Solutionbank M5
Edexcel AS and A Level Modular Mathematics

Review Exercise 2
Exercise A, Question 1

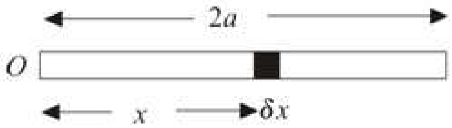
Question:

a Prove, using integration, that the moment of inertia of a uniform rod, of mass m and length $2a$, about an axis perpendicular to the rod through one end is $\frac{4}{3}ma^2$.

b Hence, or otherwise, find the moment of inertia of a uniform square lamina, of mass M and side $2a$, about an axis through one corner and perpendicular to the plane of the lamina. ***E***

Solution:

a



You consider the rod to be made up of a series of small pieces, or elements, each of length δx.

The mass per unit length of the rod is $\frac{m}{2a}$

Consider an element of length δx at a distance x from one end of the rod O.

The mass of the element is its length (δx) multiplied by the mass per unit length $\left(\frac{m}{2a}\right)$.

$$\delta I = (\delta m)x^2 = \left(\frac{m}{2a}\delta x\right)x^2 = \frac{mx^2}{2a}\delta x$$

For the whole rod

$$I = \sum \delta I = \sum \frac{mx^2}{2a}\delta x$$

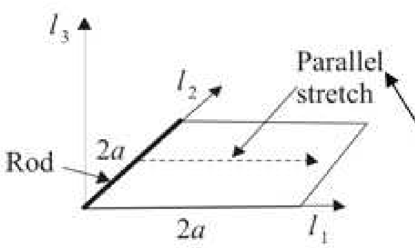
As $\delta x \to 0$

As the element ranges from O to the other end of the rod, x ranges from 0 to $2a$. So 0 and $2a$ are the limits of integration.

$$I = \int_0^{2a} \frac{mx^2}{2a}\mathrm{d}x = \frac{m}{2a}\left[\frac{x^3}{3}\right]_0^{2a}$$

$$= \frac{m}{2a}\left[\frac{8a^3}{3} - 0\right] = \frac{4}{3}ma^2\text{, as required}$$

b



Let l_1 and l_2 be axes along two of the sides of the square and l_3 be the axis through the corner perpendicular to l_1 and l_2, as shown in this sketch. The question asks you to find the moment of inertia about l_3.

By the stretching rule, the moment of inertia of the lamina about l_1 and l_2 is given by

As the lamina can be formed by taking a rod and stretching it parallel to the axis l_1, without altering the distribution of the mass relative to l_1, then the moment of inertia of the lamina about l_1 is the same as the rod, $\frac{4}{3}ma^2$.

$$I_{l_1} = I_{l_2} = \frac{4}{3}ma^2$$

By the perpendicular axes theorem

$$I_{l_3} = I_{l_1} + I_{l_2} = \frac{4}{3}ma^2 + \frac{4}{3}ma^2 = \frac{8}{3}ma^2$$

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Review Exercise 2
Exercise A, Question 2

Question:

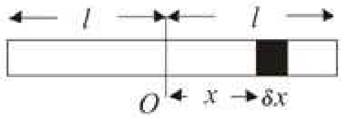
a Show, using integration, that the moment of inertia of a uniform rod, of length $2l$ and mass m, about an axis through its centre and perpendicular to the rod is $\frac{1}{3}ml^2$.

A uniform square plate, of mass M, has edges of length $2a$.

b Find the moment of inertia of the plate about an axis through its centre perpendicular to the plane of the plate. ***E***

Solution:

a



You consider the rod to be made up of a series of small pieces, or elements, each of length δx.

The mass per unit length of the rod is $\frac{m}{2l}$

When proving results you usually need to know the 'density' of the object, here the mass per unit length.

Consider an element of length δx at a distance x from the middle of the rod O.

$$\delta I = (\delta m)x^2 = \left(\frac{m}{2l}\delta x\right)x^2 = \frac{mx^2}{2l}\delta x$$

For the whole rod

$$I = \sum \delta I = \sum \frac{mx^2}{2l}\delta x$$

The whole rod is the sum of the small pieces.

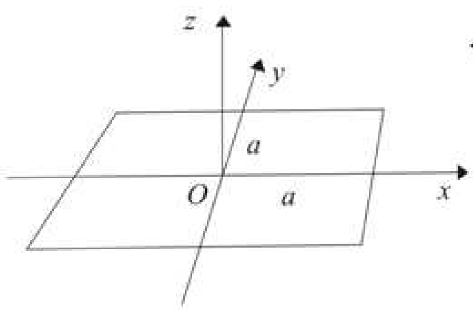
As $\delta x \to 0$

$$I = \int_{-l}^{l} \frac{mx^2}{2l}\mathrm{d}x = \frac{m}{2l}\left[\frac{x^3}{3}\right]_{-l}^{l}$$

$$= \frac{m}{2l}\left[\frac{l^3}{3} - \left(-\frac{l^3}{3}\right)\right] = \frac{1}{3}ml^2, \text{ as required}$$

As the small pieces range from one end of the rod to the other, x ranges from $-l$ at one end to l at the other. So $-l$ and l are the limits of the definite integral.

b



Let O be the centre of the plate, Ox and Oy be axes parallel to the sides of the square and Oz be the axis through O perpendicular to the plate, as shown in this sketch. The question asks you to find the moment of inertia about Oz.

By the stretching rule, the moment of inertia of the lamina about Ox and Oy is given by

$$I_{Ox} = I_{Oy} = \frac{1}{3}ma^2$$

By symmetry, the moment of inertia about Ox and Oy is the same.

By the perpendicular axes theorem

$$I_{Oz} = I_{Ox} + I_{Oy} = \frac{1}{3}ma^2 + \frac{1}{3}ma^2 = \frac{2}{3}ma^2$$

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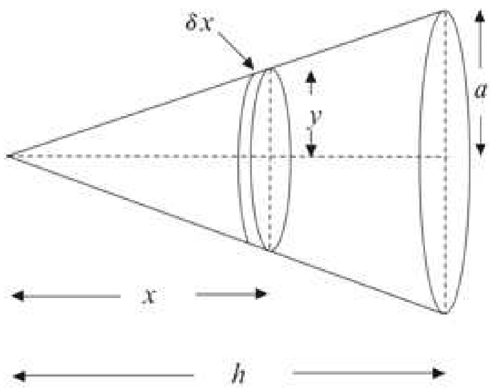
Review Exercise 2
Exercise A, Question 3

Question:

Given that the moment of inertia of a uniform disc, of mass m and radius r, about an axis through the centre perpendicular to the disc is $\frac{1}{2}mr^2$, show by integration that the moment of inertia of a uniform solid circular cone, of base radius a, height h and mass M, about its axis of symmetry is $\frac{3}{10}Ma^2$. ***E***

Solution:



You consider the cone to be made up of thin discs, each of thickness δx with the centre of the disc at a distance x from the vertex of the cone. If the radius of the disc is y, then, using the formula, $V = \pi r^2 h$, for the volume of a cylinder, the volume of a thin disc is $\pi y^2 \delta x$.

The mass per unit volume of the cone is

$$\frac{M}{\frac{1}{3}\pi a^2 h} = \frac{3M}{\pi a^2 h}$$

You are expected to know that the volume of a cone of height h and base radius r is given by $V = \frac{1}{3}\pi r^2 h$.

The moment of inertia of an elementary disc about the axis of symmetry is given by

$$\delta I = \frac{1}{2}(\delta m)y^2$$

This is the point at which you use the given result, that the moment of inertia of a disc is $\frac{1}{2}mr^2$. The mass of the disc is δm and the radius y.

By similar triangles

$$\frac{y}{x} = \frac{a}{h} \Rightarrow y = \frac{ax}{h}$$

Hence

$$\delta I = \frac{1}{2}\left(\pi y^2 \delta x\right)\left(\frac{3M}{\pi a^2 h}\right)y^2 = \frac{3M}{2a^2 h}y^4 \delta x$$

$$= \frac{3M}{2a^2 h}\left(\frac{ax}{h}\right)^4 \delta x = \frac{3Ma^2}{2h^5}x^4 \delta x$$

The mass, δm, of the disc is its volume, $\pi y^2 \delta x$, multiplied by the mass per unit length, $\frac{3M}{\pi a^2 h}$.

For the complete cone

$$I = \sum \delta I = \sum \frac{3Ma^2}{2h^5}x^4 \delta x$$

As $\delta x \to 0$

$$I = \int_0^h \frac{3Ma^2}{2h^5}x^4 \, dx = \frac{3Ma^2}{2h^5}\left[\frac{x^5}{5}\right]_0^h$$

$$= \frac{3Ma^2}{2h^5}\left(\frac{h^5}{5} - 0\right) = \frac{3}{10}Ma^2, \text{ as required}$$

As the thin, or elementary, discs range from the vertex to the base, x ranges from 0 to h. So 0 and h are the limits of integration.

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Review Exercise 2
Exercise A, Question 4

Question:

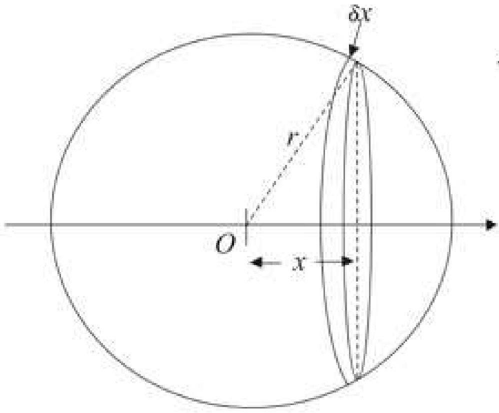
a Prove, using integration, that the moment of inertia of a uniform solid sphere, of mass M and radius r, about a diameter is $\frac{2}{5}Mr^2$.

[You may assume that the moment of inertia of a uniform disc, of mass m and radius a, about an axis through the centre perpendicular to the disc is $\frac{1}{2}ma^2$.]

b Hence obtain the moment of inertia of a solid hemisphere, of mass m and radius r, about a diameter of its plane face. ***E***

Solution:

a



You consider the cone to be made up of thin discs, each of thickness δx, with the centre of the disc at a distance x from the centre O of the sphere. If the radius of the disc is y, then, using the formula $V = \pi r^2 h$ for the volume of a cylinder, the volume of a thin disc is $\pi y^2 \delta x$.

The mass per unit volume of the sphere is

$$\frac{M}{\frac{4}{3}\pi r^3} = \frac{3M}{4\pi r^3}$$

You are expected to know that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.

The moment of inertia of an elementary disc is given by

$$\delta I = \frac{1}{2}(\delta m) y^2$$

$$y^2 = r^2 - x^2$$

You know, from module C2, that the equation of the circle is $x^2 + y^2 = r^2$.

Hence

$$\delta I = \frac{1}{2}\left(\pi y^2 \delta x\right)\left(\frac{3M}{4\pi r^3}\right) y^2 = \frac{3M}{8r^3} y^4 \delta x$$

$$= \frac{3M}{8r^3}\left(r^2 - x^2\right)^2 \delta x = \frac{3M}{8r^3}\left(r^4 - 2r^2x^2 + x^4\right)\delta x$$

For the complete sphere

$$I = \sum \delta I = \sum \frac{3M}{8r^3}\left(r^4 - 2r^2x^2 + x^4\right)\delta x$$

As $\delta x \to 0$

$$I = \int_{-r}^{r} \frac{3M}{8r^3}\left(r^4 - 2r^2x^2 + x^4\right) \mathrm{d}x$$

$$= \frac{3M}{8r^3}\left[r^4 x - \frac{2r^2x^3}{3} + \frac{x^5}{5}\right]_{-r}^{r}$$

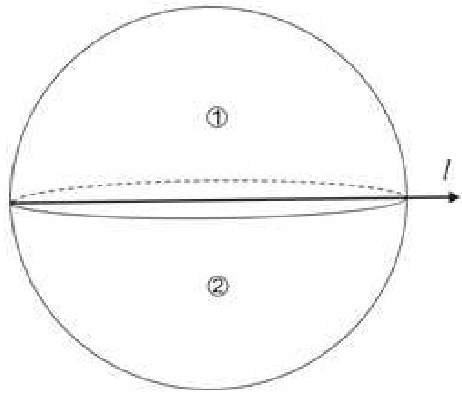
A common error is to integrate r^4 as $\frac{r^5}{5}$. With respect to x, r is a constant so $\int r^4 \, \mathrm{d}x = r^4 x$.

$$= \frac{3M}{8r^3}\left[\left(r^5 - \frac{2r^5}{3} + \frac{r^5}{5}\right) - \left(-r^5 + \frac{2r^5}{3} - \frac{r^5}{5}\right)\right]$$

$$= \frac{3M}{8r^3} \times 2r^5\left(1 - \frac{2}{3} + \frac{1}{5}\right) = \frac{3Mr^2}{4} \times \frac{8}{15}$$

$$= \frac{2}{5}Mr^2, \text{ as required}$$

b



If you consider a complete sphere as being made up to two hemispheres, then, by the addition rule, the sum of the moments of inertia of the two hemispheres about the axis l, in this diagram, must equal the moment of inertia of the whole sphere about l. So a hemisphere has half the moment of inertia of the whole sphere. However, the mass m is not now the mass of the whole sphere and you must be careful to avoid the incorrect answer $\frac{1}{5}mr^2$.

If the mass of the whole sphere is $2m$ and the radius of the sphere is r, then using the result of part **a**, the moment of inertia of the whole sphere is

$$\frac{2}{5}(2m)r^2 = \frac{4}{5}mr^2$$

By symmetry, the moment of inertia of the hemisphere, labelled ① in the diagram about l, must equal the moment of inertia of the hemisphere, labelled ② in the diagram, about the same axis.

Hence, the moment of inertia of one hemisphere is

$$\frac{1}{2} \times \frac{4}{5}mr^2 = \frac{2}{5}mr^2.$$

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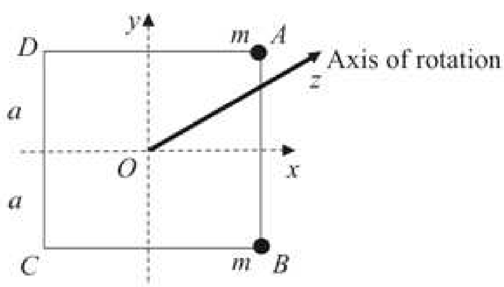
Review Exercise 2
Exercise A, Question 5

Question:

A uniform square lamina $ABCD$, of mass m and side $2a$, is free to rotate in a vertical plane about an axis through its centre O. Particles, each of mass m, are attached at the points A and B. The system is released from rest with AB vertical.

Show that the angular speed of the square when AB is horizontal is $\sqrt{\left(\frac{6g}{7a}\right)}$. ***E***

Solution:



The moment of inertia of the lamina alone about the axis of rotation is given, using the perpendicular axes theorem, by

$$I_{Oz} = I_{Ox} + I_{Oy}$$
$$= \frac{1}{3}ma^2 + \frac{1}{3}ma^2 = \frac{2}{3}ma^2$$
$$OA^2 = OB^2 = a^2 + a^2 = 2a^2$$

The moment of inertia of the lamina about Ox in the diagram is, by the stretching rule, the same as the moment of inertia of a uniform rod about its centre.

The moment of inertia of the lamina together with the particles about the axis of rotation is given by

$$I = I_{Oz} + m\left(OA^2\right) + m\left(OB^2\right)$$
$$= \frac{2}{3}ma^2 + m\left(2a^2\right) + m\left(2a^2\right) = \frac{14}{3}ma^2$$

In many questions, involving moments of inertia, you need to begin by finding the moment of inertia of the whole system, in this case the lamina with both particles, about the axis of rotation.

As the loaded plate rotates from the position with AB vertical to the position with AB horizontal
Conservation of energy
Kinetic energy gained = Potential energy lost

$$\frac{1}{2}I\dot{\theta}^2 = mg \times 2a$$
$$\dot{\theta}^2 = \frac{4mga}{I} = \frac{4mga}{\frac{14}{3}ma^2} = \frac{6g}{7a}$$
$$\dot{\theta} = \sqrt{\left(\frac{6g}{7a}\right)}, \text{ as required}$$

As AB moves from the vertical to the horizontal, the position of the centre of the lamina is unchanged and the level of the particle at B is the same in the vertical and horizontal positions. So the only potential energy lost is by the particle at A falling a distance $2a$.

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Review Exercise 2
Exercise A, Question 6

Question:

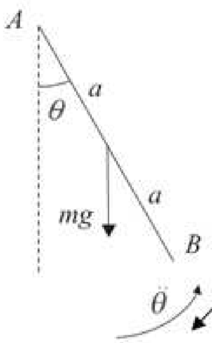
A uniform rod AB, of mass m and length $2a$, is free to rotate in a vertical plane about a smooth horizontal axis through A and perpendicular to the plane. The rod hangs in equilibrium with B below A. The rod is rotated through a small angle and released from rest at time $t = 0$.

a Show that the motion is approximately simple harmonic.

b Using this approximation, find the time t when the rod is first vertical after being released. ***E***

Solution:

a



Rates of change of the angle are always measured in the direction of θ increasing. So here the moment of the mass of the rod about A will be negative.

The equation of rotational motion about A is

$$L = I\ddot{\theta}$$

$$-mga\sin\theta = \frac{4}{3}ma^2\ddot{\theta}$$

Standard results for moments of inertia are given in the Formulae Booklet.

$$\ddot{\theta} = -\frac{3g}{4a}\sin\theta$$

For small θ, $\sin\theta \approx \theta$

Hence

$$\ddot{\theta} = -\frac{3g}{4a}\theta$$

Comparing this equation with $\ddot{\theta} = -\omega^2\theta$, $\omega^2 = \frac{3g}{4a}$ and you will need this to answer part **b**.

Comparing with the standard equation for simple harmonic motion, $\ddot{\theta} = -\omega^2\theta$, the motion is approximately simply harmonic.

b $t = \frac{1}{4}T = \frac{1}{4} \times \frac{2\pi}{\omega} = \frac{\pi}{2\omega}$

$$= \frac{\pi}{2\sqrt{\left(\frac{3g}{4a}\right)}} = \pi\sqrt{\left(\frac{a}{3g}\right)}$$

The period of the motion is the time it takes for the rod, after it is released, to return to its starting position for the first time. The time from the first release to when the rod first reaches the vertical is one quarter of this period.

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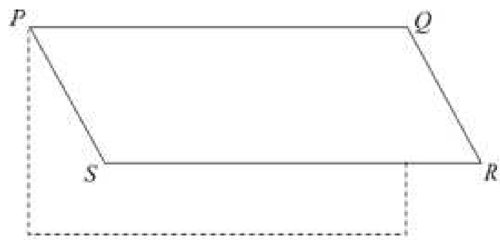
Review Exercise 2
Exercise A, Question 7

Question:

A uniform lamina of mass m is in the shape of a rectangle $PQRS$, where $PQ = 8a$ and $QR = 6a$.

a Find the moment of inertia of the lamina about the edge PQ.



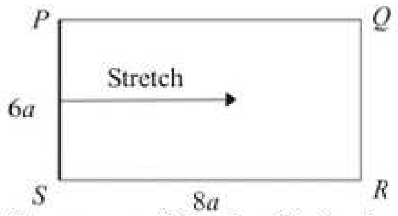
The flap on a letterbox is modelled as such a lamina. The flap is free to rotate about an axis along its horizontal edge PQ, as shown in the figure. The flap is released from rest in a horizontal position. It then swings down into a vertical position.

b Show that the angular speed of the flap as it reaches the vertical position is $\sqrt{\left(\frac{g}{2a}\right)}$.

c Find the magnitude of the vertical component of the resultant force of the axis PQ on the flap, as it reaches the vertical position. ***E***

Solution:

a

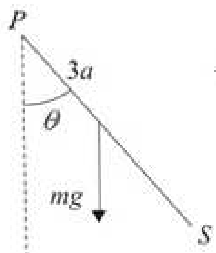


As the lamina $PQRS$ can be formed by stretching a rod PS without altering the distribution of the mass relative to PQ, then, by the stretching rule, the moment of inertia of the lamina about PQ is the same as the rod, $\frac{4}{3}ml^2$, where $2l$ is the length of PS. Here $PS = 2l = 6a$.

The moment of inertia of the lamina about PQ is given by

$$I = \frac{4}{3}ml^2 = \frac{4}{3}m(3a)^2 = 12ma^2$$

b



This diagram is drawn viewing the flap from the side. The weight acts at the centre of mass of the lamina.

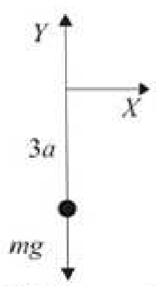
Let PS make an angle θ with the downward vertical at time t.
When PS reaches the vertical
Conservation of energy
Kinetic energy gained = Potential energy lost

$$\frac{1}{2}I\dot{\theta}^2 = mg \times 3a$$
$$6ma^2\dot{\theta}^2 = 3mga$$
$$\dot{\theta}^2 = \frac{3mga}{6ma^2} = \frac{g}{2a}$$

In falling from the horizontal to the vertical position, the centre of mass of the lamina falls a distance $3a$.

