 |
| | Critical points of denominator $x = 2, x = 3$ | B1 |
| | $x < 2 : \text{den} = (-\text{ ve})(-\text{ ve}) = +\text{ve}$ | |
| | $2 < x < 3 : \text{den} = (-\text{ ve})(+\text{ ve}) = -\text{ve}$ | |
| | $3 < x : \text{den} = (+\text{ ve})(+\text{ ve}) = +\text{ve}$ | M1 A1 |
| | Set of values $x < 2$ and $x > 3$ $\{x : x < 2\} \cup \{x : x > 3\}$ | A1 (7) |
| | | **(7 marks)** |
| **4.** | If 1 – 2i is a root, then so is 1 + 2i | B1 |
| | $(x-1+2\mathrm{i})(x-1-2\mathrm{i})$ are factors of f($x$) | M1 |
| | so $x^2-2x+5$ is a factor of f ($x$) | A1 |
| | $\mathrm{f}(x) = (x^2-2x+5)(2x-1)$ | M1 A1 ft |
| | Third root is $\frac{1}{2}$ and $p = 12$ | A1 A1 (7) |
| | | **(7 marks)** |
| **5.** (*a*) | $\dfrac{\mathrm{d}S}{\mathrm{d}t} - (0.1)S = t$ | M1 |
| | Integrating factor $\mathrm{e}^{-\int (0.1)\mathrm{d}t} = \mathrm{e}^{-(0.1)t}$ | |
| | $\dfrac{\mathrm{d}}{\mathrm{d}t}\left[S\mathrm{e}^{-(0.1)t}\right] = t\mathrm{e}^{-(0.1)t}$ | A1 A1 |
| | $\therefore S\mathrm{e}^{-(0.1)t} = \int t\mathrm{e}^{-(0.1)t}\,\mathrm{d}t$ | |
| | $= -10t\mathrm{e}^{-(0.1)t} - 100\mathrm{e}^{-(0.1)t} + C$ | M1 A1 |
| | $S = C\mathrm{e}^{(0.1)t} - 10t - 100$ | A1 (6) |
| (*b*) | $S = 200$ at $t = 0$ | |
| | $\Rightarrow 200 = C - 100$ i.e. $C = 300$ | |
| | $S = 300\mathrm{e}^{(0.1)t} - 10t - 100$ | M1 A1 |
| | At $t = 10, S = 300\mathrm{e} - 100 - 100$ | |
| | $= 615.484\ 55$ | M1 |
| | Assets £615 million | A1 ft (4) |
| | | **(10 marks)** |

| Question number | Scheme | Marks |
|---|---|---|
| **6.** (*a*) |  | B1 (Shape)<br>B1 (Labels) (2) |
| (*b*) | Tangent parallel to initial line when $y = r\sin\theta$ is stationary | |
| | Consider therefore $\frac{\mathrm{d}}{\mathrm{d}\theta}\left(a^2 \cos 2\theta \sin^2\theta\right)$ | |
| | $= -2\sin 2\theta \sin^2\theta + \cos 2\theta\left(2\sin\theta\cos\theta\right)$ | |
| | $= 0$ | M1 A1 |
| | $2\sin\theta\left[\cos 2\theta\cos\theta - \sin 2\theta \sin\theta\right] = 0$ | |
| | $\sin\theta \neq 0 \Rightarrow \cos 3\theta = 0 \Rightarrow \theta = \frac{\pi}{6}$ or $\frac{-\pi}{6}$ | M1 A1 |
| | Coordinates of the points $\left(\frac{1}{\sqrt{2}}a,\ \frac{\pi}{6}\right)\left(\frac{1}{\sqrt{2}}a,\ \frac{-\pi}{6}\right)$ | A1 A1 (6) |
| (*c*) | Area $= \frac{1}{2}\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} r^2 \ \mathrm{d}\theta = \frac{1}{2}a^2\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \cos 2\theta \ \mathrm{d}\theta$ | M1 A1 |
| | $= \frac{1}{2}a^2\left[\frac{\sin 2\theta}{2}\right]_{\frac{-\pi}{4}}^{\frac{\pi}{4}} = \frac{a^2}{4}\left[1-(-1)\right] = \frac{a^2}{2}$ | M1 A1 (4) |
| | | **(12 marks)** |

| Question number | | Scheme | Marks | |
|---|---|---|---|---|
| **7.** | (*a*) | $z = -3 + 4\text{i},\ zw = -14 + 2\text{i}$ | | |
| | | $w = \dfrac{-14+2\text{i}}{-3+4\text{i}} = \dfrac{(-14+2\text{i})(-3-4\text{i})}{(-3+4\text{i})(-3-4\text{i})}$ | M1 | |
| | | $= \dfrac{(42+8)+\text{i}(-6+56)}{9+16}$ | A1 A1 | |
| | | $= \dfrac{50+50\text{i}}{25} = 2+2\text{i}$ | A1 | (4) |
| | (*b*) | $\lvert z \rvert = \sqrt{(3^2+4^2)} = 5$ | M1 A1 | |
| | | $\arg z = \pi - \arctan\dfrac{4}{3} = 2.21$ | M1 A1 | (4) |
| | (*c*) | Equating real and imaginary parts | M1 | |
| | | $3m + 14n = 10,$ | A1 | |
| | | $4m + 2n = -20$ | A1 | |
| | | Solving to obtain | M1 | |
| | | $m = -6,\ n = 2$ | A1 | (5) |
| | | | | **(13 marks)** |

| Question number | Scheme | Marks |
|---|---|---|
| **8.** (*a*)(i) | $x = e^t,\ \dfrac{dy}{dx} = \dfrac{dy}{dt}\dfrac{dt}{dx} = e^{-t}\dfrac{dy}{dt}$ | M1 A1 |
| | $\left(\dfrac{dx}{dt} = e^t\right)$ | |
| (ii) | $\dfrac{d^2y}{dx^2} = \dfrac{dt}{dx}\dfrac{d}{dt}\left[e^{-t}\dfrac{dy}{dt}\right]$ | M1 |
| | $= e^{-t}\left[-e^{-t}\dfrac{dy}{dt} + e^{-t}\dfrac{d^2y}{dt^2}\right]$ | A1 |
| | $= e^{-2t}\left[\dfrac{d^2y}{dt^2} - \dfrac{dy}{dt}\right]$ | A1 (5) |
| (*b*) | $x^2\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + 2y = x^3$ | |
| | $e^{2t}e^{-2t}\left[\dfrac{d^2y}{dt^2} - \dfrac{dy}{dt}\right], - 2e^t e^{-t}\dfrac{dy}{dt} + 2y = e^{3t}$ | M1 A1, A1 |
| | $\dfrac{d^2y}{dt^2} - 3\dfrac{dy}{dt} + 2y = e^{3t}$ | (3) |
| (*c*) | Auxiliary equation $m^2 - 3m + 2 = 0$ | |
| | $(m-1)(m-2) = 0$ | |
| | Complementary function $y = Ae^t + Be^{2t}$ | M1 A1 |
| | Particular integral $= \dfrac{e^{3t}}{3^2 - (3\times 3) + 2} = \dfrac{1}{2}e^{3t}$ | M1 A1 |
| | General solution $y = Ae^t + Be^{2t} + \frac{1}{2}e^{3t}$ | |
| | $= Ax + Bx^2 + \frac{1}{2}x^3$ | M1 A1 ft (6) |
| | | **(14 marks)** |