The angular speed of the flap as it reaches the vertical position is , $\sqrt{\left(\frac{g}{2a}\right)}$ as required.

c



When writing down an equation of motion, you consider the whole mass of the lamina to be at the centre of mass of the lamina, which is $3a$ from the axis of rotation.

Let the magnitude of the vertical component of the resultant force of the axis PQ on the flap, as it reaches the vertical position be Y.

$$R(\uparrow) \quad \mathbf{F} = m\mathbf{a}$$
$$Y - mg = mr\dot{\theta}^2$$
$$= m(3a)\frac{g}{2a}$$
$$Y = mg + \frac{3}{2}mg = \frac{5}{2}mg$$

From part **b**, we know that $\dot{\theta}^2 = \frac{g}{2a}$.

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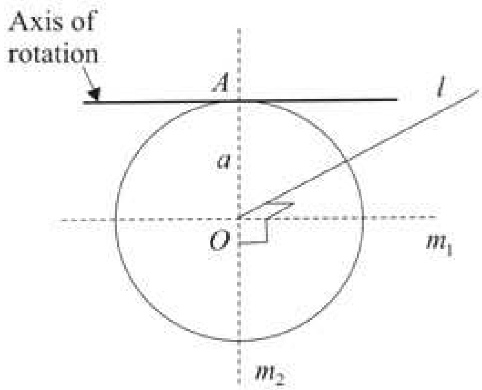
Review Exercise 2
Exercise A, Question 8

Question:

A uniform circular disc has mass m and radius a. The disc can rotate freely about an axis that is in the same plane as the disc and tangential to the disc at a point A on its circumference. The disc hangs at rest in equilibrium with its centre O vertically below A.
A particle P of mass m is moving horizontally and perpendicular to the disc with speed $\sqrt{(kga)}$, where k is a constant. The particle then strikes the disc at O and adheres to it at O.
Given that the disc rotates through an angle of $90°$ before first coming to instantaneous rest, find the value of k. ***E***

Solution:



By symmetry, the moments of inertia about the perpendicular axes m_1 and m_2, shown in the diagram, are equal.
By the perpendicular axes theorem,

$$I_l = I_{m_1} + I_{m_2}$$

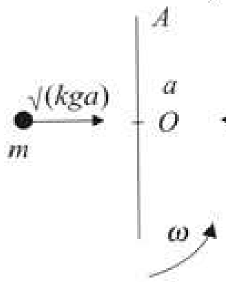
$$\frac{1}{2}ma^2 = 2I_{m_1}$$

$$I_{m_1} = \frac{1}{4}ma^2$$

Using the parallel axes theorem, the moment of inertia of the disc, I_d, about the axis of rotation is given by

$$I_d = I_{m_1} + ma^2 = \frac{1}{4}ma^2 + ma^2 = \frac{5}{4}ma^2$$

The standard result for a disc, of mass m and radius a, is that the moment of inertia of the disc about an axis, shown in this diagram as l, is $\frac{1}{2}ma^2$. From this, you find the moment of inertia about the axis of rotation using both the perpendicular and parallel axes theorems.



This sketch is drawn looking at the disc edge on.

Let the angular velocity of the disc and P immediately after impact be ω.
Conservation of angular momentum about A

$$mva = I\dot{\theta}$$

$$m\sqrt{(kga)}a = \left(\frac{5}{4}ma^2 + ma^2\right)\omega$$

$$\not{m}\sqrt{(kga)}\not{a} = \frac{9}{4}\not{m}a^{\not{2}}\omega$$

$$\omega = \frac{4}{9}\sqrt{\left(\frac{kg}{a}\right)}$$

By the addition rule, the moment of inertia of the disc and P about the axis of rotation is the moment of inertia of the disc $\left(\frac{5}{4}ma^2\right)$ added to the moment of inertia of P about the axis of rotation (ma^2).

Conservation of energy

$$\frac{1}{2}I\omega^2 = 2mg \times a$$

$$\frac{1}{2} \times \frac{9}{4}ma^2 \times \frac{16kg}{81a} = 2mga$$

$$\frac{2}{9}kmga = 2mga$$

$$k = \frac{18mga}{2mga} = 9$$

In moving from the vertical through 90° both the centre of mass of the disc and P, that is a total mass of $2m$, rise the distance a.

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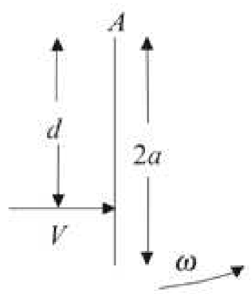
Review Exercise 2
Exercise A, Question 9

Question:

A rod AB, of length $2a$ and mass $2m$, lies at rest on a smooth horizontal table and is pivoted about a smooth vertical axis through A. A small body of mass m, moving on the table with speed V at right angles to the rod, strikes the rod at a distance d from A. Given that the body sticks to the rod after impact, find the angular speed with which the rod starts to move. ***E***

Solution:



Let the angular speed of the rod and body about A immediately after the impact be ω.

Conservation of angular momentum about A

$$mVd = I\omega$$

$$mVd = \left(\frac{4}{3}(2m)a^2 + md^2\right)\omega$$

$$\omega = \frac{mVd}{\frac{8}{3}ma^2 + md^2}$$

$$= \frac{3Vd}{8a^2 + 3d^2}$$

Before the impact, the angular momentum of the body of mass m is the linear momentum of the body (mV) multiplied by the distance d.

As the mass of the particle is $2m$, the moment of inertia of the rod about its end is $\frac{4}{3}(2m)a^2$. To this you must add the moment of inertia of the body of mass m about A.

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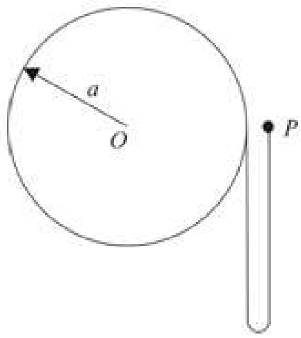
Review Exercise 2
Exercise A, Question 10

Question:



The figure shows a pulley in the form of a uniform disc of mass $2m$, centre O, and radius a. The pulley is free to rotate in a vertical plane about a fixed smooth horizontal axis through O. A light inextensible string has one end attached to a point on the rim of the pulley and is wrapped several times round the rim. The portion of the string which is not wrapped round the pulley is of length $4a$ and has a particle P of mass m attached to its free end. P is held close to the rim of the disc and level with O, with the disc at rest. The particle P is released from rest in this position.
Determine the angular speed of the disc immediately after the string becomes taut. ***E***

Solution:

Let v be the speed P immediately before the string becomes taut.

$v^2 = u^2 + 2as$

$= 0^2 + 2 \times g \times 4a$

$v = \sqrt{(8ga)}$

P falls a distance $4a$ freely under gravity before the string becomes taut.

The combined moment of inertia of the pulley and P about O is given by

$$I = \frac{1}{2}(2m)a^2 + ma^2 = 2ma^2$$

Let the angular speed of the pulley about O immediately after the impact be ω.

ω will also be the angular speed of P about O.

Conservation of angular momentum about O

$mva = I\dot{\theta}$

$m \times \sqrt{(8ag)} \times a = 2ma^2\omega$

Before the impulse, use the moment of the linear momentum of P about O.
After the impulse, use $I\omega$ for the disc and P.

$$\omega = \frac{\sqrt{(8ag)}}{2a} = \sqrt{\left(\frac{2g}{a}\right)}$$

Solutionbank M5
Edexcel AS and A Level Modular Mathematics

Review Exercise 2
Exercise A, Question 11

Question:

A uniform circular disc, of mass m and radius r, has a diameter AB. The point C on AB is such that $AC = \frac{1}{2}r$. The disc can rotate freely in a vertical plane about a horizontal axis through C, perpendicular to the plane of the disc. The disc makes small oscillations in a vertical plane about the position of equilibrium in which B is below A.

a Show that the motion is approximately simple harmonic.

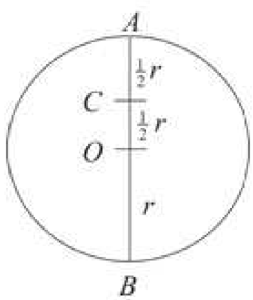
b Show that the period of this approximate simple harmonic motion is $\pi\sqrt{\frac{6r}{g}}$.

The speed of B when it is vertically below A is $\sqrt{\frac{gr}{54}}$. The disc comes to rest when CB makes an angle α with the downward vertical.

c Find an approximate value of α. ***E***

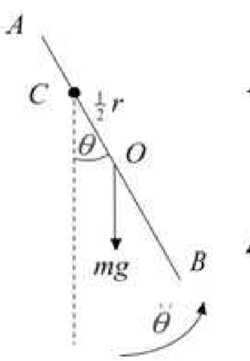
Solution:

a



Let the centre of the disc be O.
By the parallel axes theorem, the moment of inertia, I_C, of the disc about C is given by

$$I_C = I_O + m\left(\frac{1}{2}r\right)^2$$
$$= \frac{1}{2}mr^2 + \frac{1}{4}mr^2 = \frac{3}{4}mr^2$$



This diagram is drawn looking at the disc edge on.

Rates of change of the angle are always measured in the direction of θ increasing. So here the moment of the mass of the rod about A will be negative.

The equation of rotational motion about C is

$$L = I\ddot{\theta}$$

$$-mg\left(\frac{1}{2}r\sin\theta\right) = \frac{3}{4}mr^2\ddot{\theta}$$

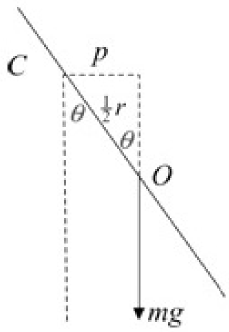
$$\ddot{\theta} = -\frac{2g}{3r}\sin\theta$$

For small $\theta, \sin\theta \approx \theta$

Hence

$$\ddot{\theta} = -\frac{2g}{3r}\theta$$

Comparing with the standard equation for simple harmonic motion, $\ddot{\theta} = -\omega^2\theta$, the motion is approximately simply harmonic, with $\omega^2 = \frac{2g}{3r}$.



The perpendicular distance from C to the line of action of the weight is given by

$\frac{p}{\frac{1}{2}r} = \sin\theta \Rightarrow p = \frac{1}{2}r\sin\theta$, so the magnitude of moment of the weight about C is $mg \times \left(\frac{1}{2}r\sin\theta\right)$. This moment is tending to make θ decrease, so it has a negative sign in this equation.

b The period of approximate simple harmonic motion is given by

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\left(\frac{2g}{3r}\right)}} = 2\pi\sqrt{\left(\frac{3r}{2g}\right)}$$

$$= \pi\sqrt{\left(\frac{6r}{g}\right)}, \text{ as required}$$

c The speed of P at B is the maximum speed during the simple harmonic motion.

Using $v = r\dot{\theta}$ with the speed of B

$$\sqrt{\left(\frac{gr}{54}\right)} = \left(\frac{3}{2}r\right)\dot{\theta}$$

$$\dot{\theta} = \sqrt{\left(\frac{gr}{54}\right)} \times \frac{2}{3r} = \frac{2}{3}\sqrt{\left(\frac{g}{54r}\right)}$$

At the maximum angular speed

$$\dot{\theta} = \omega\alpha$$

In module M3, you learnt that the maximum speed during simple harmonic motion is at the centre of the motion and is given by $v = \omega a$, where a is the amplitude. $\dot{\theta} = \omega\alpha$ is the corresponding formula for angular motion. α is the greatest angle during the motion, which is where the body is instantaneously at rest.

$$\frac{2}{3}\sqrt{\left(\frac{g}{54r}\right)} = \sqrt{\left(\frac{2g}{3r}\right)}\alpha$$

$$\alpha = \frac{2}{3}\sqrt{\left(\frac{1}{54}\right)} \times \sqrt{\left(\frac{3}{2}\right)} = \frac{2}{3}\sqrt{\left(\frac{3}{108}\right)}$$

$$= \frac{2}{3} \times \frac{1}{6} = \frac{1}{9}$$

This is an approximate answer. There are other methods of solving this question, for example using energy, which would give slightly different answers, but answers should all approximate to 0.11 radians.

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Review Exercise 2
Exercise A, Question 12

Question:

A uniform circular disc, of mass m, radius a and centre O, is free to rotate in a vertical plane about a fixed smooth horizontal axis. The axis passes through the mid-point A of a radius of the disc.

a Find an equation of motion for the disc when the line AO makes an angle θ with the downward vertical through A.

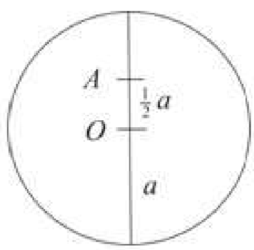
b Hence find the period of small oscillations of the disc about its position of stable equilibrium.

When the line AO makes an angle θ with the downward vertical through A, the force acting on the disc at A is $\mathbf{F}$.

c Find the magnitude of the component of $\mathbf{F}$ perpendicular to AO. ***E***

Solution:

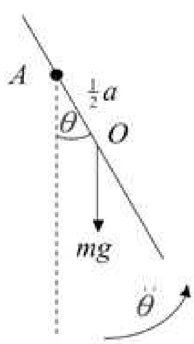
a



By the parallel axes theorem, the moment of inertia, I_A, of the disc about A is given by

$$I_A = I_O + m\left(\frac{1}{2}a\right)^2$$

$$= \frac{1}{2}ma^2 + \frac{1}{4}ma^2 = \frac{3}{4}ma^2$$



This diagram is drawn looking at the disc edge on.

The equation of motion about A is

$$L = I\ddot{\theta}$$

$$-mg\left(\frac{1}{2}a\sin\theta\right) = \frac{3}{4}ma^2\ddot{\theta}$$

$$\ddot{\theta} = -\frac{2g}{3a}\sin\theta$$

The moment of the weight about A is tending to make θ decrease so this term has a negative sign.

b For small $\theta, \sin\theta \approx \theta$

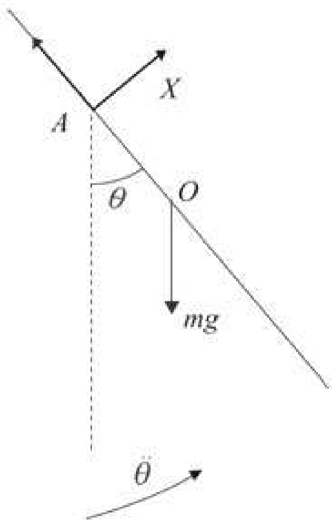
Hence

$$\ddot{\theta} = -\frac{2g}{3a}\theta$$

Comparing with the standard equation for simple harmonic motion, $\ddot{\theta} = -\omega^2\theta$, the motion is approximately simply harmonic, with $\omega^2 = \frac{2g}{3a}$. The period of small oscillations is given by

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\left(\frac{2g}{3a}\right)}} = 2\pi\sqrt{\left(\frac{3a}{2g}\right)}$$

c



Let the component of **F** perpendicular to AO be X.

$\text{R}(\perp AO)$

$$X - mg\sin\theta = mr\ddot{\theta}$$

$$= m\left(\frac{a}{2}\right)\left(-\frac{2g}{3a}\sin\theta\right) = -\frac{1}{3}mg\sin\theta$$

$$X = \frac{2}{3}mg\sin\theta$$

In part **c**, unlike part **b**, you are not told that the oscillations are small, so you must use the result of part **a** to substitute for $\ddot{\theta}$.

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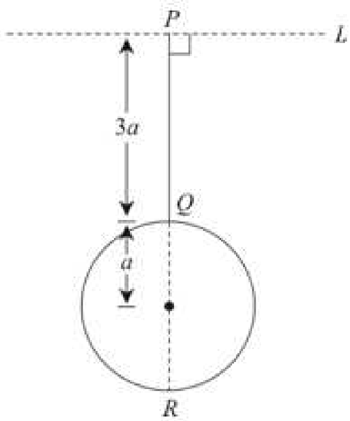
Review Exercise 2
Exercise A, Question 13

Question:



A thin uniform rod PQ has mass m and length $3a$. A thin uniform circular disc, of mass m and radius a, is attached to the rod at Q in such a way that the rod and the diameter QR of the disc are in a straight line with $PR = 5a$. The rod together with the disc form a composite body, as shown in the figure. The body is free to rotate about a fixed smooth horizontal axis L through P, perpendicular to PQ and in the plane of the disc.

a Show that the moment of inertia of the body about L is $\frac{77ma^2}{4}$.

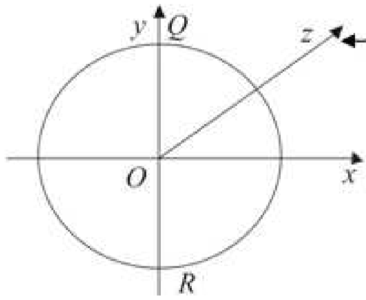
When PR is vertical, the body has angular speed ω and the centre of the disc strikes a stationary particle of mass $\frac{1}{2}m$. Given that the particle adheres to the centre of the disc,

b find, in terms of ω, the angular speed of the body immediately after the impact.

E

Solution:

a



The standard result for the moment of inertia of a disc $\left(I = \frac{1}{2}ma^2\right)$ is for an axis through its centre perpendicular to the plane of the disc. From this you need to find the moment of inertia about a diameter. You then use the moment of inertia about a diameter to find the moment of inertia of the disc about L.

Let O be the centre of the disc.
By the perpendicular axes theorem,
the moment of inertia, I_{Ox}, about the diameter through O perpendicular to PR is given by

$$I_{Ox} + I_{Oy} = I_{Oz}$$
$$2I_{Ox} = \frac{1}{2}ma^2$$
$$I_{Ox} = \frac{1}{4}ma^2$$

By the parallel axes theorem the moment of inertia of the disc, I_d, about L is given by

$$I_d = I_{Ox} + m(4a)^2$$
$$= \frac{1}{4}ma^2 + 16ma^2 = \frac{65ma^2}{4}$$

The parallel axes are Ox and L. The distance between these axes is $OP = 4a$.

The moment of inertia of the body about L is given by

$$I = I_d + \frac{4}{3}m\left(\frac{3a}{2}\right)^2$$
$$= \frac{65ma^2}{4} + 3ma^2 = \frac{77ma^2}{4}, \text{ as required}$$

Using the standard result for the moment of inertia of a rod about its end $\left(I = \frac{4}{3}ml^2\right)$. The length of the rod PQ is $3a$ giving $2l = 3a \Rightarrow l = \frac{3}{2}a$.

b The moment of inertia, I', of the body and the particle combined is given by

$$I' = I + \frac{1}{2}m(4a)^2$$
$$= \frac{77ma^2}{4} + 8ma^2 = \frac{109ma^2}{4}$$

The particle of mass $\frac{1}{2}m$ is at O, that is $4a$ from L.

Let the angular speed of the body immediately after the impact be ω'.
Conservation of angular momentum about L

$$I'\omega' = I\omega$$
$$\frac{109ma^2}{4}\omega' = \frac{77ma^2}{4}\omega$$
$$\omega' = \frac{77}{109}\omega$$

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Review Exercise 2
Exercise A, Question 14

Question:

A thin uniform rod AB, of mass M and length $2L$, is freely pivoted at A. The rod hangs vertically with B below A. A particle of mass $\frac{1}{2}M$, travelling horizontally with speed u, strikes the rod at B. After this impact the particle is at rest and the rod starts to move with angular speed ω.

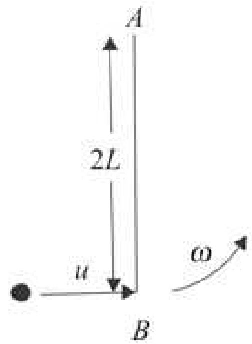
a Show that $\omega = \frac{3u}{4L}$.

The rod comes to instantaneous rest when AB is inclined at an angle $\arccos\left(\frac{1}{3}\right)$ to the downward vertical.

b Find u in terms of L and g. ***E***

Solution:

a



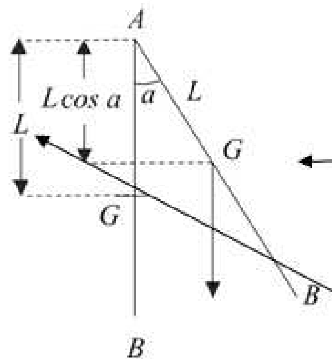
Conservation of angular momentum about A.

$$\frac{1}{2}Mu \times 2L = \frac{4}{3}ML^2\omega$$

$$\omega = \frac{6MLu}{8ML^2} = \frac{3u}{4L}, \text{ as required}$$

After the impact the particle is at rest, so the only angular momentum is that of the rod AB about its end A.

b



In this diagram, G is the centre of mass of the rod and $\alpha = \arccos\left(\frac{1}{3}\right)$.