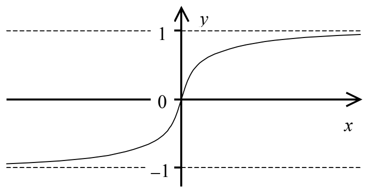
| Question Number | Scheme | Marks |
|---|---|---|
| **1.** | $\cosh^2 T = 1 + \sinh^2 T = 1 + \frac{16}{9} = \frac{25}{9}$ | M1 A1 |
| | $\therefore \cosh T = \pm\frac{5}{3} = \frac{5}{3}$ since $\cosh T > 1$ | (2) |
| | $e^T = \cosh T + \sinh T = \frac{4}{3} + \frac{5}{3} = 3$ | M1 A1 |
| | Hence $T = \ln 3$ | A1 ft (3) |
| | | **(5 marks)** |
| **2.** (a) | $y = \arcsin x$ | |
| | $\Rightarrow \sin y = x$ | M1 |
| | $\cos y \frac{dy}{dx} = 1$ | |
| | $\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$ | M1 A1 (3) |
| (b) | $\frac{d^2 y}{dx^2} = -\frac{1}{2}(1-x^2)^{\frac{-3}{2}}(-2x)$ | |
| | $= x(1-x^2)^{\frac{-3}{2}}$ | M1 A1 |
| | $(1-x^2)\frac{d^2 y}{dx^2} - x\frac{dy}{dx} = (1-x^2)x(1-x^2)^{\frac{-3}{2}} - x(1-x^2)^{-\frac{1}{2}} = 0$ | M1 A1 (4) |
| | | **(7 marks)** |

| Question Number | Scheme | Marks |
|---|---|---|
| **3.** (a) | $s = \int_0^x \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}} dx$ | |
| | $y = \cosh x, \frac{dy}{dx} = \sinh x$ | B1 |
| | $s = \int_0^x \left[1 + \sinh^2 x\right]^{\frac{1}{2}} dx$ | |
| | $= \int_0^x \cosh x \, dx = \sinh x$ | M1 A1 (3) |
| (b) | Gradient of tangent | |
| | $\frac{dy}{dx} = \tan\psi = \sinh x = s$ | |
| | $\therefore s = \tan\psi$ | M1 A1 (2) |
| (c) | $\rho = \frac{ds}{d\psi} = \sec^2\psi$ | M1 A1 |
| | At $\psi = \frac{\pi}{4}$, $\rho = \sec^2\frac{\pi}{4} = 2$ | A1 (3) |
| | | **(8 marks)** |

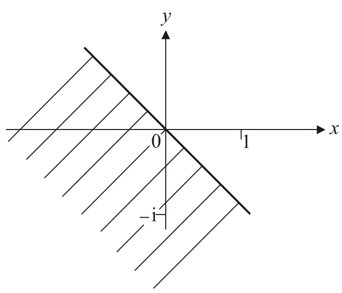
| Question Number | Scheme | Marks |
|---|---|---|
| **4.** (a) | $I_n = \int_0^{\frac{\pi}{2}} x^n \sin x \, dx$ | |
| | $= \left[x^n(-\cos x)\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} nx^{n-1}(-\cos x)dx$ | M1 A1 |
| | $= 0 + n\left\{\left[x^{n-1}\sin x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}}(n-1)x^{n-2}\sin x\, dx\right\}$ | A1 |
| | $= n\left[\left(\frac{\pi}{2}\right)^{n-1} - (n-1)I_{n-2}\right]$ | |
| | So $I_n = n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)I_{n-2}$ | A1 (4) |
| (b) | $I_3 = 3\left(\frac{\pi}{2}\right)^2 - 3.2I_1$ | |
| | $I_1 = \int_0^{\frac{\pi}{2}} x\sin x\, dx = \left[x(-\cos x)\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}}\cos x\, dx$ | M1 |
| | $= \left[\sin x\right]_0^{\frac{\pi}{2}} = 1$ | A1 |
| | $I_3 = (3)\left(\frac{\pi}{2}\right)^2 - 6 = \frac{3\pi^2}{4} - 6$ | M1 A1 (4) |
| | | **(8 marks)** |

| Question Number | Scheme | Marks |
| --- | --- | --- |
| **5.** (a) | $x = 2\sinh t$ | B1 |
| | $\sqrt{(x^2+4)} = (4\sinh^2 t + 4)^{\frac{1}{2}} = 2\cosh t$ | |
| | $\mathrm{d}x = 2\cosh t \,\mathrm{d}t$ | |
| | $I = \int \sqrt{(x^2+4)}\,\mathrm{d}x = 4\int \cosh^2 t\,\mathrm{d}t$ | M1 A1 |
| | $= 2\int (\cosh 2t + 1)\,\mathrm{d}t$ | |
| | $= \sinh 2t + 2t + c$ | M1 A1 |
| | $= \frac{1}{2}x\sqrt{(x^2+4)} + 2\operatorname{arsinh}\left(\frac{x}{2}\right) + c$ | M1 A1 ft |
| | | (7) |
| (b) | $\text{Area} = \int_0^2 y\,\mathrm{d}x = \int_0^2 \sqrt{(x^2+4)}\,\mathrm{d}x$ | M1 |
| | $= \left[\frac{1}{2}x\sqrt{(x^2+4)}\right]_0^2 + \left[2\operatorname{arsinh}\frac{x}{2}\right]_0^2$ | |
| | $= \frac{1}{2}2\sqrt{8} + 2\operatorname{arsinh}(1)$ | A1 |
| | $2\sqrt{2} + 2\ln[1+\sqrt{2}] = 2\sqrt{2} + \ln(3+2\sqrt{2})$ | M1 A1 (4) |
| | | **(11 marks)** |

| Question Number | Scheme | Marks |
|---|---|---|
| **6.** (a) | $s = \int_0^{2\pi} \left[ \dot{x}^2 + \dot{y}^2 \right]^{\frac{1}{2}} \mathrm{d}t$ | |
| | $\frac{\mathrm{d}x}{\mathrm{d}t} = \dot{x} = a(1 - \cos t); \frac{\mathrm{d}y}{\mathrm{d}t} = \dot{y} = a \sin t$ | M1 A1; A1 |
| | $s = \int_0^{2\pi} a\left[(1 - \cos t)^2 + \sin^2 t\right]^{\frac{1}{2}} \mathrm{d}t = a \int_0^{2\pi} [2 - 2\cos t]^{\frac{1}{2}} \mathrm{d}t$ | |
| | $= 2a \int_0^{\frac{\pi}{2}} \sin\left(\frac{t}{2}\right) \mathrm{d}t, = -4a\left[\cos\left(\frac{t}{2}\right)\right]_0^{2\pi} = 8a$ | M1 A1, A1 ft |
| | | (6) |
| (b) | $s = 2\pi \int_0^{2\pi} y\left(\dot{x}^2 + \dot{y}^2\right)^{\frac{1}{2}} \mathrm{d}t$ | |
| | $= 2\pi \int_0^{2\pi} 2^{\frac{1}{2}} a^2 (1 - \cos t)^{\frac{3}{2}} \mathrm{d}t$ | |
| | $= 8\pi a^2 \int_0^{2\pi} \sin^3\left(\frac{t}{2}\right) \mathrm{d}t$ | M1 A1 |
| | $= 8\pi a^2 \int_0^{2\pi} \left[1 - \cos^2\left(\frac{t}{2}\right)\right] \sin\frac{t}{2} \mathrm{d}t$ | M1 |
| | $= 8\pi a^2 \left[-2\cos\frac{t}{2} + \frac{2}{3}\cos^3\frac{t}{2}\right]_0^{2\pi} = \frac{64\pi a^2}{3}$ | A1 A1 ft (5) |
| | | **(11 marks)** |

| Question Number | Scheme | Marks |
|---|---|---|
| **7.** (a) | $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ | B1 (1) |
| (b) |  | B1<br>B1 (2) |
| (c) | $\text{artanh}x = z \Rightarrow \tanh z = x$ | |
| | $\dfrac{e^z - e^{-z}}{e^z + e^{-z}} = x$ | M1 A1 |
| | $e^z - e^{-z} = x(e^z + e^{-z})$ | |
| | $(1-x)e^z = (1+x)e^{-z}$ | |
| | $e^{2z} = \dfrac{1+x}{1-x}$ | |
| | $z = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right) = \text{artanh } x$ | M1 A1 (4) |
| (d) | $\dfrac{dz}{dx} = \dfrac{1}{2}\left(\dfrac{1}{1+x} + \dfrac{1}{1-x}\right) = \dfrac{1}{1-x^2}$ | M1 A1 |
| | $\int \text{artanh } x \, dx = (x \text{ artanh } x) - \int \dfrac{1}{1-x^2} x \, dx$ | M1 A1 |
| | $= (x \text{ artanh } x) + \dfrac{1}{2}\ln(1-x^2) + \text{constant}$ | A1 (5) |
| | | **(10 marks)** |