Conservation of energy

Kinetic energy lost = Potential energy gained

$$\frac{1}{2}I\omega^2 = Mg(L - L\cos\alpha)$$

As the rod swings through an angle α, the centre of mass of the rod G rises a distance $L - L\cos\alpha$.

$$\frac{1}{2} \times \frac{4}{3}ML^2\left(\frac{3u}{4L}\right)^2 = Mg\left(L - \frac{1}{3}L\right)$$

Using the result of part **a**.

$$\frac{3}{8}Mu^2 = \frac{2}{3}MgL$$

$$u^2 = \frac{16}{9}gL$$

$$u = \frac{4}{3}\sqrt{(gL)}$$

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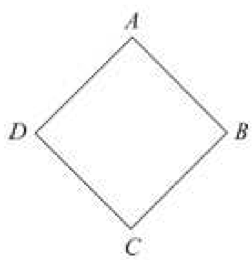
Review Exercise 2
Exercise A, Question 15

Question:



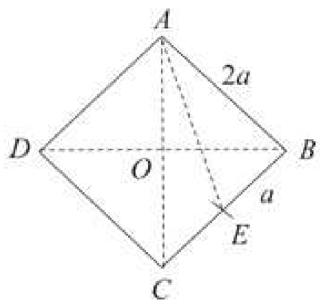
The figure shows four uniform rods, each of mass m and length $2a$, rigidly fixed together to form a square framework $ABCD$. The framework is free to rotate about a fixed smooth horizontal axis which passes through A and is perpendicular to the plane $ABCD$.

a Find the moment of inertia of the framework about this axis.

b Show, that for small oscillations of the framework about its position of equilibrium with C below A, the period of oscillation of the motion is $2\pi\left(\frac{(5\sqrt{2})a}{3g}\right)^{\frac{1}{2}}$.

Solution:

a



Let E be the mid-point of BC.

$AE^2 = a^2 + (2a)^2 = 5a^2$

Using Pythagoras' Theorem.

By the parallel axes theorem, the moment of inertia of the rod BC about the axis through A is given by

$$I_{BC} = \frac{1}{3}ma^2 + mAE^2$$

$$= \frac{1}{3}ma^2 + 5ma^2 = \frac{16}{3}ma^2$$

The moment of inertia of the framework about the axis through A is given by

$$I = I_{AB} + I_{BC} + I_{CD} + I_{DA}$$

$$= \frac{4}{3}ma^2 + \frac{16}{3}ma^2 + \frac{16}{3}ma^2 + \frac{4}{3}ma^2$$

$$= \frac{40}{3}ma^2$$

By symmetry, the moment of inertia of the rod CD about the axis through A is the same as the moment of inertia of the rod BC.

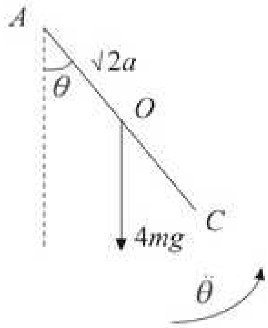
b Let O be the centre of the framework

$AO^2 + BO^2 = AB^2 = 4a^2$

$2AO^2 = 4a^2 \Rightarrow AO^2 = 2a^2$

$AO = \sqrt{2}a$

The total weight of the framework, $4mg$, acts at the centre of the framework O and, to form the equation of motion, you need to find the distance of the centre of mass from the axis of rotation.



This diagram is drawn looking at the oscillation of the framework from the side.

The equation of angular motion about A is

$$L = I\ddot{\theta}$$

$$-4mg\left(\sqrt{2}a\right)\sin\theta = \frac{40}{3}ma^2\ddot{\theta}$$

$$\ddot{\theta} = -\frac{3\sqrt{2}g}{10a}\sin\theta = -\frac{3g}{5\sqrt{2}a}\sin\theta$$

The moment of the weight about the axis through A is making θ decrease and so has a negative sign in the equation of motion.

For small θ, $\sin\theta \approx \theta$

Hence

$$\ddot{\theta} = -\frac{3g}{5\sqrt{2}a}\theta$$

Comparing with the standard equation for simple harmonic motion, $\ddot{\theta} = -\omega^2\theta$, the motion is approximately simply harmonic, with $\omega^2 = \frac{3g}{5\sqrt{2}a}$.

The period of small oscillations is given by

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\left(\frac{3g}{5\sqrt{2}a}\right)^{\frac{1}{2}}} = 2\pi\left(\frac{5\sqrt{2}a}{3g}\right)^{\frac{1}{2}}$$, as required

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Edexcel AS and A Level Modular Mathematics

Review Exercise 2
Exercise A, Question 16

Question:

A uniform square lamina $ABCD$, of mass m and side $2a$, is free to rotate in a vertical plane about a fixed smooth horizontal axis L which passes through A and is perpendicular to the plane of the lamina. The moment of inertia of the lamina about L is $\frac{8ma^2}{3}$.

Given that the lamina is released from rest when the line AC makes an angle of $\frac{\pi}{3}$ with the downward vertical,

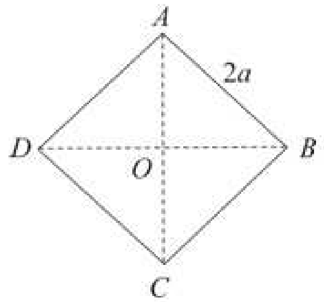
a find the magnitude of the vertical component of the force acting on the lamina at A when the line AC is vertical.

Given instead that the lamina now makes small oscillations about its position of stable equilibrium,

b find the period of these oscillations. ***E***

Solution:

a



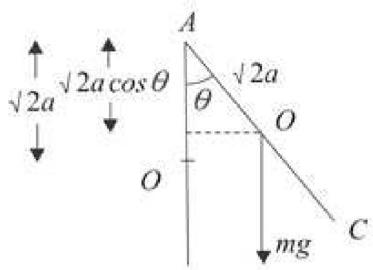
Let O be the centre of mass of the lamina

$$AO^2 + OB^2 = AB^2$$

$$2AO^2 = 4a^2 \Rightarrow AO^2 = 2a^2$$

$$AO = \sqrt{2}a$$

The weight of the lamina, mg, acts at the centre of the lamina O and you need to find the distance of the centre of mass from the axis of rotation.



As the lamina swings through an angle θ with the vertical, the centre of mass falls a distance of $\sqrt{2}a - \sqrt{2}a\cos\theta$.

Conservation of energy

$$\frac{1}{2}I\dot{\theta}^2 = mg\left(\sqrt{2}a - \sqrt{2}a\cos\theta\right)$$

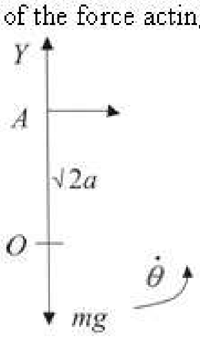
$$\frac{1}{2} \times \frac{8ma^2}{3}\dot{\theta}^2 = mg\sqrt{2}a\left(1 - \cos\frac{\pi}{3}\right) = \frac{mg\sqrt{2}a}{2}$$

$$\dot{\theta}^2 = \frac{3\sqrt{2}g}{8a}$$

The equation of motion needed to find the vertical component will contain a term for the radial acceleration $r\dot{\theta}^2$ and $\dot{\theta}^2$, when AO is vertical, can be found by conservation of energy.

$$1 - \cos\frac{\pi}{3} = 1 - \frac{1}{2} = \frac{1}{2}$$

When AC is vertical let the vertical component of the force acting on the lamina at A be Y.



$\text{R}(\uparrow)\quad \mathbf{F} = m\mathbf{a}$

$$Y - mg = m(\sqrt{2}a)\dot{\theta}^2$$

$$Y = mg + m\sqrt{2}a \times \frac{3\sqrt{2}g}{8a}$$

$$= mg + \frac{3}{4}mg = \frac{7}{4}mg$$

b The equation of angular motion about A is

$$L = I\ddot{\theta}$$

$$-mg(\sqrt{2}a)\sin\theta = \frac{8}{3}ma^2\ddot{\theta}$$

$$\ddot{\theta} = -\frac{3\sqrt{2}g}{8a}\sin\theta$$

For small θ, $\sin\theta \approx \theta$

This approximation, in radians, is accurate to within 1% up to 0.24 radians or up to 13.75°. This is accurate enough for many practical purposes.

Hence

$$\ddot{\theta} = -\frac{3\sqrt{2}g}{8a}\theta$$

Comparing with the standard equation for simple harmonic motion, $\ddot{\theta} = -\omega^2\theta$, the motion is approximately simply harmonic, with $\omega^2 = \frac{3\sqrt{2}g}{8a}$. The period of small oscillations is given by

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\left(\frac{3\sqrt{2}g}{8a}\right)^{\frac{1}{2}}} = 2\pi\left(\frac{8a}{3\sqrt{2}g}\right)^{\frac{1}{2}} = 2\pi\left(\frac{4\sqrt{2}a}{3g}\right)^{\frac{1}{2}}$$

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Review Exercise 2
Exercise A, Question 17

Question:

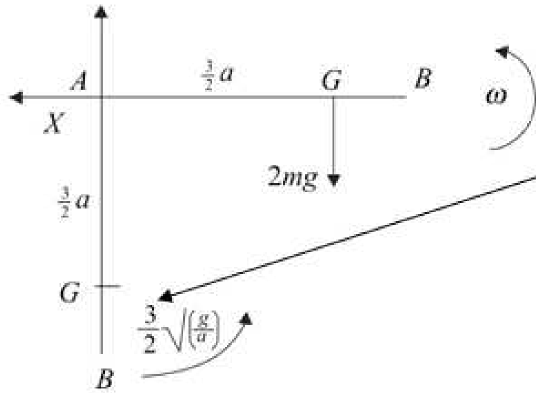
A uniform rod AB, of length $2a$ and mass m, can rotate freely about a fixed horizontal axis through A. A particle of mass m is attached at B. When AB is vertical, with B below A, the system has angular speed $\frac{3}{2}\sqrt{\left(\frac{g}{a}\right)}$.

a Show that, when AB is horizontal, its angular speed is $\frac{3}{4}\sqrt{\left(\frac{2g}{a}\right)}$.

b Find the horizontal component of the force exerted by the rod on the axis when AB is horizontal. ***E***

Solution:

a



The centre of mass, G, of the combined rod of mass m and particle of mass m will be half way between the middle of the rod and the particle at B, so $AG = \frac{3}{2}a$.

The moment of inertia of the combined rod and particle about A is given by

$$I = \frac{4}{3}ma^2 + m(2a)^2 = \frac{16}{3}ma^2$$

Let the angular speed of the system when AB is horizontal be ω.

Conservation of energy

$$\frac{1}{2}I\omega_0^2 - \frac{1}{2}I\omega^2 = 2mg \times \frac{3}{2}a$$

As the rod moves from the vertical to the horizontal the centre of mass of the system rises a distance $\frac{3}{2}a$.

$$\frac{1}{2}\times\frac{16}{3}ma^2\times\left(\frac{9g}{4a}\right) - \frac{1}{2}\times\frac{16}{3}ma^2\omega^2 = 3mga$$

$$\frac{8}{3}ma^2\omega^2 = 6mga - 3mga$$

$$\omega^2 = \frac{9g}{8a} = \frac{9}{16}\times\frac{2g}{a}$$

$$\omega = \frac{3}{4}\sqrt{\left(\frac{2g}{a}\right)}, \text{ as required}$$

b When AB is horizontal, let the horizontal component of the force exerted by the rod on the axis be X

When AB is horizontal

$$R(\leftarrow) \quad X = (2m)r\dot{\theta}^2$$

$$= 2m\left(\frac{3}{2}a\right)\left(\frac{9g}{8a}\right) = \frac{27}{8}mg$$

The weight has no component in the horizontal direction.

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Review Exercise 2
Exercise A, Question 18

Question:

Particles P and Q have mass $3m$ and m respectively. Particle P is attached to one end of a light inextensible string and Q is attached to the other end. The string passes over a circular pulley which can freely rotate in a vertical plane about a fixed horizontal axis through its centre O. The pulley is modelled as a uniform circular disc of mass $2m$ and radius a. The pulley is sufficiently rough to prevent the string slipping. The system is at rest with the string taut. A third particle R of mass m falls freely under gravity from rest for a distance a before striking and adhering to Q. Immediately before R strikes Q, particles P and Q are at rest with the string taut.

a Show that, immediately after R strikes Q, the angular speed of the pulley is $\frac{1}{3}\sqrt{\left(\frac{g}{2a}\right)}$.

When R strikes Q, there is an impulse in the string attached to Q.

b Find the magnitude of this impulse.

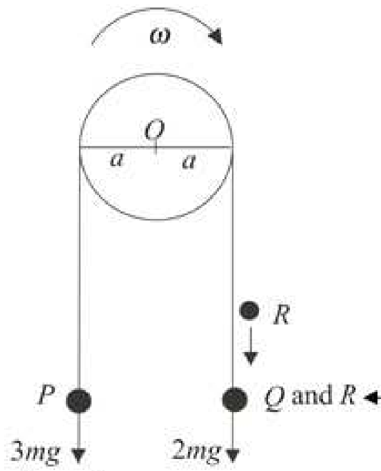
Given that P does not hit the pulley,

c find the distance that P moves upwards before first coming to instantaneous rest.

E

Solution:

a



> R adheres to Q and, after the impact, they form a single particle of mass $2m$.

$v^2 = u^2 + 2as$

$\quad = 0^2 + 2ga$

$v = \sqrt{(2ag)}$

> You first need to find the speed of R immediately before it strikes Q.

Let the angular speed of the disc immediately after impact be ω.
Conservation of angular momentum about the centre of the pulley O

> As the string does not slip, ω is also the angular speed of the whole system, consisting of the pulley, three particles and string.

$$m\sqrt{(2ga)} \times a = \left(3ma^2 + 2ma^2 + \frac{1}{2}(2m)a^2\right)\omega$$

$$= 6ma^2\omega$$

$$\omega = \frac{\sqrt{(2ga)}}{6a} = \frac{1}{3}\sqrt{\left(\frac{g}{2a}\right)}, \text{ as required}$$

> The moment of inertia of the system about O is given by
> $I = I_P + I_{R \text{ and } Q} + I_{\text{pulley}}$
> $= (3m)a^2 + (2m)a^2 + \frac{1}{2}(2ma^2)$
> $= 6ma^2$.

b The impulse J in the string attached to Q is given by

$$J = 2ma\omega - m\sqrt{(2ga)}$$

$$= 2ma \times \frac{1}{3}\sqrt{\left(\frac{g}{2a}\right)} - m\sqrt{(2ga)}$$

$$= \frac{1}{3}m\sqrt{(2ga)} - m\sqrt{(2ga)} = -\frac{2}{3}m\sqrt{(2ga)}$$

> Immediately before impact, only R is moving and has linear momentum $mv = m\sqrt{(2ga)}$.
> Immediately after impact, Q and R are combined and have linear momentum $2mv$, where $v = a\omega$.

The magnitude of the impulse is $\frac{2}{3}m\sqrt{(2ga)}$.

c Let the distance that P moves upwards before first coming to instantaneous rest be s.
Conservation of energy
Kinetic energy lost = Potential energy gained

$$\frac{1}{2}I\omega^2 = 3mgs - 2mgs$$

$$\frac{1}{2}(6ma^2)\frac{g}{18a} = mgs$$

$$s = \frac{1}{6}a$$

P has risen a distance of s and gained potential energy $3mgs$. Q and R have fallen a distance s and have lost potential energy $2mgs$. The net gain in potential energy is mgs.
All of the kinetic energy of the system has been lost.

Using the expression for the angular speed found in part **a**.

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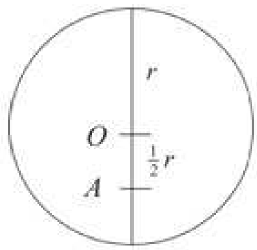
Review Exercise 2
Exercise A, Question 19

Question:

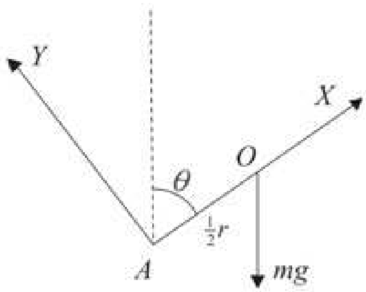
A uniform circular disc of mass m and radius r is free to rotate about a fixed smooth horizontal axis perpendicular to the plane of the disc and at a distance $\frac{1}{2}r$ from the centre of the disc. The disc is held at rest with the centre of the disc vertically above the axis.
Given that the disc is slightly disturbed from its position of rest, find the magnitude of the force on the axis when the centre of the disc is in the horizontal plane of the axis. ***E***

Solution:



Let O be the centre of the disc and the point where the axis of rotation meets the disc be A.
By the parallel axes theorem, the moment of inertia, I_A, of the disc about A is given by

$$I_A = I_O + m\left(\frac{1}{2}r\right)^2$$
$$= \frac{1}{2}mr^2 + \frac{1}{4}mr^2 = \frac{3}{4}mr^2$$



When AO has rotated through an angle θ, let the components of the force parallel and perpendicular to AO be X and Y respectively.
Conservation of energy

$$\frac{1}{2}I\dot{\theta}^2 = mg\left(\frac{1}{2}r - \frac{1}{2}r\cos\theta\right)$$
$$\frac{1}{2}\times\frac{3}{4}mr^2\dot{\theta}^2 = \frac{1}{2}mgr(1-\cos\theta)$$
$$r\dot{\theta}^2 = \frac{4}{3}g(1-\cos\theta) \qquad ①$$

The centre of mass O of the disc has fallen a distance $\frac{1}{2}r - \frac{1}{2}r\cos\theta$.

R(∥ AO) $\mathbf{F} = m\mathbf{a}$

$$mg\cos\theta - X = m\left(\frac{1}{2}r\right)\dot{\theta}^2$$
$$X = mg\cos\theta - \frac{1}{2}m\times\frac{4}{3}g(1-\cos\theta)$$
$$= \frac{5}{3}mg\cos\theta - \frac{2}{3}mg$$

When AO is horizontal, $\theta = \frac{\pi}{2}$

$X = -\frac{2}{3}mg$

The component of the force parallel to AO is in the opposite direction to that indicated in the diagram. When drawing diagrams of the forces acting on the axis of rotation, you need not worry about the directions of the components. If you have them the wrong way round, this comes out in the working.

$\mathrm{R}(\perp AO)$ $\quad \mathbf{F} = m\mathbf{a}$

$$mg\sin\theta - Y = m\left(\frac{1}{2}r\right)\ddot{\theta}$$

$$Y = mg\sin\theta - \frac{1}{2}mr\ddot{\theta} \quad ②$$

Differentiating ① with respect to t

$$2r\dot{\theta}\,\ddot{\theta} = \frac{4}{3}g\sin\theta\,\dot{\theta}$$

$$r\ddot{\theta} = \frac{2}{3}g\sin\theta \quad ③$$

Using the chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\dot{\theta}^2) = \frac{\mathrm{d}}{\mathrm{d}\dot{\theta}}(\dot{\theta}^2)\times\frac{\mathrm{d}\dot{\theta}}{\mathrm{d}t} = 2\dot{\theta}\ddot{\theta} \text{ and}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\cos\theta) = \frac{\mathrm{d}}{\mathrm{d}\theta}(\cos\theta)\times\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\sin\theta\,\dot{\theta}$$

.

Substituting ③ into ②

$$Y = mg\sin\theta - \frac{1}{3}mg\sin\theta = \frac{2}{3}mg\sin\theta$$

When AO is horizontal, $\theta = \frac{\pi}{2}$

$$Y = \frac{2}{3}mg$$

Let the magnitude of the force on the axis be R

$$R^2 = X^2 + Y^2 = \frac{4}{9}m^2g^2 + \frac{4}{9}m^2g^2 = \frac{8}{9}m^2g^2$$

$$R = \frac{2\sqrt{2}}{3}mg$$

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Review Exercise 2
Exercise A, Question 20

Question:

A uniform rod AB, of mass m and length $2a$, is free to rotate in a vertical plane about a fixed smooth horizontal axis through A. The rod is hanging in equilibrium with B below A when it is hit by a particle of mass m moving horizontally with speed v in a vertical plane perpendicular to the axis. The particle strikes the rod at B and immediately adheres to it.

a Show that the angular speed of the rod immediately after the impact is $\frac{3v}{8a}$.

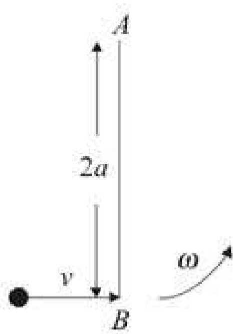
Given that the rod rotates through $120°$ before first coming to instantaneous rest,

b find v in terms of a and g,

c find, in terms of m and g, the magnitude of the vertical component of the force acting on the rod at A immediately after the impact. ***E***

Solution:

a



Let the angular speed of the rod immediately after the impact be ω.
The moment of inertia, I, of the rod combined with the particle about A is given by

$$I = I_{rod} + I_{particle}$$
$$= \frac{4}{3}ma^2 + m(2a)^2 = \frac{16}{3}ma^2$$

Conservation of angular momentum about A

$$mv \times 2a = \frac{16}{3}ma^2\omega$$
$$\omega = \frac{6mva}{16ma^2} = \frac{3v}{8a}, \text{ as required}$$

Before the impact, the angular momentum of the particle about A is its linear momentum (mv) multiplied by the distance AB ($2a$).

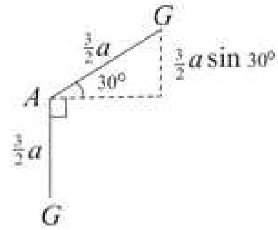
b Let the centre of mass of the particle combined with the rod be G, then

$$AG = \frac{3}{2}a$$

The centre of mass, G, of the combined rod of mass m and particle of mass m will be half way between the middle of the rod and the particle at B, so $AG = \frac{3}{2}a$.