| Question Number | Scheme | Marks |
|---|---|---|
| **8.** (a) | $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ | |
| | $\frac{2x}{a^2}-\frac{2y}{b^2}\frac{\mathrm{d}y}{\mathrm{d}x}=0$ | M1 A1 |
| | $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2x}{a^2}\frac{b^2}{2y}=\frac{b^2}{a^2}\frac{a\sec\theta}{b\tan\theta}=\frac{b}{a\sin\theta}$ | M1 A1 |
| | Gradient of normal is then $-\frac{a}{b}\sin\theta$ | |
| | Equation of normal: $(y-b\tan\theta)=-\frac{a}{b}\sin\theta(x-a\sec\theta)$ | |
| | $ax\sin\theta+by=(a^2+b^2)\tan\theta$ | M1 A1 (6) |
| (b) | M: A normal cuts $x=0$ at $y=\frac{(a^2+b^2)}{b}\tan\theta$ | M1 A1 |
| | B normal cuts $y=0$ at $x=\frac{a^2+b^2}{a\sin\theta}\tan\theta$ | |
| | $=\frac{(a^2+b^2)}{a\cos\theta}$ | A1 |
| | Hence M is $\left[\frac{(a^2+b^2)}{2a}\sec\theta, \frac{(a^2+b^2)}{2b}\tan\theta\right]$ | M1 |
| | Eliminating $\theta$ | M1 |
| | $\sec^2\theta=1+\tan^2\theta$ | |
| | $\left[\frac{2aX}{a^2+b^2}\right]^2=1+\left[\frac{2bY}{a^2+b^2}\right]^2$ | A1 |
| | $4a^2X^2-4b^2Y^2=\left[a^2+b^2\right]^2$ | A1 (7) |
| | | **(15 marks)** |

| Question Number | Scheme | Marks |
|---|---|---|
| **1.** | $x_0 = 0,\ y_0 = 1, \left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)_0 = 0 - 1 = -1$ | B1 |
| | $y_1 - y_0 = h\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)_0 \Rightarrow y_1 = 1 + (0.1)(-1) = 0.9$ | M1 A1 ft |
| | $x_1 = 0.1,\ y_1 = 0.9, \left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)_1 = (0.1)^2 - 0.9$ | A1 |
| | $= -0.89$ | |
| | $y_2 = y_1 + h\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)_1$ | |
| | $= 0.9 + (0.1)(-0.89) = 0.811 \approx \underline{0.81}$ | M1 A1 (6) |
| | | **(6 marks)** |
| **2.** (*a*) | $w = \dfrac{z - \mathrm{i}}{z + 1} \Rightarrow w(z + 1) = (z - \mathrm{i})$ | |
| | $z(w - 1) = -\mathrm{i} - w$ | |
| | $z = \dfrac{-\mathrm{i} - w}{w - 1}$ | M1 A1 |
| | $\lvert z \rvert = 1 \Rightarrow \left\lvert \dfrac{-\mathrm{i} - w}{w - 1} \right\rvert = 1$ | |
| | i.e. $\lvert w - 1 \rvert = \lvert w + \mathrm{i} \rvert$ | M1 A1 (4) |
| (*b*) | $\lvert z \rvert \le 1 \Rightarrow \lvert w + \mathrm{i} \rvert \le \lvert w - 1 \rvert$ | |
| |  | B1 (line) B1 (shading) (2) |
| | | **(6 marks)** |

| Question Number | Scheme | Marks |
|---|---|---|
| **3.** | For $n = 1$, LHS $=1$, RHS $=\frac{1}{2}$ | |
| | So result is true for $n = 1$ | M1 A1 |
| | Assume true for $n = k$. Then | |
| | $\sum_{r=1}^{k+1} r > \frac{1}{2}k^2 + k + 1$ | M1 A1 |
| | $= \frac{1}{2}(k^2 + 2k + 1) + \frac{1}{2}$ | |
| | $= \frac{1}{2}(k+1)^2 + \frac{1}{2}$ | M1 |
| | If true for $k$, true for $k$+1 | A1 |
| | So true for all positive integral $n$ | A1 (7) |
| | | **(7 marks)** |
| **4.** | $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + y\frac{\mathrm{d}y}{\mathrm{d}x} = x,\ y = 0,\ \frac{\mathrm{d}y}{\mathrm{d}x} = 2$ at $x = 1$ | |
| | $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 0 + 1 = 1$ | B1 |
| | Differentiating with respect to $x$ | |
| | $\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + y\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 1$ | M1 A1 |
| | $\left.\frac{\mathrm{d}^3 y}{\mathrm{d}x^3}\right\|_{x=1} = -(2)^2 + 0 + 1 = -3$ | A1 |
| | By Taylor's Theorem | |
| | $y = 0 + 2(x-1) + \frac{1}{2!}1(x-1)^2 + \frac{1}{3!}(-3)(x-1)^3$ | M1 A1 |
| | $= 2(x-1) + \frac{1}{2}(x-1)^2 - \frac{1}{2}(x-1)^3$ | A1 (7) |
| | | **(7 marks)** |

| Question Number | Scheme | Marks |
|---|---|---|
| **5.** (*a*) | $\lvert \mathbf{A} - \lambda\mathbf{I} \rvert = 0$ | |
| | $\begin{vmatrix} (7-\lambda) & 6 \\ 6 & (2-\lambda) \end{vmatrix} = 0$ | |
| | $(7-\lambda)(2-\lambda) - 36 = 0$ | M1 A1 |
| | $\lambda^2 - 9\lambda + 14 - 36 = 0$ | |
| | $\lambda^2 - 9\lambda - 22 = 0$ | |
| | $(\lambda - 11)(\lambda + 2) = 0 \Rightarrow \lambda_1 = -2, \lambda_2 = 11$ | M1 A1 (4) |
| (*b*) | $\lambda = -2$ Eigenvector obtained from | |
| | $\begin{pmatrix} 7-(-2) & 6 \\ 6 & 2-(-2) \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ | |
| | $3x_1 + 2y_1 = 0$ | M1 A1 |
| | e.g. $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ normalised $\frac{1}{\sqrt{13}}\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ | M1 A1 ft |
| | $\lambda = 11 \quad \begin{pmatrix} -4 & 6 \\ 6 & -9 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ | |
| | $-2x_2 + 3y_2 = 0$ | A1 |
| | e.g. $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ normalised $\frac{1}{\sqrt{13}}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ | A1 ft (6) |
| | | **(10 marks)** |

| Question Number | Scheme | Marks |
|---|---|---|
| **6.** (*a*) | $\overrightarrow{AB} = (-1,\ 3, -1)$; $\overrightarrow{AC} = (-1, 3, 1)$. | M1 A1 |
| | $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 3 & -1 \\ -1 & 3 & 1 \end{vmatrix}$ | |
| | $= \mathbf{i}\,(3+3) + \mathbf{j}\,(1+1) + \mathbf{k}\,(-3+3)$ | |
| | $= 6\mathbf{i} + 2\mathbf{j}$ | M1 A1 A1 |
| | Area of $\Delta\, ABC = \frac{1}{2}\left\vert \overrightarrow{AB} \times \overrightarrow{AC} \right\vert$ | |
| | $= \frac{1}{2}\sqrt{36+4} = \sqrt{10}$ square units | M1 A1 ft (7) |
| (*b*) | Volume of tetrahedron $= \frac{1}{6}\left\vert \overrightarrow{AD} \,.\, \left(\overrightarrow{AB} \times \overrightarrow{AC}\right) \right\vert$ | |
| | $= \frac{1}{6}\left\vert -12+8 \right\vert$ | |
| | $= \frac{2}{3}$ cubic units | M1 A1 (2) |
| (*c*) | Unit vector in direction $\overrightarrow{AB} \times \overrightarrow{AC}$ i.e. perpendicular to plane containing *A*, *B*, and *C* is | |
| | $\mathbf{n} = \frac{1}{\sqrt{40}}(6\mathbf{i} + 2\mathbf{j}) = \frac{1}{\sqrt{10}}(3\mathbf{i} + \mathbf{j})$ | M1 |
| | $p = \left\vert \mathbf{n} \cdot \overrightarrow{AD} \right\vert = \frac{1}{\sqrt{10}}\left\vert (3\mathbf{i} + \mathbf{j}) \cdot (-2\mathbf{i} + 4\mathbf{j}) \right\vert$ | |
| | $= \frac{1}{\sqrt{10}}\left\vert -6+4 \right\vert = \frac{2}{\sqrt{10}}$ units. | M1 A1 (3) |
| | | **(12 marks)** |