Conservation of energy

$$\frac{1}{2}I\omega^2 = 2mg \times \left(\frac{3}{2}a + \frac{3}{2}a\sin 30^\circ\right)$$



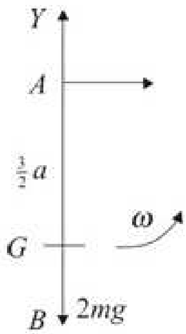
As AB rotates through 120°, G rises a distance of $\frac{3}{2}a + \frac{3}{2}a\sin 30^\circ = \frac{9}{4}a$.

$$\frac{8}{3}ma^2\left(\frac{3v}{8a}\right)^2 = \frac{9}{2}mga$$

$$v^2 = \frac{9}{2} \times \frac{8}{3}ga = 12ga$$

$$v = \sqrt{(12ga)} = 2\sqrt{(3ga)}$$

c



Let the magnitude of the vertical component of the force acting on the rod at A immediately after the impact be Y.

Immediately after the impact

R(↑) $\mathbf{F} = m\mathbf{a}$

The radial component of the acceleration is $r\dot{\theta}^2$.

$$Y - 2mg = 2mr\dot{\theta}^2$$

$$= 2m\left(\frac{3}{2}a\right)\left(\frac{3v}{8a}\right)^2 = \frac{27m}{64a}v^2$$

Using the answer to part **a**.

$$= \frac{27m}{64a} \times 12ga = \frac{81}{16}mg$$

Using the answer to part **b**.

$$Y = 2mg + \frac{81}{16}mg = \frac{113}{16}mg$$

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Review Exercise 2
Exercise A, Question 21

Question:

A uniform lamina, of mass m, has the form of a quadrant of a circle radius a.

a Show, by integration, that the moment of inertia of the lamina about an axis l perpendicular to the plane of the lamina and through the centre of the circle of which it is part, is $\frac{1}{2}ma^2$.

The lamina is free to rotate about l, which is horizontal, and when the centre of mass of the lamina is immediately below the axis of rotation the angular speed is Ω.

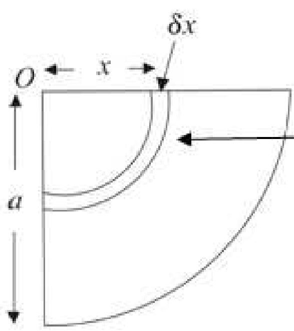
b Determine whether the lamina makes complete revolutions in the cases

i $\Omega = 2\sqrt{\left(\frac{g}{a}\right)}$,

ii $\Omega = 3\sqrt{\left(\frac{g}{a}\right)}$. ***E***

Solution:

a



You consider the quadrant to be made up of thin 'quarter rings', each of thickness δx.

The mass per unit area of the quadrant is $\frac{m}{\frac{1}{4}\pi a^2} = \frac{4m}{\pi a^2}$

The moment of inertia of an element of radius x and thickness δx is given by

$$\delta I = (\delta m)x^2 = \left(\frac{\pi x}{2} \times \frac{4m}{\pi a^2}\delta x\right)x^2 = \frac{2mx^3}{a^2}\delta x$$

Each 'quarter ring' has length one quarter of the circumference of the corresponding complete circle, that is $\frac{1}{4} \times 2\pi x = \frac{\pi x}{2}$.

For the whole quadrant

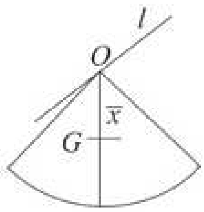
$$I = \sum \delta I = \sum \frac{2mx^3}{a^2}\delta x$$

As $\delta x \to 0$

$$I = \int_0^a \frac{2mx^3}{a^2}\mathrm{d}x = \left[\frac{2mx^4}{4a^2}\right]_0^a$$

$$= \frac{2ma^4}{4a^2} = \frac{1}{2}ma^2, \text{ as required}$$

b



Let G be the centre of mass of the quadrant, O the centre of the circle of which the quadrant is part, and $OG = \bar{x}$.

$$\bar{x} = \frac{2a \sin \alpha}{3\alpha}$$

This formula for the centre of mass of a sector of a circle, radius r, angle at centre 2α, is one of the formulae for module M2 given in the Formulae Booklet. For module M5 you are expected to know the specifications for modules M1, M2, M3 and M4 together with their associated formulae.

With $\alpha = \frac{\pi}{4}$

$$\bar{x} = \frac{2a \sin \frac{\pi}{4}}{3 \times \frac{\pi}{4}} = \frac{8a \times \frac{1}{\sqrt{2}}}{3\pi} = \frac{4\sqrt{2}a}{3\pi}$$

For complete revolutions, by energy,

$$\frac{1}{2} I\Omega^2 \geq mg \times 2\bar{x}$$

For complete revolutions, the kinetic energy at the lowest point must be sufficient to allow the centre of mass of the lamina, G, to rise from the point where G is vertically below O to the point where G is vertically above O; that is a distance of $2\bar{x}$.

$$\frac{1}{4} ma^2\Omega^2 \geq mg \frac{8\sqrt{2}a}{3\pi}$$

$$\Omega^2 \geq \frac{32\sqrt{2}g}{3\pi a}$$

$$\Omega \geq \left(\frac{32\sqrt{2}}{3\pi}\right)^{\frac{1}{2}} \sqrt{\left(\frac{g}{a}\right)} \approx 2.19\sqrt{\left(\frac{g}{a}\right)}$$

i If $\Omega = 2\sqrt{\left(\frac{g}{a}\right)}$, as $2 < 2.19$, the lamina does not make complete revolutions.

ii If $\Omega = 3\sqrt{\left(\frac{g}{a}\right)}$, as $3 > 2.19$, the lamina does make complete revolutions.

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Review Exercise 2
Exercise A, Question 22

Question:

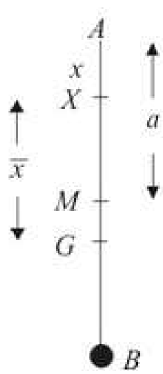
A uniform rod AB, of length $2a$ and mass $6m$, has a particle of mass $2m$ attached at B. The rod is free to rotate in a vertical plane about a smooth fixed vertical axis perpendicular to the rod and passing through a point X of the rod so that $AX = x$, where $x < a$.

a Show that the moment of inertia of the system about this axis is $4m(4a^2 - 5ax + 2x^2)$.

b Find the period of small oscillations of the system about its equilibrium position with B below A. ***E***

Solution:

a



Let the mid-point of the rod be M.
The moment of inertia of the rod, I_R, about X is given by

$$I_R = \frac{1}{3}(6m)a^2 + 6m \times XM^2$$

Using the parallel axes theorem.

$$= 2ma^2 + 6m(a-x)^2$$

The moment of inertia, I, of the rod and particle combined is given by

$$I = I_R + I_P$$
$$= 2ma^2 + 6m(a-x)^2 + 2m(2a-x)^2$$
$$= 2ma^2 + 6ma^2 - 12max + 6mx^2 + 8ma^2 - 8max + 2mx^2$$
$$= 16ma^2 - 20max + 8mx^2$$
$$= 4m(4a^2 - 5ax + 2x^2), \text{as required}$$

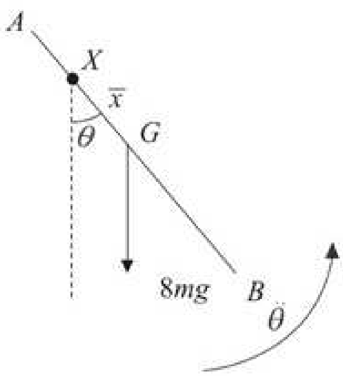
b Let the centre of mass of the rod and particle combined be G and $GX = \bar{x}$.

M(X) $8mg\bar{x} = 6mg(a-x) + 2mg(2a-x)$

$$8\bar{x} = 6a - 6x + 4a - 2x = 10a - 8x$$

$$\bar{x} = \frac{5}{4}a - x$$

The total weight of the rod and particle acts at G and you can find the position of G by taking moments about X. You can then use this distance to write down the equation of rotational motion.



The equation of rotational motion about X is

$$L = I\ddot{\theta}$$

$$-8mg\bar{x}\sin\theta = I\ddot{\theta}$$

$$\ddot{\theta} = -\frac{8mg\bar{x}}{I}\sin\theta$$

The moment of the weight about the axis through X is tending to make θ decrease and so has a negative sign in the equation of rotational motion.

For small θ, $\sin\theta \approx \theta$

Hence

$$\ddot{\theta} = -\frac{8mg\bar{x}}{I}\theta = -\frac{8mg\left(\frac{5}{4}a - x\right)}{4m\left(4a^2 - 5ax + 2x^2\right)}\theta$$

$$= -\frac{g(5a-4x)}{2\left(4a^2 - 5ax + 2x^2\right)}\theta$$

Comparing with the standard equation for simple harmonic motion, $\ddot{\theta} = -\omega^2\theta$, the motion is approximately simply harmonic, with

$$\omega^2 = \frac{g(5a-4x)}{2\left(4a^2 - 5ax + 2x^2\right)}.$$

So $\omega = \left(\frac{g(5a-4x)}{2\left(4a^2 - 5ax + 2x^2\right)}\right)^{\frac{1}{2}}$ and you use this expression in the formula for the period of simple harmonic motion $T = \frac{2\pi}{\omega}$.

The period of small oscillations is given by

$$T = \frac{2\pi}{\omega} = 2\pi\left(\frac{2\left(4a^2 - 5ax + 2x^2\right)}{g(5a-4x)}\right)^{\frac{1}{2}}$$

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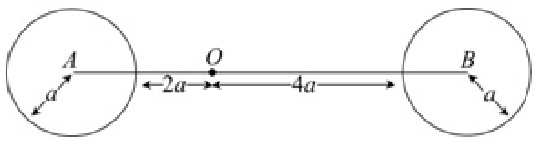
Review Exercise 2
Exercise A, Question 23

Question:



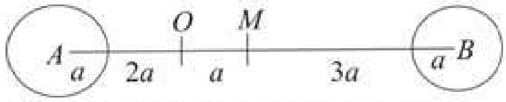
A body consists of two uniform circular discs, each of mass m and radius a, with a uniform rod. The centres of the discs are fixed to the ends A and B of the rod, which has mass $3m$ and length $8a$. The discs and the rod are coplanar, as shown in the figure. The body is free to rotate in a vertical plane about a smooth fixed horizontal axis. The axis is perpendicular to the plane of the discs and passes through the point O of the rod, where $AO = 3a$.

a Show that the moment of inertia of the body about the axis is $54ma^2$.
The body is held at rest with AB horizontal and is then released. When the body has turned through an angle of $30°$, the rod AB strikes a small fixed smooth peg P where $OP = 3a$. Given that the body rebounds from the peg with its angular speed halved by the impact,

b show that the magnitude of the impulse exerted on the body by the peg at the impact is $9m\sqrt{\left(\frac{5ga}{6}\right)}$. ***E***

Solution:

a



In this diagram, M is the mid-point of AB. By symmetry, M is the centre of mass of the body.

The moment of inertia of the rod AB, I_{AB}, about O is given by

$$I_{AB} = \frac{1}{3}(3m)(4a)^2 + (3m)a^2 = 19ma^2$$

The moment of inertia of the disc centre A, I_A, about O is given by

$$I_A = \frac{1}{2}ma^2 + m(3a)^2 = \frac{19}{2}ma^2$$

The moment of inertia of the disc centre B, I_B, about O is given by

$$I_B = \frac{1}{2}ma^2 + m(5a)^2 = \frac{51}{2}ma^2$$

You use the parallel axes theorem for all three of the component parts which make up the body.

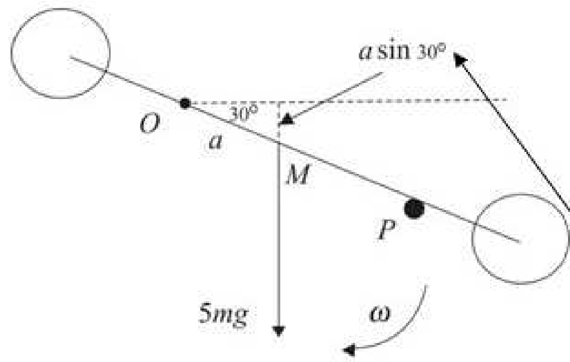
The moment of inertia of the whole body, I, about O is given by

$$I = I_{AB} + I_A + I_B$$

$$= 19ma^2 + \frac{19}{2}ma^2 + \frac{51}{2}ma^2$$

$$= 54ma^2$$, as required

b



As the body rotates through 30°, the centre of mass of the body, M, falls vertically a distance $a\sin 30° = \frac{1}{2}a$.

Let the angular speed of the body immediately before the impact be ω.
Conservation of energy
Kinetic energy gained = Potential energy lost

$$\frac{1}{2}I\omega^2 = 5mg \times \frac{1}{2}a$$

$$\omega^2 = \frac{5mga}{I} = \frac{5g}{54a}$$

$$\omega = \sqrt{\left(\frac{5g}{54a}\right)} = \frac{1}{3}\sqrt{\left(\frac{5g}{6a}\right)}$$

Let the angular speed of the body immediately after the impact be ω'.

$$\omega' = \frac{1}{6}\sqrt{\left(\frac{5g}{6a}\right)}$$

The question gives you that the angular speed is halved by the impact.

Let J be the magnitude of the impulse exerted on the body by the peg at the impact.

Moment of impulse = change in angular momentum

$$-3aJ = I(-\omega') - I\omega$$

The impulse is in a direction which decreases the angle the rod makes with the horizontal.

$$3aJ = I(\omega + \omega')$$

$$= 54ma^2\left[\frac{1}{3}\sqrt{\left(\frac{5g}{6a}\right)} + \frac{1}{6}\sqrt{\left(\frac{5g}{6a}\right)}\right]$$

The angular velocities before and after impact are in opposite senses. As drawn, ω is clockwise and ω' is anti-clockwise.

$$= 54ma^2 \times \frac{1}{2}\sqrt{\left(\frac{5g}{6a}\right)}$$

$$J = \frac{27ma^2}{3a}\sqrt{\left(\frac{5g}{6a}\right)} = 9m\sqrt{\left(\frac{5ga}{6}\right)}, \text{ as required}$$

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Review Exercise 2
Exercise A, Question 24

Question:

a Show that the moment of inertia of a uniform solid right circular cone, of mass m, height h and base radius a, about a line through its vertex and perpendicular to its axis of symmetry is

$$\frac{3}{20}m(a^2+4h^2)$$

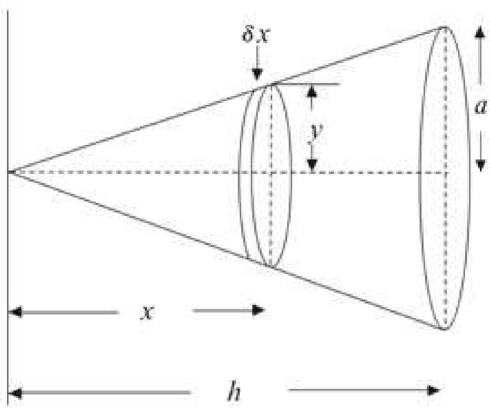
[You may assume that the moment of inertia of a uniform circular disc, of mass M and radius R, about a diameter is $\frac{1}{4}MR^2$.]

A cone, with $h=\frac{2}{3}a$, is free to rotate about a smooth horizontal axis through its vertex.

b Find the period of small oscillations under gravity about the stable position of equilibrium. ***E***

Solution:

a Axis of rotation



You consider the cone to be made up of thin discs, each of thickness δx with the centre of the disc at a distance x from the vertex of the cone. If the radius of the disc is y, then, using the formula, $V = \pi r^2 h$, for the volume of a cylinder, the volume of a thin disc is $\pi y^2 \delta x$.

The mass per unit volume of the cone is

$$\frac{m}{\frac{1}{3}\pi a^2 h} = \frac{3m}{\pi a^2 h}$$

The moment of inertia of an elementary disc about the axis of rotation is given by

$$\delta I = \frac{1}{4}(\delta m)y^2 + (\delta m)x^2$$

The question specifies that you can use the formula $I = \frac{1}{4}MR^2$ for the thin disc. You use this, with $M = \delta m$ and $R = y$, and the parallel axes theorem to form an expression for the moment of inertia, δI, of the thin disc about the axis of rotation.

By similar triangles

$$\frac{y}{x} = \frac{a}{h} \Rightarrow y = \frac{ax}{h}$$

Hence

$$\delta I = \frac{1}{4}(\pi y^2 \delta x)\left(\frac{3m}{\pi a^2 h}\right)y^2 + (\pi y^2 \delta x)\left(\frac{3m}{\pi a^2 h}\right)x^2$$

$$= \frac{3m}{a^2 h}\left(\frac{y^4}{4} + y^2 x^2\right)\delta x = \frac{3m}{a^2 h}\left(\frac{a^4 x^4}{4h^4} + \frac{a^2 x^4}{h^2}\right)\delta x$$

$$= \frac{3m}{4h^5}(a^2 + 4h^2)x^4 \delta x$$

The mass, δm, of the disc is its volume, $\pi y^2 \delta x$, multiplied by the mass per unit length $\frac{3m}{\pi a^2 h}$.

For the complete cone

$$I = \sum \delta I = \sum \frac{3m}{4h^5}(a^2 + 4h^2)x^4 \delta x$$

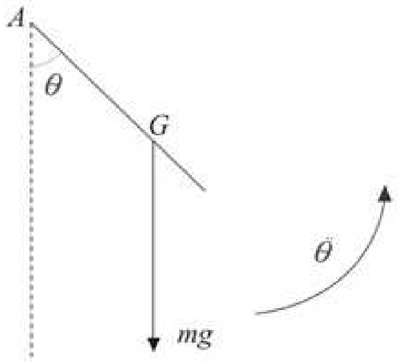
As $\delta x \to 0$

$$I = \int_0^h \frac{3m}{4h^5}(a^2 + 4h^2)x^4 \, dx = \frac{3m}{4h^5}(a^2 + 4h^2)\left[\frac{x^5}{5}\right]_0^h$$

$$= \frac{3m}{4h^5}(a^2 + 4h^2)\left(\frac{h^5}{5} - 0\right) = \frac{3}{20}m(a^2 + 4h^2), \text{ as required}$$

As the thin, or elementary, discs range from the vertex to the base, x ranges from 0 to h. So 0 and h are the limits of integration.

b



Let A be the vertex of the cone and G the centre of mass of the cone.

$AG = \frac{3}{4}h = \frac{3}{4} \times \frac{2}{3}a = \frac{1}{2}a$

A formula for the centre of mass of a cone is given among the formulae for module M3 in the Formulae Booklet. For module M5 you are expected to know the specifications for modules M1, M2, M3 and M4 together with their associated formulae.

As $h = \frac{2}{3}a$,

$$I = \frac{3}{20}m\left(a^2 + 4h^2\right) = \frac{3}{20}m\left(a^2 + 4 \times \frac{4a^2}{9}\right)$$

$$= \frac{3}{20}m \times \frac{25}{9}a^2 = \frac{5}{12}ma^2$$

The equation of motion about A is

$$L = I\ddot{\theta}$$

$$-mg\left(\frac{a}{2}\right)\sin\theta = \frac{5}{12}ma^2\ddot{\theta}$$

$$\ddot{\theta} = -\frac{6g\sin\theta}{5a}$$

The moment of the weight about A is tending to make θ decrease and so has a negative sign in the equation of rotational motion.

For small $\theta, \sin\theta \approx \theta$

Hence

$$\ddot{\theta} = -\frac{6g}{5a}\theta$$

Comparing with the standard equation for simple harmonic motion, $\ddot{\theta} = -\omega^2\theta$, the motion is approximately simply harmonic, with $\omega^2 = \frac{6g}{5a}$.

The period of small oscillations is given by

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{6g}{5a}}} = 2\pi\sqrt{\frac{5a}{6g}}$$

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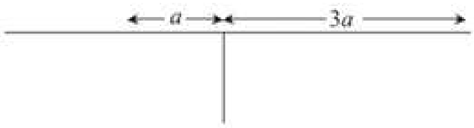
Edexcel AS and A Level Modular Mathematics

Review Exercise 2
Exercise A, Question 25

Question:



A rough uniform rod, of mass m and length $4a$, is held on a rough horizontal table. The rod is perpendicular to the edge of the table and a length $3a$ projects horizontally over the edge, as shown in the figure.

a Show that the moment of inertia of the rod about the edge of the table is $\frac{7}{3}ma^2$.

The rod is released from rest and rotates about the edge of the table. When the rod has turned through an angle θ, its angular speed is $\dot{\theta}$. Assuming that the rod has not started to slip,

b show that $\dot{\theta}^2 = \frac{6g\sin\theta}{7a}$,

c find the angular acceleration of the rod,

d find the normal reaction of the table on the rod.