| Question Number | Scheme | Marks |
|---|---|---|
| **7.** (*a*) | $\mathbf{A}(x)=\begin{pmatrix}1 & x & -1\\ 3 & 0 & 2\\ 1 & 1 & 0\end{pmatrix}$ | |
| | Cofactors $\begin{pmatrix}-2 & 2 & 3\\ -1 & 1 & x-1\\ 2x & -5 & -3x\end{pmatrix}$ | M1 A1<br>A1<br>A1 |
| | Determinant $=2x-3-2=2x-5$ | M1 A1 |
| | $\mathrm{A}^{-1}(x)=\dfrac{1}{2x-5}\begin{pmatrix}-2 & -1 & 2x\\ 2 & 1 & -5\\ 3 & (x-1) & -3x\end{pmatrix}$ | M1 A1<br>(8) |
| (*b*) | $\begin{pmatrix}p\\ q\\ r\end{pmatrix}=\mathrm{B}^{-1}\begin{pmatrix}2\\ 3\\ 4\end{pmatrix}=\dfrac{1}{1}\begin{pmatrix}-2 & -1 & 6\\ 2 & 1 & -5\\ 3 & 2 & -9\end{pmatrix}\begin{pmatrix}2\\ 3\\ 4\end{pmatrix}$ | M1 A1 ft |
| | $=(17,-13,-24)$ | M1<br>A1 (4) |
| | | **(12 marks)** |

| Question Number | Scheme | Marks |
|---|---|---|
| **8.** (*a*) | $z^p + \frac{1}{z^p} = e^{ip\theta} + \frac{1}{e^{ip\theta}}$ | |
| | $= \left(e^{ip\theta} + e^{-ip\theta}\right)$ | |
| | $= 2\cos p\theta$ | M1 A1 (2) |
| (*b*) | By De Moivre if $z = e^{i\theta}$ | |
| | $z^p + \frac{1}{z^p} = 2\cos p\theta$ | |
| | $p = 1$: $(2\cos\theta)^4 = \left(z + \frac{1}{z}\right)^4$ | M1 A1 |
| | $= z^4 + 4z^3 . \frac{1}{z} + 6z^2 \frac{1}{z^2} + 4z . \frac{1}{z^3} + \frac{1}{z^4}$ | M1 A1 |
| | $= \left(z^4 + \frac{1}{z^4}\right) + 4\left(z^2 + \frac{1}{z^2}\right) + 6$ | |
| | $= 2\cos 4\theta + 8\cos 2\theta + 6$ | M1 A1 |
| | $\therefore \cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$ | A1 ft (7) |
| (c) | $V = \pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} y^2 dx = \pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4 x \, dx$ | |
| | $= \pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}\right) d\theta$ | M1 A1 ft |
| | $= \pi \left[\frac{1}{32}\sin 4\theta + \frac{1}{4}\sin 2\theta + \frac{3}{8}\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$ | M1 A1 ft |
| | $= \frac{3}{8}\pi^2$ | M1 A1 (6) |
| | | **(15 marks)** |