The coefficient of friction between the rod and the edge of the table is μ.

e Show that the rod starts to slip when $\tan\theta = \frac{4}{13}\mu$. ***E***

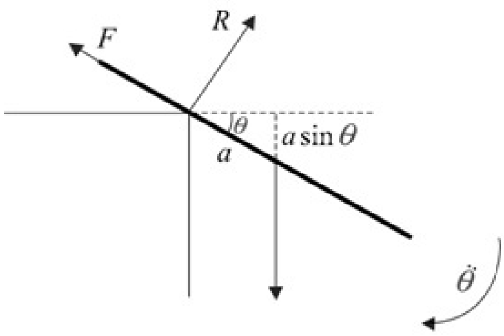
Solution:

a Using the parallel axes theorem, the moment of inertia, I, of the rod about the edge of the table is given by

$I = \frac{1}{3}m(2a)^2 + ma^2 = \frac{7}{3}ma^2$, as required

Using the standard formula for the moment of inertia of a rod about its centre $\frac{1}{3}ml^2$ with $l = 2a$. The centre of mass of the rod is the distance a from the edge of the table.

b



Conservation of energy

Kinetic energy gained = Potential energy lost

$$\frac{1}{2}I\dot{\theta}^2 = mga\sin\theta$$

$$\frac{7}{6}ma^2\dot{\theta}^2 = mga\sin\theta$$

$$\dot{\theta}^2 = \frac{6g\sin\theta}{7a}\text{, as required}$$

As the rod rotates through θ, its centre of mass falls a vertical distance $a\sin\theta$.

c Differentiate the result of **b** implicitly throughout with respect to t

$$2\dot{\theta}\ddot{\theta} = \frac{6g\cos\theta}{7a}\dot{\theta}$$

$$\ddot{\theta} = \frac{3g\cos\theta}{7a}$$

Using the chain rule,

$\frac{d}{dt}(\dot{\theta}^2) = \frac{d}{d\dot{\theta}}(\dot{\theta}^2) \times \frac{d\dot{\theta}}{dt} = 2\dot{\theta}\ddot{\theta}$ and

$\frac{d}{dt}(\sin\theta) = \frac{d}{d\theta}(\sin\theta) \times \frac{d\theta}{dt} = \cos\theta\,\dot{\theta}$

d Let the reaction on the rod, normal to the rod, at the edge of the table be R.

R($\perp AB$) $\mathbf{F} = m\mathbf{a}$

$mg\cos\theta - R = ma\ddot{\theta}$

The component of the weight is in the direction of θ increasing and R is in the direction of θ decreasing.

$$R = mg\cos\theta - ma\ddot{\theta} = mg\cos\theta - ma\frac{3g}{7a}\cos\theta$$

Using the result of part **c**.

$$= \frac{4mg\cos\theta}{7}$$

e Let F be the frictional force at the edge of the table

R($\parallel AB$) $\mathbf{F} = m\mathbf{a}$

$F - mg\sin\theta = ma\dot{\theta}^2$

The radial component of the acceleration is $r\dot{\theta}^2$.

$$F = mg\sin\theta + ma\dot{\theta}^2 = mg\sin\theta + ma\frac{6g\sin\theta}{7a}$$

$$= \frac{13mg\sin\theta}{7}$$

Using the result of part **b**.

As the rod starts to slip

$F = \mu R$

$$\frac{13mg\sin\theta}{7} = \mu\frac{4mg\cos\theta}{7}$$

$\tan\theta = \frac{4}{13}\mu$, as required

As $\frac{\sin\theta}{\cos\theta} = \tan\theta$.

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Review Exercise 2
Exercise A, Question 26

Question:

a Show, by integration, that the moment of inertia of a uniform circular disc, of mass M and radius a, about an axis which passes through its centre and is perpendicular to its plane is $\frac{1}{2}Ma^2$.

b Without further integration, deduce the moment of inertia of the disc

i about an axis perpendicular to its plane and passing through a point on its circumference,

ii about a diameter.

A uniform disc, of mass M and radius a, is suspended from a smooth pivot on its circumference and rests in equilibrium.

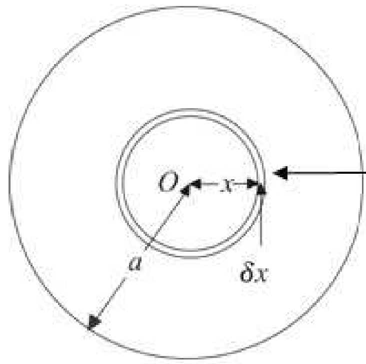
c Calculate the period of small oscillations when the centre of the disc is slightly displaced

i in the plane of the disc,

ii perpendicular to the plane of the disc. ***E***

Solution:

a



You consider the disc as made up a series of concentric rings of thickness δx. The area of each ring is $2\pi x\delta x$.

Let the centre of the disc be O.

The mass per unit area of the disc is $\dfrac{M}{\pi a^2}$.

The moment of inertia, δI, of a ring is given by

$$\delta I = (\delta m)x^2 = \left(2\pi x\delta x \times \frac{M}{\pi a^2}\right)x^2 = \frac{2M}{a^2}x^3\delta x$$

The moment of inertia of the disc, I, is given by

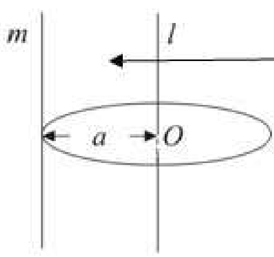
$$I = \sum \delta I = \sum \frac{2M}{a^2}x^3\delta x$$

As $\delta x \to 0$

$$I = \int_0^a \frac{2M}{a^2}x^3 \, dx = \frac{2M}{a^2}\left[\frac{x^4}{4}\right]_0^a = \frac{2M}{a^2}\left(\frac{a^4}{4} - 0\right)$$

$$= \frac{1}{2}Ma^2, \text{ as required}$$

b i



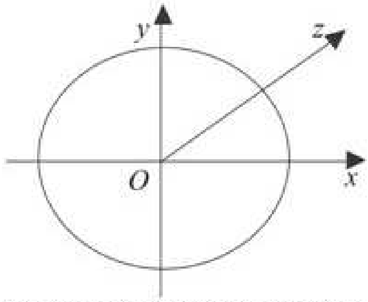
In part **a**, you have shown that the moment of inertia about the axis l, an axis through the centre perpendicular to the plane of the disc, is $\frac{1}{2}Ma^2$. You find the moment of inertia about the axis m, an axis parallel to l, through a point on the circumference of the disc, using the result of part **a** and the parallel axes theorem.

By the parallel axes theorem

$$I_m = I_l + Ma^2$$

$$= \frac{1}{2}Ma^2 + Ma^2 = \frac{3}{2}Ma^2$$

ii



By the perpendicular axes theorem, the moment of inertia, I_{Ox}, about a diameter Ox through O is given by

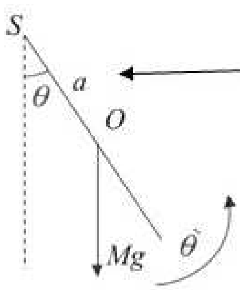
$$I_{Ox} + I_{Oy} = I_{Oz}$$

$$2I_{Ox} = \frac{1}{2}Ma^2$$

$$I_{Ox} = \frac{1}{4}Ma^2$$

By symmetry the moment of inertia about the axis Ox equals the moment of inertia about the axis Oy.

c



In this diagram, the smooth pivot is S and the angle SO makes with the downward vertical is θ.

The equation of rotational motion about S is

$$L = I\ddot{\theta}$$

$$-Mga\sin\theta = I\ddot{\theta}$$

$$\ddot{\theta} = -\frac{Mga}{I}\sin\theta$$

By leaving the moment of inertia as I, you can find the equations of angular motion for both parts **c i** and **c ii** together.

For small $\theta, \sin\theta \approx \theta$

Hence

$$\ddot{\theta} = -\frac{Mga}{I}\theta$$

Comparing with the standard equation for simple harmonic motion, $\ddot{\theta} = -\omega^2\theta$, the motion is approximately simply harmonic, with

$$\omega^2 = \frac{Mga}{I}.$$

i $I = \frac{3}{2}Ma^2$

The axis of rotation is the same as in part **b i**.

$$\omega^2 = \frac{Mga}{I} = \frac{Mga}{\frac{3}{2}Ma^2} = \frac{2g}{3a}$$

Hence the period of small oscillations is given by

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\left(\frac{3a}{2g}\right)} = \pi\sqrt{\left(\frac{6a}{g}\right)}$$

ii By the parallel axes theorem the moment of inertia, I, about a tangent to the disc is given by

$$I = \frac{1}{4}Ma^2 + Ma^2 = \frac{5}{4}Ma^2$$

The axis of rotation in this part is a tangent to the disc which is parallel to a diameter and, so, you use the result of part **b ii** together with the parallel axes theorem.

$$\omega^2 = \frac{Mga}{I} = \frac{Mga}{\frac{5}{4}Ma^2} = \frac{4g}{5a}$$

Hence the period of small oscillations is given by

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\left(\frac{5a}{4g}\right)} = \pi\sqrt{\left(\frac{5a}{g}\right)}$$

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Edexcel AS and A Level Modular Mathematics

Review Exercise 2
Exercise A, Question 27

Question:

A uniform plane circular disc, of mass m and radius a, hangs in equilibrium from a point B on its circumference. The disc is free to rotate about a fixed smooth horizontal axis which is in the plane of the disc and tangential to the disc at B. A particle P, of mass m, is moving horizontally with speed u in a direction which is perpendicular to the plane of the disc. At time $t = 0, P$ strikes the disc at its centre and adheres to the disc.

a Show that the angular speed of the disc immediately after it has been struck by P is $\frac{4u}{9a}$.

It is given that $u^2 = \frac{1}{10}ag$, and that air resistance is negligible.

b Find the angle through which the disc turns before it first comes to instantaneous rest.

The disc first returns to its initial position at time $t = T$.

c **i** Write down an equation of motion for the disc.

ii Hence find T in terms of a, g and m, using a suitable approximation which should be justified. ***E***

Solution:

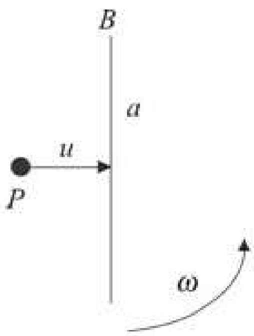
a The moment of inertia of the disc about a diameter is $\frac{1}{4}ma^2$.

This formula is given the Formulae Booklet and, if you are not asked to prove it, it can be quoted.

The moment of inertia, I, of the disc about a tangent is given, using the parallel axes theorem, by

$$I = \frac{1}{4}ma^2 + ma^2 = \frac{5}{4}ma^2$$

A tangent is parallel to a diameter and the distance between the tangent and the diameter is the radius a.



This diagram is drawn looking at the disc edge on. ω is the angular speed of the disc and particle immediately after impact.

The moment of inertia, I', of the disc and P about the axis through B is given by

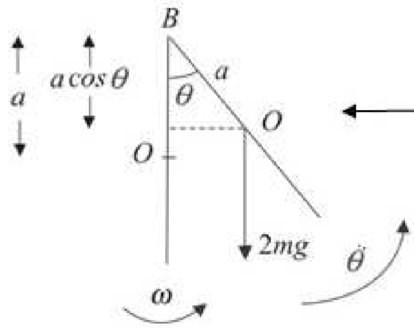
$$I' = \frac{5}{4}ma^2 + ma^2 = \frac{9}{4}ma^2$$

Conservation of angular momentum about B

$$mu \times a = I'\omega = \frac{9}{4}ma^2\omega$$

$$\omega = \frac{4u}{9a}, \text{ as required}$$

b



This diagram is drawn looking at the disc edge on. O is the centre of the disc.

As $u^2 = \frac{1}{10}ag$,

$$\omega^2 = \frac{16u^2}{81a^2} = \frac{16}{81a^2} \times \frac{1}{10}ag = \frac{8g}{405a}$$

Using the answer to part **a**.

Let the disc first come to rest when $\theta = \alpha$

Conservation of energy

$$\frac{1}{2}I\omega^2 = 2mg(a - a\cos\alpha)$$

From the diagram, as the disc swings through θ, the centre of the disc rises a vertical distance of $a - a\cos\theta$.

$$\frac{1}{2} \times \frac{9ma^2}{4} \times \frac{8g}{405a} = 2mga(1 - \cos\alpha)$$

$$\frac{1}{45} = 2(1 - \cos\alpha) \Rightarrow \cos\alpha = \frac{89}{90}$$

$$\alpha = \arccos\left(\frac{89}{90}\right) = 8.5^\circ, \text{ to the nearest } 0.1^\circ.$$

c **i** The equation of rotational motion about B is

$$L = I\ddot{\theta}$$

$$-2mga\sin\theta = I\ddot{\theta}$$

$$\ddot{\theta} = -\frac{2mga}{I}\sin\theta = -\frac{2mga}{\frac{9}{4}ma^2}\sin\theta = -\frac{8g}{9a}\sin\theta$$

ii For small θ, $\sin\theta \approx \theta$

Hence

$$\ddot{\theta} = -\frac{8g}{9a}\theta$$

Comparing with the standard equation for simple harmonic motion, $\ddot{\theta} = -\omega^2\theta$, the motion is approximately simply harmonic, with

$$\omega^2 = \frac{8g}{9a}.$$

Hence

$$T = \frac{1}{2} \times \frac{2\pi}{\omega} = \pi\sqrt{\left(\frac{9a}{8g}\right)} = \frac{3\pi}{2}\sqrt{\left(\frac{a}{2g}\right)}$$

The time the disc takes to moves from the centre of its motion to its amplitude and then back to the centre of its motion is one half of a complete period.

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Edexcel AS and A Level Modular Mathematics

Review Exercise 2
Exercise A, Question 28

Question:

Four uniform rods, each of mass m and length $2a$, are joined together at their ends to form a plane rigid square framework $ABCD$ of side $2a$. The framework is free to rotate in a vertical plane about a fixed smooth horizontal axis through A. The axis is perpendicular to the plane of the framework.

a Show that the moment of inertia of the framework about the axis is $\frac{40ma^2}{3}$.

The framework is slightly disturbed from rest when C is vertically above A. Find

b the angular acceleration of the framework when AC is horizontal,

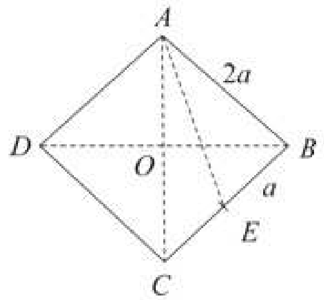
c the angular speed of the framework when AC is horizontal,

d the magnitude of the force acting on the framework at A when AC is horizontal.

E

Solution:

a



Let E be the mid-point of BC.

$AE^2 = a^2 + (2a)^2 = 5a^2$ ← Using Pythagoras' Theorem.

By the parallel axes theorem, the moment of inertia of the rod BC about the axis through A is given by

$$I_{BC} = \frac{1}{3}ma^2 + mAE^2$$

$$= \frac{1}{3}ma^2 + 5ma^2 = \frac{16}{3}ma^2$$

The moment of inertia of the framework about the axis through A is given by

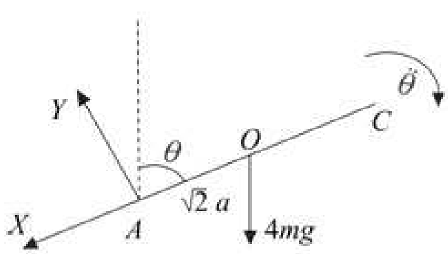
$$I = I_{AB} + I_{BC} + I_{CD} + I_{DA}$$

$$= \frac{4}{3}ma^2 + \frac{16}{3}ma^2 + \frac{16}{3}ma^2 + \frac{4}{3}ma^2$$

$$= \frac{40}{3}ma^2, \text{ as required}$$

By symmetry, the moment of inertia of the rod CD about the axis through A will be the same as the moment of inertia of the rod BC.

b Let O be the centre of mass of the lamina



$AO^2 + BO^2 = AB^2 = (2a)^2 = 4a^2$

$2AO^2 = 4a^2 \Rightarrow AO^2 = 2a^2 \Rightarrow AO = \sqrt{2}a$ ← By symmetry, $AO = BO$.

Equation of angular motion about A

$$L = I\ddot{\theta}$$

$$4mg\sqrt{2}a\sin\theta = \frac{40}{3}ma^2\ddot{\theta}$$

$$\ddot{\theta} = \frac{3g\sqrt{2}}{10a}\sin\theta$$

When $\theta = \frac{\pi}{2}$

$$\ddot{\theta} = \frac{3g\sqrt{2}}{10a}$$

When AC is horizontal, $\theta = \frac{\pi}{2}$, $\sin\theta = 1$ and $\cos\theta = 0$.

c Conservation of energy

$$\frac{1}{2}I\dot{\theta}^2 = 4mg(\sqrt{2}a - \sqrt{2}a\cos\theta)$$

$$\frac{1}{2} \times \frac{40}{3}ma^2\dot{\theta}^2 = 4\sqrt{2}mga(1-\cos\theta)$$

$$\dot{\theta}^2 = \frac{3g\sqrt{2}}{5a}(1-\cos\theta)$$

Initially the framework has no kinetic energy. As the framework rotates through θ, O falls a vertical distance $\sqrt{2}a - \sqrt{2}a\cos\theta$.

When $\theta = \frac{\pi}{2}$

$$\dot{\theta}^2 = \frac{3g\sqrt{2}}{5a}$$

$$\dot{\theta} = \left(\frac{3g\sqrt{2}}{5a}\right)^{\frac{1}{2}}$$

d Let the magnitude of the force acting on the framework at A be R and the components of this force parallel and perpendicular to AO be X and Y respectively.

R(|| AO)

$$X + 4mg\cos\theta = 4m(\sqrt{2}a)\dot{\theta}^2$$

When $\theta = \frac{\pi}{2}$

$$X + 4\sqrt{2}ma \times \frac{3g\sqrt{2}}{5a} = \frac{24}{5}mg$$

Using the result of part **c** and $\cos\frac{\pi}{2} = 0$.

R($\perp AO$)

$$4mg\sin\theta - Y = 4m(\sqrt{2}a)\ddot{\theta}$$

When $\theta = \frac{\pi}{2}$

$$Y = 4mg - 4\sqrt{2}ma \times \frac{3g\sqrt{2}}{10a}$$

$$= 4mg - \frac{12}{5}mg = \frac{8}{5}mg$$

Using the result of part **b** and $\sin\frac{\pi}{2} = 1$.

$$R^2 = X^2 + Y^2$$

$$= \left(\frac{24}{5}mg\right)^2 + \left(\frac{8}{5}mg\right)^2 = \frac{640}{25}m^2g^2$$

$$R = \frac{8\sqrt{10}}{5}mg$$

Solutionbank M5
Edexcel AS and A Level Modular Mathematics

Review Exercise 2
Exercise A, Question 29

Question:

a Prove, using integration, that the moment of inertia of a uniform circular disc, of mass m and radius a, about an axis through its centre O perpendicular to the plane of the disc is $\frac{1}{2}ma^2$.

The line AB is a diameter of the disc and P is the mid-point of OA. The disc is free to rotate about a fixed smooth horizontal axis L. The axis lies in the plane of the disc, passes through P and is perpendicular to OA. A particle of mass m is attached to the disc at A and a particle of mass $2m$ is attached to the disc at B.

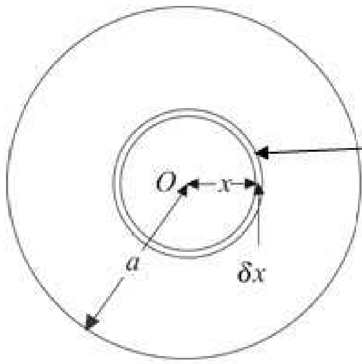
b Show that the moment of inertia of the loaded disc about L is $\frac{21}{4}ma^2$.

At time $t = 0, PB$ makes a small angle with the downward vertical through P and the loaded disc is released from rest. By obtaining an equation of motion for the disc and using a suitable approximation,

c find the time when the loaded disc first comes to instantaneous rest. ***E***

Solution:

a



You consider the disc as made up a series of concentric rings of thickness δx. The area of each ring is $2\pi x \delta x$.

The mass per unit area of the disc is $\frac{m}{\pi a^2}$.

The moment of inertia, δI, of a ring is given by

$$\delta I = (\delta m)x^2 = \left(2\pi x \delta x \times \frac{m}{\pi a^2}\right)x^2 = \frac{2m}{a^2}x^3 \delta x$$

The moment of inertia of the disc, I, is given by

$$I = \sum \delta I = \sum \frac{2m}{a^2}x^3 \delta x$$

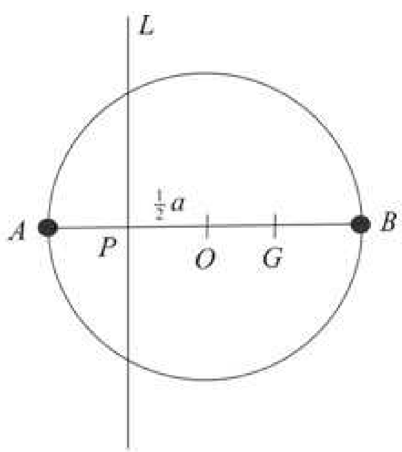
As $\delta x \to 0$

$$I = \int_0^a \frac{2m}{a^2}x^3 \, dx = \frac{2m}{a^2}\left[\frac{x^4}{4}\right]_0^a = \frac{2m}{a^2}\left(\frac{a^4}{4} - 0\right)$$

$= \frac{1}{2}ma^2$, as required

This is a standard result which you should be able to prove. You are expected to be able to prove all of the standard results given in the Formulae Booklet and you should practise writing these out.

b



The moment of inertia of the disc about L is given by

$$I_{\text{disc}} = \frac{1}{4}ma^2 + m\left(\frac{1}{2}a\right)^2 = \frac{1}{2}ma^2$$

Using the standard result for the moment of inertia of a disc about a diameter and the parallel axes theorem.

The moment of inertia of the loaded disc, I, is given by

$I = I_{\text{disc}} + I_A + I_B$

$= \frac{1}{2}ma^2 + m\left(\frac{1}{2}a\right)^2 + 2m\left(\frac{3}{2}a\right)^2$

$= \frac{1}{2}ma^2 + \frac{1}{4}ma^2 + \frac{9}{2}ma^2 = \frac{21}{4}ma^2$, as required

$AP = \frac{1}{2}a$ and $BP = \frac{3}{2}a$.

c Let G be the centre of mass of the loaded plate.

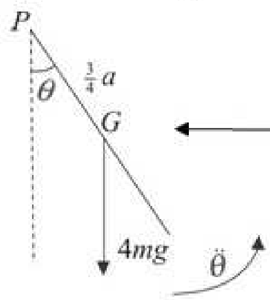
M(L)

$4m \times PG = m \times \frac{1}{2}a + 2m \times \frac{3}{2}a - m \times \frac{1}{2}a$

$4mPG = 3ma$

$PG = \frac{3}{4}a$

You take moments about L to locate the position of G. As A is on the other side of L from O and B, the moment of the particle of mass m has a negative sign in this equation.



The total weight, $4mg$, of the loaded plate acts at G.

The equation of rotational motion about P is

$L = I\ddot{\theta}$

$-4mg \times \frac{3}{4}a \sin\theta = \frac{21}{4}ma^2\ddot{\theta}$

$\ddot{\theta} = -\frac{4g}{7a}\sin\theta$

For small θ, $\sin\theta \approx \theta$

Hence

$\ddot{\theta} = -\frac{4g}{7a}\theta$

Comparing with the standard equation for simple harmonic motion, $\ddot{\theta} = -\omega^2\theta$, the motion is approximately simply harmonic, with

$\omega^2 = \frac{4g}{7a}$.

The time, after release, for the loaded disc to first come to instantaneous rest is given by

$t = \frac{1}{2}T = \frac{\pi}{\omega} = \pi\sqrt{\left(\frac{7a}{4g}\right)} = \frac{\pi}{2}\sqrt{\left(\frac{7a}{g}\right)}$

This time is the time for one half of a complete oscillation.

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Edexcel AS and A Level Modular Mathematics

Review Exercise 2
Exercise A, Question 30

Question:

A uniform lamina of mass m is in the shape of an equilateral triangle ABC of perpendicular height h. The lamina is free to rotate in a vertical plane about a fixed smooth horizontal axis L through A and perpendicular to the plane of the lamina.

a Show, by integration, that the moment of inertia of the lamina about L is $\frac{5}{9}mh^2$.

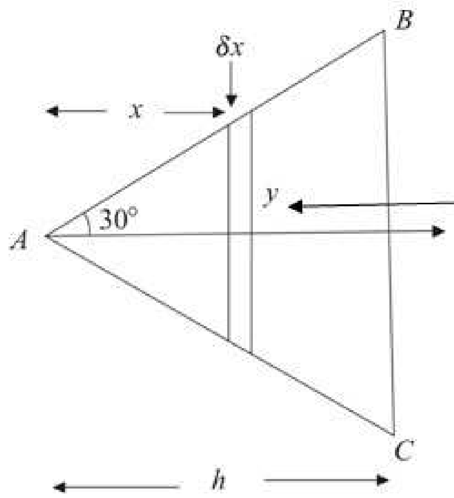
The centre of mass of the lamina is G. The lamina is in equilibrium, with G below A, when it is given an angular speed $\sqrt{\left(\frac{6g}{5h}\right)}$.

b Find the angle between AG and the downward vertical, when the lamina first comes to rest.

c Find the greatest magnitude of the angular acceleration during the motion. ***E***

Solution:

a



You consider the triangle as made up of thin rods of length $2y$, thickness δx and mass δm. A rod is a distance x from A.

By trigonometry

$$\frac{\frac{1}{2}BC}{h} = \tan 30°$$

$$BC = 2h\tan 30° = \frac{2}{\sqrt{3}}h$$

To find an expression for the mass per unit area of the triangle, you have to first obtain an expression for the area of the triangle in terms of h.

The area of the triangle is given by

$$\frac{1}{2}BC \times h = \frac{1}{\sqrt{3}}h \times h = \frac{1}{\sqrt{3}}h^2$$

The mass per unit area of the triangle is

$$\frac{m}{\frac{1}{\sqrt{3}}h^2} = \frac{\sqrt{3}m}{h^2}$$

The moment of inertia, δI, of one elementary rod about L is given by

$$\delta I = \frac{1}{3}(\delta m)y^2 + (\delta m)x^2$$

$$= (\delta m)\left(\frac{1}{3}y^2 + x^2\right)$$

$$= \left(2y\delta x \times \frac{\sqrt{3}m}{h^2}\right)\left(\frac{1}{3}y^2 + x^2\right)$$

You use the standard formula for the moment of inertia of a rod about its centre and the parallel axes theorem to find the moment of inertia of a rod about axis L through A.

The mass, δm, of an elementary rod is its area, $2y\delta x$, multiplied by the mass per unit area.

By trigonometry

$$\frac{y}{x} = \tan 30° = \frac{1}{\sqrt{3}} \Rightarrow y = \frac{1}{\sqrt{3}}x$$

Hence

$$\delta I = \left(2\frac{1}{\sqrt{3}}x\delta x \times \frac{\sqrt{3}m}{h^2}\right)\left(\frac{1}{3}\left(\frac{1}{\sqrt{3}}x\right)^2 + x^2\right)$$

$$= \frac{2mx}{h^2}\delta x \times \left(\frac{1}{9}x^2 + x^2\right) = \frac{20m}{9h^2}x^3\delta x$$

$$I = \sum \delta I = \sum \frac{20m}{9h^2} x^3 \delta x$$

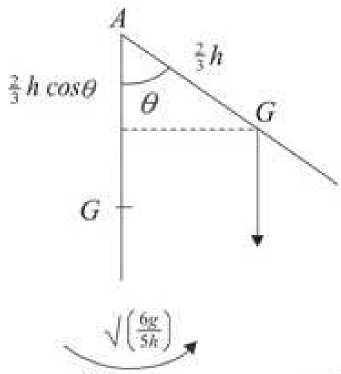
As $\delta x \to 0$

$$I = \frac{20m}{9h^2}\int_0^h x^3\,dx = \frac{20m}{9h^2}\left[\frac{x^4}{4}\right]_0^h$$

$$= \frac{20m}{9h^2} \times \frac{h^4}{4} = \frac{5}{9}mh^2\text{, as required}$$

As the rods range from A to BC, x ranges from 0 to h, so 0 and h are the limits of integration.

b



$$AG = \frac{2}{3}h$$

Using the standard result that the centre of mass of a triangle is $\frac{2}{3}$ along the median from the vertex.

Conservation of energy

$$\frac{1}{2}I\omega^2 = mg\left(\frac{2}{3}h - \frac{2}{3}h\cos\theta\right)$$

When the lamina comes to rest, all of the original kinetic energy has been converted to potential energy

$$\frac{5}{18}mh^2 \times \frac{6g}{5h} = \frac{2}{3}mgh(1-\cos\theta)$$

$$\frac{1}{3} = \frac{2}{3}(1-\cos\theta) \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$$

The angle between AG and the downward vertical when the lamina first comes to rest is $\frac{\pi}{3}$.

c Equation of angular motion about A is

$$L = I\ddot{\theta}$$

$$-mg\frac{2}{3}h\sin\theta = \frac{5}{9}mh^2\ddot{\theta}$$

$$\ddot{\theta} = -\frac{6g}{5h}\sin\theta$$

The greatest magnitude of the angular acceleration corresponds to the greatest possible value of $\sin\theta$. In part **b**, you established that this is when $\theta = \frac{\pi}{3}$.

When

$$\theta = \frac{\pi}{3}$$

$$|\ddot{\theta}| = \frac{6g}{5h} \times \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}g}{5h}$$

The greatest magnitude of the angular acceleration during the motion is $\frac{3\sqrt{3}g}{5h}$.

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Review Exercise 2
Exercise A, Question 31

Question:

To the end B of a thin uniform rod AB, of length $3a$ and mass m, is attached a thin uniform circular disc, of radius a and mass m, so that the rod and the diameter BC of the disc are in a straight line and $AC = 5a$.

a Show that the moment of inertia of this composite body, about an axis through A and perpendicular to AB and in the plane of the disc, is $\frac{77}{4}ma^2$.

The body is held at rest with the end A smoothly hinged to a fixed pivot and with the plane of the disc horizontal. The body is released and has angular speed ω when AC is vertical.

b Find ω in terms of a and g.

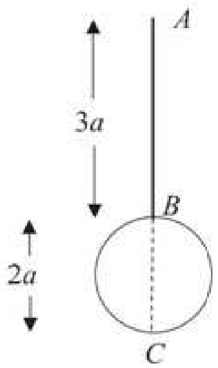
When AC is vertical, the centre of the disc strikes a stationary particle of mass $\frac{1}{2}m$.

Given that the particle adheres to the centre of the disc,

c show that the angular speed of the body immediately after impact is $\frac{77}{109}\omega$. ***E***

Solution:

a



The moment of inertia of the rod about the axis through A is given by

$$I_{\text{rod}} = \frac{4}{3}m\left(\frac{3a}{2}\right)^2 = 3ma^2$$

Using the standard result, $I = \frac{4}{3}ml^2$, for a rod of length $2l$ about an axis through its end, with $2l = 3a$.

By the parallel axes theorem, the moment of inertia of the disc about the axis through A is given by

$$I_{\text{disc}} = \frac{1}{4}ma^2 + m(4a)^2 = \frac{65}{4}ma^2$$

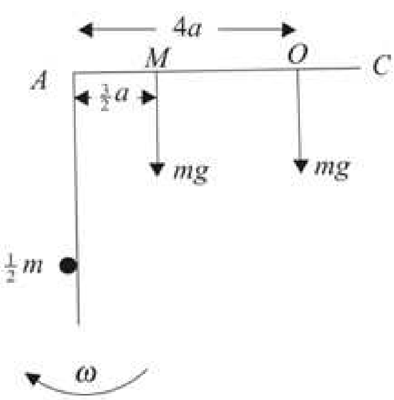
You can quote the result for the moment of inertia of a disc about its diameter and the centre of the disc is $4a$ from A.

The moment of inertia of the composite body is given by

$$I = I_{\text{rod}} + I_{\text{disc}}$$

$$= 3ma^2 + \frac{65}{4}ma^2 = \frac{77}{4}ma^2, \text{ as required}$$

b



If M is the centre of mass of the rod and O is the centre of the mass of the disc, then $AM = \frac{3}{2}a$ and $AO = 4a$.

Conservation of energy

$$\frac{1}{2}I\omega^2 = mg \times \frac{3}{2}a + mg \times 4a$$

$$\frac{77}{8}ma^2\omega^2 = \frac{11}{2}mga$$

$$\omega^2 = \frac{11mga}{2} \times \frac{8}{77ma^2} = \frac{4g}{7a}$$

$$\omega = \sqrt{\left(\frac{4g}{7a}\right)}$$

As the composite body moves from the horizontal to the vertical, the centre of mass, M, of the rod falls a vertical distance $\frac{3}{2}a$ and the centre of mass of the disc, O, falls a vertical distance $4a$.

c The moment of inertia, I', of the composite body and the particle of mass $\frac{1}{2}m$ about the axis through A is given by

$$I' = \frac{77}{4}ma^2 + \frac{1}{2}m(4a)^2 = \frac{109}{4}ma^2$$

Let ω' be the angular speed of the body immediately after impact.
Conservation of linear momentum about A

$$I'\omega' = I\omega$$

$$\frac{109}{4}ma^2\omega' = \frac{77}{4}ma^2\omega$$

$$\omega' = \frac{77}{109}\omega, \text{ as required}$$

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Review Exercise 2
Exercise A, Question 32

Question:

a Prove, by integration, that the moment of inertia of a uniform rod, of mass m and length a, about an axis through its mid-point and perpendicular to the rod is $\frac{ma^2}{12}$.

Four uniform rods AB, BC, CD and DA, each of length a, are rigidly joined to form a square $ABCD$. Each of the rods AB, CD and DA has mass m and the rod BC has mass $3m$. The rods are free to rotate about a smooth horizontal axis L which passes through A and is perpendicular to the plane of the square.

b Show that the moment of inertia of the system about L is $6ma^2$ and find the distance of the centre of mass of the system from A.

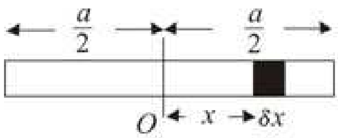
The system is released from rest with AB horizontal and C vertically below B.

c Find the greatest value of the angular speed of the system in the subsequent motion.

d Find the period of small oscillations of the system about the position of stable equilibrium. ***E***

Solution:

a



You consider the rod to be made up of a series of small pieces, or elements, each of length δx.

The mass per unit length of the rod is $\frac{m}{a}$.

When proving results you usually need to know the 'density' of the object, here the mass per unit length.

Consider an element of length δx at a distance x from the middle of the rod O.

$$\delta I = (\delta m)x^2 = \left(\frac{m}{a}\delta x\right)x^2 = \frac{mx^2}{a}\delta x$$

For the whole rod

$$I = \sum \delta I = \sum \frac{mx^2}{a}\delta x$$

The whole rod is the sum of the small pieces.

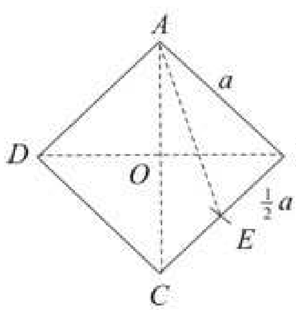
As $\delta x \to 0$

$$I = \int_{-\frac{a}{2}}^{\frac{a}{2}} \frac{mx^2}{a}\,dx = \frac{m}{a}\left[\frac{x^3}{3}\right]_{-\frac{a}{2}}^{\frac{a}{2}}$$

$$= \frac{m}{a}\left[\frac{a^3}{24} - \left(-\frac{a^3}{24}\right)\right] = \frac{ma^2}{12}\text{, as required}$$

As the small pieces range from one end of the rod to the other, x ranges from $-\frac{a}{2}$ at one end to $\frac{a}{2}$ at the other. So $-\frac{a}{2}$ and $\frac{a}{2}$ are the limits of the definite integral.

b



Let E be the mid-point of BC.

$$AE^2 = a^2 + \left(\frac{1}{2}a\right)^2 = \frac{5}{4}a^2$$

By the parallel axes theorem, the moment of inertia of the rod BC about the axis through A is given by

$$I_{BC} = \frac{1}{12}(3m)a^2 + (3m)AE^2$$

$$= \frac{1}{4}ma^2 + \frac{15}{4}ma^2 = 4ma^2$$

The mass of BC is $3m$.

Similarly for the rod CD

$$I_{CD} = \frac{1}{12}ma^2 + m \times \frac{5}{4}a^2 = \frac{4}{3}ma^2$$

As the mass of CD is one third of the mass of BC, this could just be written down.

The moments of inertia of the rods AB and AD about the axis through A are given by

$$I_{AB} = I_{AD} = \frac{1}{12}ma^2 + m\left(\frac{1}{2}a\right)^2 = \frac{1}{3}ma^2$$

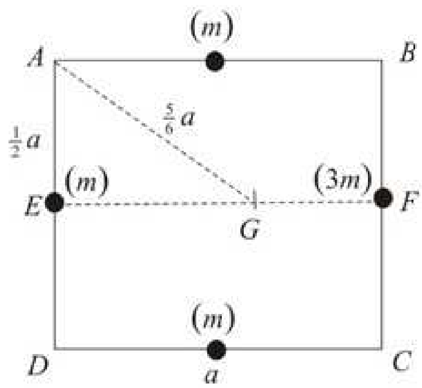
Using the parallel axes theorem.

The moment of inertia of the framework about the axis through A is given by

$$I = I_{AB} + I_{BC} + I_{CD} + I_{DA}$$

$$= \frac{1}{3}ma^2 + 4ma^2 + \frac{4}{3}ma^2 + \frac{1}{3}ma^2$$

$$= 6ma^2\text{, as required}$$



If E is the mid-point of AD and F is the mid-point of BC, then, by symmetry, the centre of mass of the framework G lies on EF. The location of G is found by taking moments about E.

M(E)

$$6m \times EG = m \times \frac{a}{2} + 3m \times a + m \times \frac{a}{2} = 4ma$$

The weight of each rod acts at the mid-point of the rod.

$$EG = \frac{2}{3}a$$

$$AG^2 = AE^2 + EG^2 = \frac{1}{4}a^2 + \frac{4}{9}a^2 = \frac{25}{36}a^2$$

$$AG = \frac{5}{6}a$$

The distance of the centre of mass of the system from A is $\frac{5}{6}a$.

c Let ω be the maximum angular speed of the system.

Conservation of energy

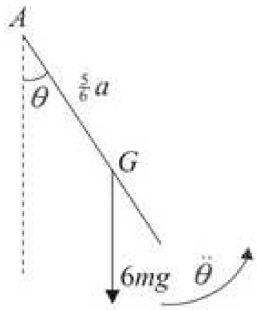
$$\frac{1}{2}I\omega^2 = 6mg\left(\frac{5}{6}a - \frac{1}{2}a\right)$$

$$3ma^2\omega^2 = 2mga$$

$$\omega^2 = \frac{2g}{3a} \Rightarrow \omega = \sqrt{\left(\frac{2g}{3a}\right)}$$

The maximum angular speed occurs when G is vertically below A. At that point G is $\frac{5}{6}a$ below A. Initially G is $\frac{1}{2}a$ below A. So G falls $\frac{5}{6}a - \frac{1}{2}a = \frac{1}{3}a$.

d



The equation of angular motion about A is

$$L = I\ddot{\theta}$$

$$-6mg\left(\frac{5}{6}a\right)\sin\theta = 6ma^2\ddot{\theta}$$

$$\ddot{\theta} = -\frac{5g}{6a}\sin\theta$$

For small θ, $\sin\theta \approx \theta$

Hence

$$\ddot{\theta} = -\frac{5g}{6a}\theta$$

Comparing with the standard equation for simple harmonic motion, $\ddot{\theta} = -\omega^2\theta$, the motion is approximately simply harmonic, with $\omega^2 = \frac{5g}{6a}$.

The period of small oscillations is given by

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\left(\frac{6a}{5g}\right)}$$

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Review Exercise 2
Exercise A, Question 33

Question:

A compound pendulum consists of a thin uniform rod AB, of length $2a$ and mass $3m$, with a particle of mass $2m$ attached at B. The pendulum is free to rotate in a vertical plane about a horizontal axis l which is perpendicular to the rod through a point C of the rod, where $AC = x, x < a$.

a Show that the moment of inertia of the pendulum about l is $(5x^2 - 14ax + 12a^2)m$.

b Find the square of the period of small oscillations of the pendulum about l.

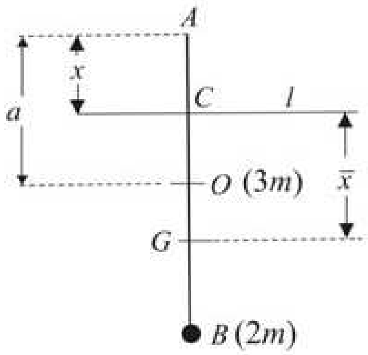
c Show that, as x varies, the period takes its minimum value when

$$x = \frac{(7-\sqrt{11})a}{5}.$$

E

Solution:

a



Let O be the centre of the rod.
Using the parallel axes theorem, the moment of inertia of the rod about l is given by

$$I_{\text{rod}} = \frac{1}{3}(3m)a^2 + 3mOC^2$$

$$= ma^2 + 3m(a-x)^2$$

$OC = a - x$

The moment of inertia of the compound pendulum about l is given by

$$I = I_{\text{rod}} + I_{\text{particle}}$$

$$= ma^2 + 3m(a-x)^2 + 2m(2a-x)^2$$

$$= ma^2 + 3ma^2 - 6max + 3mx^2 + 8ma^2 - 8max + 2mx^2$$

$$= 5mx^2 - 14max + 12ma^2$$

$$= \left(5x^2 - 14ax + 12a^2\right)m, \text{ as required}$$

$BC = 2a - x$

b Let G be the centre of mass of the compound pendulum and $CG = \bar{x}$.

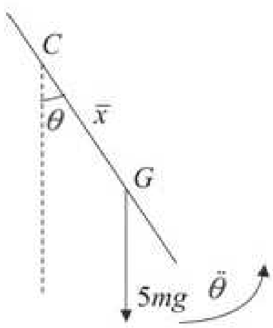
M(C)

$$5m\bar{x} = 3m(a-x) + 2m(2a-x)$$

$$= 7ma - 5mx$$

$$\bar{x} = \frac{7}{5}a - x$$

You locate the position of G by taking the moments of mass about C.



The equation of angular motion about A is

$$L = I\ddot{\theta}$$

$$-5mg\bar{x}\sin\theta = I\ddot{\theta}$$

$$\ddot{\theta} = -\frac{5mg\bar{x}}{I}\sin\theta = -\frac{5mg\left(\frac{7}{5}a - x\right)}{\left(5x^2 - 14ax + 12a^2\right)m}\sin\theta$$

For small θ, $\sin\theta \approx \theta$

Hence

$$\ddot{\theta} = -\frac{(7a-5x)g}{5x^2-14ax+12a^2}\theta$$

Comparing with the standard equation for simple harmonic motion, $\ddot{\theta} = -\omega^2\theta$, the motion is approximately simply harmonic, with $\omega^2 = \dfrac{(7a-5x)g}{5x^2-14ax+12a^2}$.

The period of small oscillations, T, is given by

$$T = \frac{2\pi}{\omega} \Rightarrow T^2 = \frac{4\pi^2}{\omega^2}$$

$$= \frac{4\pi^2}{g}\left(\frac{5x^2-14ax+12a^2}{7a-5x}\right)$$

c

$$T^2 = \frac{4\pi^2}{g}\left(\frac{5x^2-14ax+12a^2}{7a-5x}\right)$$

$$2T\frac{\mathrm{d}T}{\mathrm{d}x} = \frac{4\pi^2}{g}\left[\frac{(7a-5x)(10x-14a)+5\left(5x^2-14ax+12a^2\right)}{(7a-5x)^2}\right]$$

Differentiate this equation throughout with respect to x, using implicit differentiation on the left hand side and the quotient rule on the right hand side.

At a minimum value $\dfrac{\mathrm{d}T}{\mathrm{d}x} = 0$

Hence

$$(7a-5x)(10x-14a)+5\left(5x^2-14ax+12a^2\right) = 0$$

$$70ax-98a^2-50x^2+70ax+25x^2-70ax+60a^2 = 0$$

$$-25x^2+70ax-38a^2 = 0$$

$$25x^2-70ax+38a^2 = 0$$

Using the quadratic formula $x = \dfrac{-b \pm \sqrt{\left(b^2-4ac\right)}}{2a}$.

$$x = \frac{70 \pm \sqrt{1100}}{50}a = \frac{7 \pm \sqrt{11}}{5}a$$

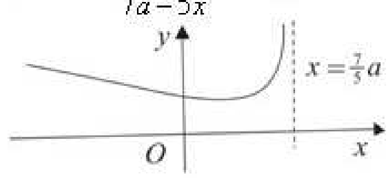
$x = \dfrac{7+\sqrt{11}}{5}a \approx 2.06a$ is impossible

This value is longer than the length of the rod, $2a$, and can be rejected.

The period takes its minimum value when

$x = \dfrac{\left(7-\sqrt{11}\right)a}{5}$, as required

Unless the question specifically asks you do to so, you are not expected to show that the stationary point is a minimum. A sketch of $y = \dfrac{5x^2-14ax+12a^2}{7a-5x}$ is



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Review Exercise 2
Exercise A, Question 34

Question:

a Show, using integration, that the moment of inertia of a uniform equilateral triangular lamina, of side $2a$, and of mass m, about an axis through a vertex perpendicular to its plane is $\frac{5}{3}ma^2$.

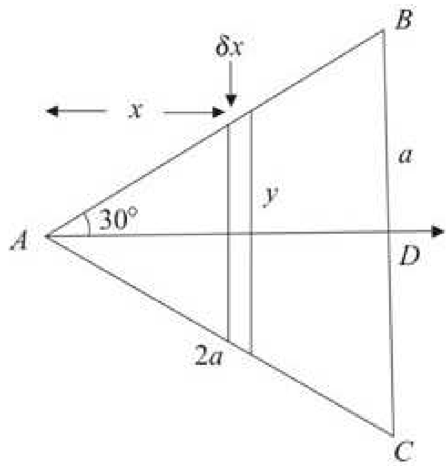
b Deduce that the moment of inertia of a uniform regular hexagonal lamina, of side $2a$ and mass M, about an axis through a vertex perpendicular to the plane of the lamina is $\frac{17}{3}Ma^2$.

A compound pendulum consists of a uniform regular hexagonal lamina $ABCDEF$, of side $2a$ and mass M, with a particle of mass $\frac{1}{2}M$ attached at the vertex D. The pendulum oscillates about a smooth horizontal axis which passes through the vertex A and is perpendicular to the plane of the lamina.

c Show that the period of small oscillations is $\pi\sqrt{\left(\frac{41a}{3g}\right)}$. ***E***

Solution:

a



You consider the triangle as made up of thin rods of length $2y$, thickness δx and mass δm. The rod is a distance x from A.

Let the triangle be ABC and D the mid-point of BC as shown in the diagram above.

By Pythagoras' theorem

$AD^2 = AB^2 - BD^2 = 4a^2 - a^2 = 3a^2$

$AD = \sqrt{3}a$

The area of the triangle is given by

$\frac{1}{2}BC \times AD = a \times \sqrt{3}a = \sqrt{3}a^2$

To find an expression for the mass per unit area of the triangle, you have to first obtain an expression for the area of the triangle in terms of a.

The mass per unit area of the triangle is $\frac{m}{\sqrt{3}a^2}$

The moment of inertia, δI, of one elementary rod about the axis is given by

$$\delta I = \frac{1}{3}(\delta m)y^2 + (\delta m)x^2$$

$$= (\delta m)\left(\frac{1}{3}y^2 + x^2\right)$$

$$= \left(2y\delta x \times \frac{m}{\sqrt{3}a^2}\right)\left(\frac{1}{3}y^2 + x^2\right)$$

You use the standard formula for the moment of inertia of a rod about its centre and the parallel axes theorem to find the moment of inertia of a rod about the axis through the vertex A.

The mass, δm, of an elementary rod is its area, $2y\delta x$, multiplied by the mass per unit area.

By trigonometry

$$\frac{y}{x} = \tan 30^\circ = \frac{1}{\sqrt{3}} \Rightarrow y = \frac{1}{\sqrt{3}}x$$

Hence

$$\delta I = \left(2\frac{1}{\sqrt{3}}x\delta x \times \frac{m}{\sqrt{3}a^2}\right)\left(\frac{1}{3}\left(\frac{1}{\sqrt{3}}x\right)^2 + x^2\right)$$

$$= \frac{2mx}{3a^2}\delta x \times \left(\frac{1}{9}x^2 + x^2\right) = \frac{20m}{27a^2}x^3\delta x$$

$$I = \sum \delta I = \sum \frac{20m}{27a^2}x^3\delta x$$

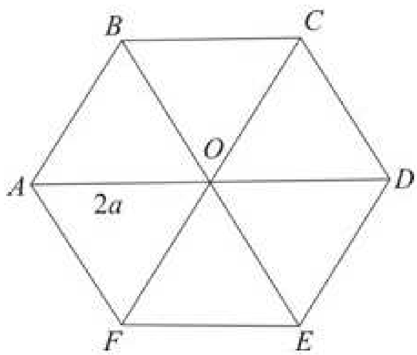
As $\delta x \to 0$

$$I = \frac{20m}{27a^2}\int_0^{\sqrt{3}a} x^3\,dx = \frac{20m}{27a^2}\left[\frac{x^4}{4}\right]_0^{\sqrt{3}a}$$

$$= \frac{20m}{27a^2} \times \frac{9a^4}{4} = \frac{5}{3}ma^2, \text{ as required}$$

The upper limit corresponds to the distance AD which is $\sqrt{3}a$.

b



You consider the hexagon to be made up of 6 triangles. Each of the six triangles has the moment of inertia found in part **a** about the centre of the hexagon O.

The hexagon is made up of 6 triangles of mass m, where $M = 6m$.
The moment of inertia of the hexagon about an axis through O, I_O, is given by

$$I_O = 6 \times \frac{5}{3}ma^2 = \frac{5}{3}Ma^2$$

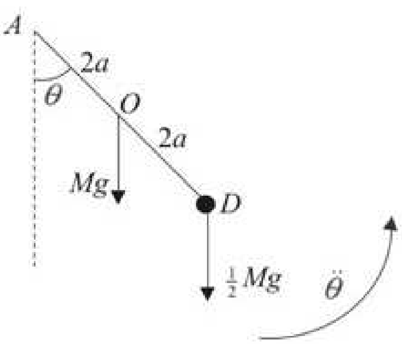
As $M = 6m$.

By the parallel axes theorem, the moment of inertia of the hexagon about an axis through A, I_A, is given by

$$I_A = I_O + M \times OA^2$$

$$= \frac{5}{3}Ma^2 + M \times 4a^2 = \frac{17}{3}Ma^2, \text{ as required}$$

c



The moment of inertia of the compound pendulum, I, about the axis through A is given by

$$I = I_{\text{hexagon}} + I_{\text{particle}}$$

$$= \frac{17}{3}Ma^2 + \frac{1}{2}M(4a)^2 = \frac{41}{3}Ma^2$$

The equation of angular motion about A is

$$L = I\ddot{\theta}$$

$$-Mg \times 2a\sin\theta - \frac{1}{2}Mg \times 4a\sin\theta = \frac{41}{3}Ma^2\ddot{\theta}$$

$$-4Mga\sin\theta = \frac{41}{3}Ma^2\ddot{\theta}$$

$$\ddot{\theta} = -\frac{12g}{41a}\sin\theta$$

For small θ, $\sin\theta \approx \theta$

Hence

$$\ddot{\theta} = -\frac{12g}{41a}\theta$$

Comparing with the standard equation for simple harmonic motion, $\ddot{\theta} = -\omega^2\theta$, the motion is approximately simply harmonic, with $\omega^2 = \frac{12g}{41a}$.

The period of small oscillations is given by

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\left(\frac{41a}{12g}\right)} = \pi\sqrt{\left(\frac{41a}{3g}\right)}$$, as required

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Review Exercise 2
Exercise A, Question 35

Question:

a Show, by integration, that the moment of inertia of a uniform circular disc, of mass m and radius a, about an axis through the centre and perpendicular to the plane of the disc is $\frac{1}{2}ma^2$.

b Deduce the moment of inertia of the disc about a diameter.

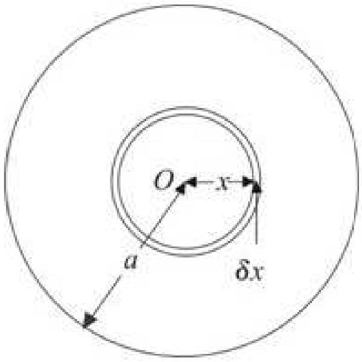
c Show that the moment of inertia of a uniform right circular cone, of height r, base radius r and mass M about an axis through its vertex and parallel to a diameter of the base is $\frac{3}{4}Mr^2$.

The above cone is free to turn about a fixed smooth pivot at its vertex and is released from rest with its axis horizontal.

d Find the angular speed of the cone when its axis is vertical. ***E***

Solution:

a



You consider the disc as made up a series of concentric rings of thickness δx. The area of each ring is $2\pi x \delta x$.

The mass per unit area of the disc is $\frac{m}{\pi a^2}$.

The moment of inertia, δI, of a ring is given by

$$\delta I = (\delta m)x^2 = \left(2\pi x \delta x \times \frac{m}{\pi a^2}\right)x^2 = \frac{2m}{a^2}x^3 \delta x$$

The moment of inertia of the disc, I, is given by

$$I = \sum \delta I = \sum \frac{2m}{a^2}x^3 \delta x$$

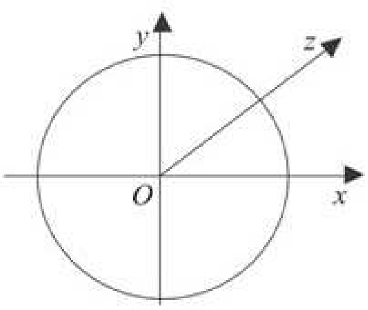
As $\delta x \to 0$

$$I = \int_0^a \frac{2m}{a^2}x^3 \, dx = \frac{2m}{a^2}\left[\frac{x^4}{4}\right]_0^a = \frac{2m}{a^2}\left(\frac{a^4}{4} - 0\right)$$

$$= \frac{1}{2}ma^2, \text{ as required}$$

This is a standard result which you should be able to prove. You are expected to be able to prove all of the standard results given in the Formulae Booklet and you should practise writing these out.

b



Let the centre of the disc be O and Ox and Oy be perpendicular axes through O.
By the perpendicular axes theorem, the moment of inertia, I_{Ox}, about a diameter Ox through O is given by

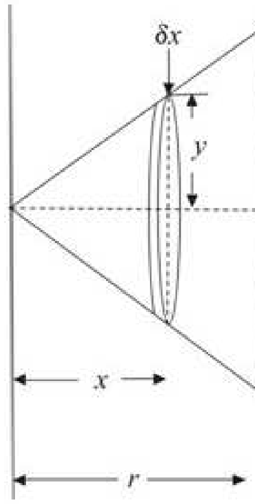
$$I_{Ox} + I_{Oy} = I_{Oz}$$

$$2I_{Ox} = \frac{1}{2}ma^2$$

$$I_{Ox} = \frac{1}{4}ma^2$$

By symmetry the moment of inertia about the axis Ox equals the moment of inertia about the axis Oy.

c Axis of rotation



You consider the cone to be made up of thin discs, each of thickness δx with the centre of the disc at a distance x from the vertex of the cone. If the radius of the disc is y, then, using the formula, $V = \pi r^2 h$, for the volume of a cylinder, the volume of a thin disc is $\pi y^2 \delta x$.

The mass per unit volume of the cone is

$$\frac{M}{\frac{1}{3}\pi r^2 \times r} = \frac{3M}{\pi r^3}$$

Using the formula for the volume of a cone, $V = \frac{1}{3}\pi r^2 h$, with $h = r$.

The moment of inertia of an elementary disc about the axis of rotation is given by

$$\delta I = \frac{1}{4}(\delta m)y^2 + (\delta m)x^2$$

You use the answer to part **b** with $m = \delta m$ and $a = y$, and the parallel axes theorem to form an expression for the moment of inertia, δI, of the thin disc.

By similar triangles

$$\frac{y}{x} = \frac{r}{r} \Rightarrow y = x$$

Hence

$$\delta I = \frac{1}{4}(\pi y^2 \delta x)\left(\frac{3M}{\pi r^3}\right)y^2 + (\pi y^2 \delta x)\left(\frac{3M}{\pi r^3}\right)x^2$$

$$= \frac{3M}{r^3}\left(\frac{y^4}{4} + y^2 x^2\right)\delta x = \frac{3M}{r^3}\left(\frac{x^4}{4} + x^4\right)\delta x$$

$$= \frac{15M}{4r^3}x^4 \delta x$$

The mass, δm, of the disc is its volume, $\pi y^2 \delta x$, multiplied by the mass per unit volume $\frac{3M}{\pi r^3}$.

As $y = x$.

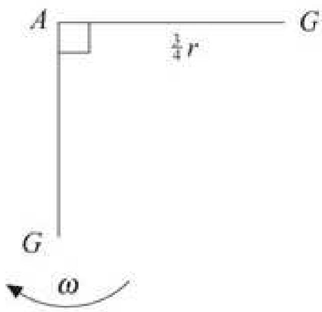
For the complete cone

$$I = \sum \delta I = \sum \frac{15M}{4r^3} x^4 \delta x$$

As $\delta x \to 0$

$$I = \frac{15M}{4r^3}\int_0^r x^4 \, dx = \frac{15M}{4r^3}\left[\frac{x^5}{5}\right]_0^r$$

$$= \frac{15M}{4r^3}\left(\frac{r^5}{5} - 0\right) = \frac{3}{4}Mr^2, \text{ as required}$$

d 

Let the vertex of the cone be A and the centre of mass of the cone be G, then

$AG = \frac{3}{4}r$.

The standard result for the centre of mass of a cone can be found among the formulae for module M3 in the Formulae Booklet.

Let the angular speed of the cone when its axis is vertical be ω.

Conservation of energy

$$\frac{1}{2}I\omega^2 = Mg \times \frac{3}{4}r$$

$$\frac{3}{8}Mr^2\omega^2 = \frac{3}{4}Mgr \Rightarrow \omega^2 = \frac{2g}{r}$$

$$\omega = \sqrt{\left(\frac{2g}{r}\right)}$$

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Review Exercise 2
Exercise A, Question 36

Question:

a Find the moment of inertia of a uniform square lamina $ABCD$, of side $2a$ and mass m, about an axis through A perpendicular to the plane of the lamina.

The lamina is free to rotate about a fixed smooth horizontal axis through A perpendicular to the plane of the lamina.

b Show that the period of small oscillations about the stable equilibrium position is

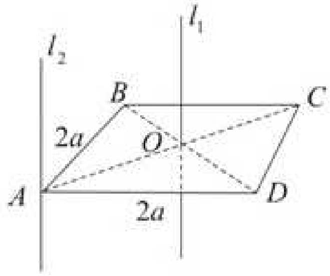
$$2\pi\left(\frac{8a}{3g\sqrt{2}}\right)^{\frac{1}{2}}.$$

The lamina is rotating with angular speed ω when C is vertically below A.

c Determine the components, along and perpendicular to AC, of the reaction of the lamina on the axis when AC makes an angle θ with the downward vertical through A. ***E***

Solution:

a



Let O be the centre of the lamina

By Pythagoras

$$AO^2 + BO^2 = (2a)^2$$

$$2AO^2 = 4a^2 \Rightarrow AO^2 = 2a^2 \Rightarrow AO = \sqrt{2}a$$

As $AO = BO$.

Let l_1 be the axis through O perpendicular to the plane of the lamina and l_2 be the axis through A perpendicular to the plane of the lamina.

The moment of inertia of the lamina about l_1 is given by

$$I_{l_1} = \frac{1}{3}m\left(a^2 + a^2\right) = \frac{2}{3}ma^2.$$

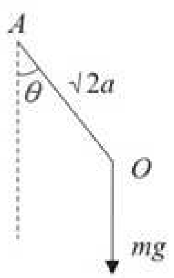
The Formulae Booklet gives you that the moment of inertia of a rectangle, mass m, sides $2a$ and $2b$, about a perpendicular axis through its centre is $\frac{1}{3}m\left(a^2 + b^2\right)$ and, for a square, $a = b$.

By the parallel axis theorem, the moment of inertia of the lamina about l_2 is given by

$$I_{l_2} = I_{l_1} + mAO^2$$

$$= \frac{2}{3}ma^2 + m \times 2a^2 = \frac{8}{3}ma^2$$

b



Equation of angular motion about l_2

$$L = I\ddot{\theta}$$

$$-mg \times \sqrt{2}a\sin\theta = \frac{8}{3}ma^2\ddot{\theta}$$

$$\ddot{\theta} = -\frac{3g\sqrt{2}}{8a}\sin\theta \quad ①$$

You will need this equation to find the component of the reaction perpendicular to AC in part **c**.

For small θ, $\sin\theta \approx \theta$

Hence

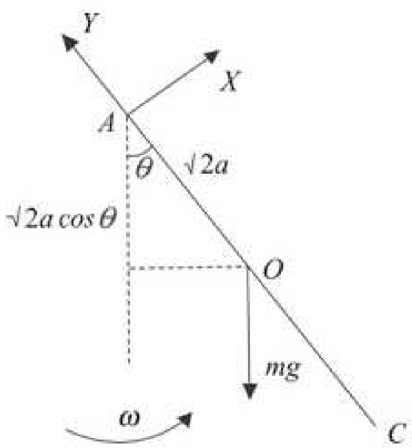
$$\ddot{\theta} = -\frac{3g\sqrt{2}}{8a}\theta$$

Comparing with the standard equation for simple harmonic motion, $\ddot{\theta} = -\omega^2\theta$, the motion is approximately simply harmonic, with $\omega^2 = \frac{3g\sqrt{2}}{8a}$.

The period of small oscillations is given by

$T = \frac{2\pi}{\omega} = 2\pi\left(\frac{8a}{3g\sqrt{2}}\right)^{\frac{1}{2}}$, as required

c



Let X and Y be the components, perpendicular to and along AC, of the reaction of the lamina on the axis.

Conservation of energy

$$\frac{1}{2}I\omega^2 - \frac{1}{2}I\dot{\theta}^2 = mg(\sqrt{2}a - \sqrt{2}a\cos\theta)$$

$$\frac{4}{3}ma^2\left(\omega^2 - \dot{\theta}^2\right) = mg\sqrt{2}a(1-\cos\theta)$$

$$\omega^2 - \dot{\theta}^2 = \frac{3g\sqrt{2}}{4a}(1-\cos\theta)$$

$$\dot{\theta}^2 = \omega^2 - \frac{3g\sqrt{2}}{4a}(1-\cos\theta) \qquad ②$$

R(∥AC)

$$Y - mg\cos\theta = m\left(\sqrt{2}a\right)\dot{\theta}^2$$

$$Y = mg\cos\theta + m\sqrt{2}a\left(\omega^2 - \frac{3g\sqrt{2}}{4a}(1-\cos\theta)\right)$$

Using equation ②.

$$= mg\cos\theta + m\sqrt{2}a\omega^2 - \frac{6mg}{4}(1-\cos\theta)$$

$$= ma\sqrt{2}\omega^2 + \frac{5mg}{2}\cos\theta - \frac{3mg}{2}$$

No further simplification of this expression is possible.

R(⊥AC)

$$X - mg\sin\theta = m(\sqrt{2}a)\ddot{\theta}$$

$$= -m(\sqrt{2}a)\frac{3g\sqrt{2}}{8a}\sin\theta = -\frac{3}{4}mg\sin\theta$$

Using equation ① in part **b**.

$$X = \frac{1}{4}mg\sin\theta$$

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Review Exercise 2
Exercise A, Question 37

Question:

A uniform lamina of mass M is in the form of an isosceles triangle ABC, with $AB = AC = 5l$ and $BC = 6l$.

a Show, by integration, that the moment of inertia of the lamina about an axis which passes through A and is parallel to BC is $8Ml^2$.

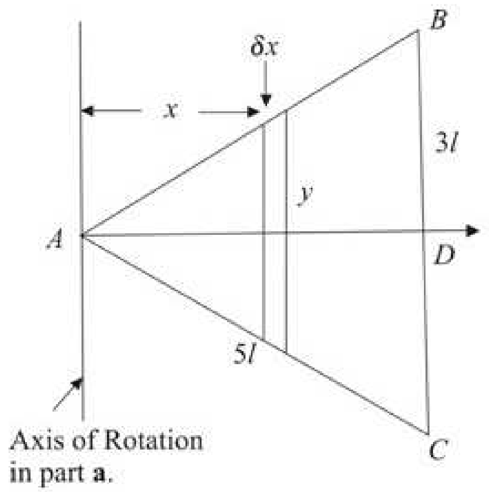
b Find also the moment of inertia of the lamina about an axis that passes through A and the mid-point of BC.

A particle of mass M is attached to the lamina at the mid-point of BC. The system is free to rotate about a smooth horizontal axis through A perpendicular to the plane of the triangle.

c Find the period of small oscillations about the position of equilibrium in which BC is below A. ***E***

Solution:

a



You consider the triangle as made up of thin rods of length $2y$, thickness δx and mass δm. Each rod is a distance x from A.

Let the mid-point of BC be D.
By Pythagoras
$AD^2 = AB^2 - BD^2 = (5l)^2 - (3l)^2 = 16l^2 \Rightarrow AD = 4l$

The area of $\Delta ABC = \frac{1}{2}BC \times AD = 3l \times 4l = 12l^2$

The mass per unit area of the triangle is $\frac{M}{12l^2}$

The moment of inertia, δI, of one elementary rod about the axis is given by

$$\delta I = (\delta m)x^2$$

$$= \left(2y\delta x \times \frac{M}{12l^2}\right)x^2 = \frac{M}{6l^2}x^2 y\delta x$$

All points on the rod are a distance x from the axis of rotation.

The mass, δm, of an elementary rod is its area, $2y\delta x$, multiplied by the mass per unit area.

By similar triangles

$$\frac{y}{x} = \frac{3l}{4l} \Rightarrow y = \frac{3}{4}x$$

Hence

$$\delta I = \frac{M}{6l^2}x^2 \times \frac{3}{4}x\delta x = \frac{M}{8l^2}x^3\delta x$$

$$I = \sum \delta I = \sum \frac{M}{8l^2}x^3\delta x$$

As $\delta x \to 0$

$$I = \frac{M}{8l^2}\int_0^{4l} x^3 \, dx = \frac{M}{8l^2}\left[\frac{x^4}{4}\right]_0^{4l}$$

The upper limit corresponds to the distance AD which is $4l$.

$$= \frac{M}{8l^2} \times \frac{256l^4}{4} = 8Ml^2\text{, as required}$$

b The moment of inertia, δI, of one elementary rod about AD is given by

$$\delta I = \frac{1}{3}(\delta m)y^2$$

Using the standard result for the moment of inertia of a rod about an axis through its centre.

$$= \frac{1}{3}\left(2y\delta x \times \frac{M}{12l^2}\right)y^2 = \frac{M}{18l^2}y^3\delta x$$

$$= \frac{M}{18l^2} \times \frac{27}{64}x^3\delta x = \frac{3M}{128l^2}x^3\delta x$$

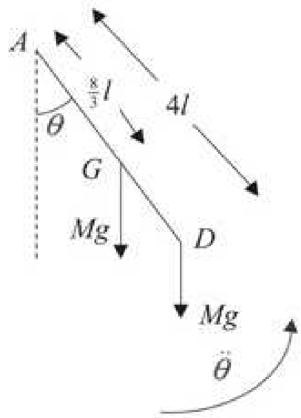
$$I = \sum \delta I = \sum \frac{3M}{128l^2}x^3\delta x$$

As $\delta x \to 0$

$$I = \frac{3M}{128l^2}\int_0^{4l} x^3 \, dx = \frac{3M}{128l^2}\left[\frac{x^4}{4}\right]_0^{4l}$$

$$= \frac{3M}{128l^2} \times \frac{256l^4}{4} = \frac{3}{2}Ml^2$$

c



Using the standard result that the centre of mass of a triangle is $\frac{2}{3}$ along the median from the vertex, if G is the centre of mass of the triangle then $AG = \frac{2}{3} \times 4l = \frac{8}{3}l$.

The moment of inertia, I, of the triangle about a smooth horizontal axis through A perpendicular to the plane of the triangle is given by

$$I = 8Ml^2 + \frac{3}{2}Ml^2 = \frac{19Ml^2}{2}$$

The axis of rotation in part **c** is perpendicular to both the axes in part **a** and part **b**. So you find the moment of inertia required for part **c** using the perpendicular axes theorem.

The moment of inertia, I', of the triangle and particle about the axis is given by

$$I' = \frac{19}{2}Ml^2 + M(4l)^2 = \frac{51}{2}Ml^2$$

Equation of angular motion about the axis through A perpendicular to the plane of the triangle

$$L = I\ddot{\theta}$$

You can take the moments of the weights for the triangle and particle about the axis separately.

$$-Mg \times \frac{8}{3}l\sin\theta - Mg \times 4l\sin\theta = \frac{51}{2}Ml^2\ddot{\theta}$$

$$-\frac{20}{3}Mgl\sin\theta = \frac{51}{2}Ml^2\ddot{\theta}$$

$$\ddot{\theta} = -\frac{40g}{153l}\sin\theta$$

For small $\theta, \sin\theta \approx \theta$

Hence

$$\ddot{\theta} = -\frac{40g}{153l}\theta$$

Comparing with the standard equation for simple harmonic motion, $\ddot{\theta} = -\omega^2\theta$, the motion is approximately simply harmonic, with $\omega^2 = \frac{40g}{153l}$.

The period of small oscillations is given by

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\left(\frac{153l}{40g}\right)} = \pi\sqrt{\left(\frac{153l}{10g}\right)}$$

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Review Exercise 2
Exercise A, Question 38

Question:

A body consists of 2 uniform discs, each of mass m and radius a, the centres of which are fixed to the ends A and B of a uniform rod of mass m and length $5a$. The discs and the rod are coplanar. The body is free to rotate about a fixed smooth horizontal axis which is perpendicular to the plane of the dics and which passes through O on the rod where $OA = 2a$.

a Show that the moment of inertia of the body about this axis is $\frac{49}{3}ma^2$.

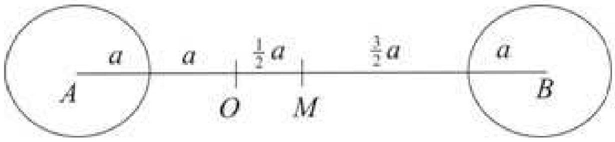
The body is initially at rest with B vertically below O. A particle of mass $6m$ is moving horizontally in the plane of the disc with speed u. It strikes the rod at a point P below O, where $OP = x$, and adheres to the rod.

b Find, in terms of a, x and u, the angular speed with which the system starts to move immediately after impact.

c Show that this angular speed is a maximum when $x = \frac{7}{6}a\sqrt{2}$.

d Given that $x = \frac{4}{3}a$, find the value of u for which the rod just reaches the horizontal position in the subsequent motion. ***E***

Solution:



Let the mid-point of AB be M.
By the parallel axes theorem, the moment of inertia, I_{rod}, about O is given by

$$I_{\text{rod}} = \frac{1}{3}m\left(\frac{5a}{2}\right)^2 + mOM^2 = \frac{25}{12}ma^2 + m\left(\frac{a}{2}\right)^2$$
$$= \frac{7}{3}ma^2$$

Using the standard result, $I = \frac{1}{3}ml^2$, for a rod of length $2l$ about an axis through its centre, with $2l = 5a$.

By the parallel axes theorem, the moment of inertia, I_A, of the disc centre A about O is given by

$$I_A = \frac{1}{2}ma^2 + mAO^2 = \frac{1}{2}ma^2 + m(2a)^2$$
$$= \frac{9}{2}ma^2$$

By the parallel axes theorem, the moment of inertia, I_B, of the disc centre B about O is given by

$$I_B = \frac{1}{2}ma^2 + mBO^2 = \frac{1}{2}ma^2 + m(3a)^2$$
$$= \frac{19}{2}ma^2$$

$BO = 5a - OA = 5a - 2a = 3a$

The moment of inertia, I, of the body about O is given by
$$I = I_{\text{rod}} + I_A + I_B$$
$$= \frac{7}{3}ma^2 + \frac{9}{2}ma^2 + \frac{19}{2}ma^2 = \frac{49}{3}ma^2, \text{ as required}$$

b The moment inertia, I', of the body together with the particle about O is given by
$$I' = I + 6mx^2 = \frac{49}{3}ma^2 + 6mx^2$$
Let the angular speed with which the system starts to move be ω.
Conservation of linear momentum about O
$$6mux = I'\omega = \left(\frac{49}{3}ma^2 + 6mx^2\right)\omega$$
$$\omega = \frac{6mux}{\frac{49}{3}ma^2 + 6mx^2} = \frac{18ux}{49a^2 + 18x^2}$$

c $\frac{d\omega}{dx} = 18u\left[\frac{49a^2 + 18x^2 - x \times 36x}{(49a^2 + 18x^2)^2}\right] = \frac{18u}{(49a^2 + 18x^2)^2}(49a^2 - 18x^2)$

$\frac{d\omega}{dx} = 0 \Rightarrow 49a^2 - 18x^2 = 0$

$x^2 = \frac{49}{18}a^2 = \frac{49 \times 2}{36}a^2$

$x = \frac{7}{6}a\sqrt{2}$, as required

> Unless you are asked specifically to do so, you are not expected to show that the stationary point is a maximum. If you were asked to do so, you could argue that as $18x^2$ ranges from less than $49a^2$ to more than $49a^2$, $\frac{d\omega}{dx}$ changes sign from positive to negative and, so, the point is a maximum.

d If $x = \frac{4}{3}a$, $\omega = \frac{18u \times \frac{4}{3}a}{49a^2 + 18\left(\frac{4}{3}a\right)^2} = \frac{24ua}{81a^2} = \frac{8u}{27a}$

and $I' = \frac{49}{3}ma^2 + 6m\left(\frac{4}{3}a\right)^2$

$= \frac{49}{3}ma^2 + \frac{32}{3}ma^2 = 27ma^2$

Conservation of energy

$\frac{1}{2}I\omega^2 = 3mg \times \frac{1}{2}a + 6mgx$

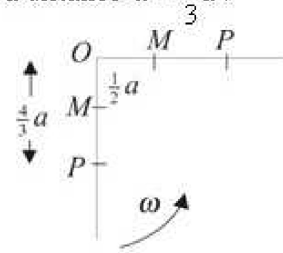
$\frac{1}{2} \times 27ma^2\left(\frac{8u}{27a}\right)^2 = 3mg\left(\frac{1}{2}a\right) + 6mg\left(\frac{4}{3}a\right) = \frac{19}{2}mga$

$\frac{32}{27}mu^2 = \frac{19}{2}mga$

$u^2 = \frac{513}{64}ga = \frac{9 \times 57}{64}ga$

$u = \frac{3}{8}\sqrt{(57ga)}$

> To reach the horizontal, the centre of mass M of the body must rise a distance $\frac{1}{2}a$ and the particle must rise a distance $x = \frac{4}{3}a$.



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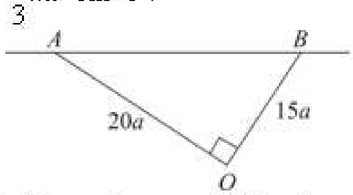
Review Exercise 2
Exercise A, Question 39

Question:

a Use integration to show that the moment of inertia of a uniform rod of mass m and length l, about an axis through one end and inclined at an angle θ to the rod is $\frac{1}{3}ml^2\sin^2\theta$.



The figure shows a rigid body consisting of two uniform rods. Rod AO has mass m and length $20a$ and the rod BO has mass m and length $15a$. They are rigidly joined together at O so that angle AOB is a right angle. The body is free to rotate about a fixed horizontal axis AB and hangs in equilibrium with O below AB. A particle of mass $\frac{1}{3}m$ is moving horizontally at right angles to the plane OAB with speed u. It collides with, and adheres to, the body at O.

b Show that the moment of inertia of the composite body, consisting of the two rods and the particle, about AB is $144ma^2$.

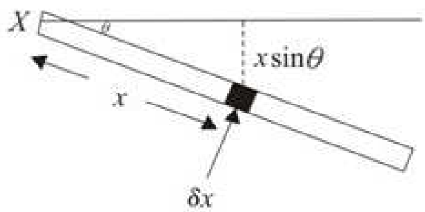
c Find the range of values of u for which the composite body will make complete revolutions.

Given that the composite body does make complete revolutions,

d find the value of u for which the greatest angular speed during the subsequent motion is twice the smallest angular speed. ***E***

Solution:

a



The mass per unit area is $\frac{m}{l}$.

Consider an element of length δx at a distance x from one end of the rod X.

$$\delta I = (\delta m)(x\sin\theta)^2 = \left(\frac{m}{l}\delta x\right)x^2\sin^2\theta$$

$$= \frac{m\sin^2\theta}{l}x^2\delta x$$

Each elementary particle, of mass δm, is at a distance $x\sin\theta$ from the axis of rotation.

For the whole rod

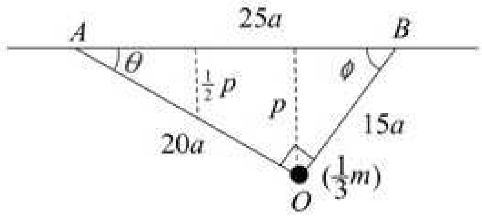
$$I = \sum \delta I = \sum \frac{m\sin^2\theta}{l}x^2\delta x$$

As $\delta x \to 0$

$$I = \int_0^l \frac{m\sin^2\theta}{l}x^2\,\mathrm{d}x = \frac{m\sin^2\theta}{l}\left[\frac{x^3}{3}\right]_0^l$$

$$= \frac{m\sin^2\theta}{l}\left[\frac{l^3}{3} - 0\right] = \frac{1}{3}ml^2\sin^2\theta, \text{ as required}$$

b



$$AB^2 = (20a)^2 + (15a)^2 = (25a)^2 \Rightarrow AB = 25a$$

$$\sin\theta = \frac{15a}{25a} = \frac{3}{5}, \sin\phi = \frac{20a}{25a} = \frac{4}{5}$$

$$\frac{p}{20a} = \sin\theta = \frac{3}{5} \Rightarrow p = 12a$$

Part **a** gives you the clue that you need to start by finding the sines of the angles the rods make with AB.

The moment of inertia of the composite body is given by

$$I = I_{\text{rod } AO} + I_{\text{rod } OB} + I_{\text{particle}}$$

$$= \frac{1}{3}m(20a)^2\sin^2\theta + \frac{1}{3}m(15a)^2\sin^2\phi + \left(\frac{1}{3}m\right)p^2$$

$$= \frac{1}{3}m \times 400a^2 \times \frac{9}{25} + \frac{1}{3}m \times 225a^2 \times \frac{16}{25} + \frac{1}{3}m \times 144a^2$$

$$= 48ma^2 + 48ma^2 + 48ma^2 = 144ma^2, \text{ as required}$$

c Let ω be the angular speed of the composite body immediately after impact

Conservation of angular momentum about AB

$$\frac{1}{3}mu \times 12a = I\omega$$

$$4mua = 144ma^2\omega$$

$$\omega = \frac{4mua}{144ma^2} = \frac{u}{36a}$$

Using energy, for complete revolutions

$$\frac{1}{2}I\omega^2 > 2mgp + \frac{2}{3}mgp$$

$$72ma^2\omega^2 > \frac{8}{3}mgp$$

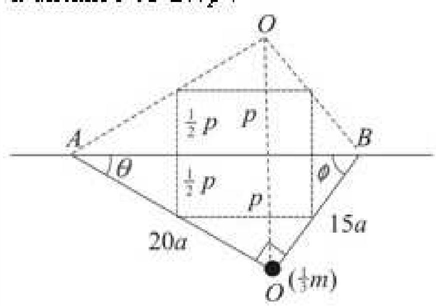
$$72ma^2\left(\frac{u}{36a}\right)^2 > \frac{8}{3}mg \times 12a$$

$$\frac{ma^2u^2}{18a^2} > 32mga$$

$$u^2 > 576ga$$

$$u > 24\sqrt{(ga)}$$

For complete revolutions, there must be enough initial kinetic energy to raise each of the centres of mass of the rods a vertical distance of $2 \times \frac{1}{2}p$ and the particle a distance of $2 \times p$.



d The greatest angular speed is immediately after the impact and, from part **c**, is $\frac{u}{36a}$.

The least angular speed is when O is vertically above AB and is $\frac{1}{2} \times \frac{u}{36a} = \frac{u}{72a}$

Conservation of energy

$$\frac{1}{2}I\left(\frac{u}{36a}\right)^2 - \frac{1}{2}I\left(\frac{u}{72a}\right)^2 = 32mga$$

The increase in potential energy needed for complete revolutions is the same as in part **c**.

$$72ma^2\left(\frac{u^2}{36^2a^2} - \frac{u^2}{72^2a^2}\right) = 32mga$$

$$\frac{1}{24}mu^2 = 32mga$$

$$u^2 = 24 \times 32ga = 256 \times 3ga$$

$$u = 16\sqrt{(3ga)}$$

Solutionbank M5
Edexcel AS and A Level Modular Mathematics

Review Exercise 2
Exercise A, Question 40

Question:

A uniform right angled triangular lamina ABM of mass m is such that $\angle AMB = 90°$, $AM = 3a$ and $BM = a$.

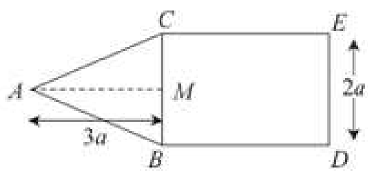
a Find the moment of inertia of the lamina about

i BM,

ii an axis through A parallel to BM,

iii AM.

b Deduce that the moment of inertia of a uniform triangular lamina ABC of mass $2m$ such that $AB = AC, BC = 2a$ and $AM = 3a$, where M is the mid-point of BC, about an axis through A perpendicular to the plane of the lamina is $\frac{28}{3}ma^2$.



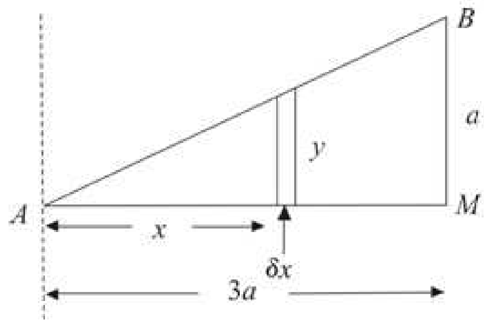
The figure shows a uniform plane lamina of mass $6m$ formed by joining a triangular lamina ABC to a rectangular lamina $BDEC$ along a common line BC. The sides BD and CE are each of length $3a$. The side DE is of length $2a$. $AB = AC$ and $AM = 3a$, where M is the mid-point of BC. The lamina can rotate about a smooth fixed axis through A, perpendicular to the plane of the lamina.

c Show that the moment of inertia of the lamina about this axis is $\frac{284}{3}ma^2$.

d Determine the period of small oscillations of the lamina about its position of stable equilibrium. ***E***

Solution:

a



Area of triangle $AMB = \frac{1}{2} 3a \times a = \frac{3}{2}a^2$

Mass per unit area is $\frac{m}{\frac{3}{2}a^2} = \frac{2m}{3a^2}$

By similar triangles

$\frac{y}{x} = \frac{a}{3a} \Rightarrow y = \frac{1}{3}x$

i About BM

$$\delta I = (\delta m)(3a - x)^2 = \left(y\delta x \times \frac{2m}{3a^2}\right)(3a - x)^2$$

Each point on the thin strip is a distance $(3a - x)$ from BM.

$$= \frac{2m}{9a^2} x(3a - x)^2 \delta x$$

Using $y = \frac{1}{3}x$.

Hence

$$I = \frac{2m}{9a^2}\int_0^{3a} x(3a-x)^2\,dx = \frac{2m}{9a^2}\int_0^{3a}\left(9a^2x - 6ax^2 + x^3\right)dx$$

$$= \frac{2m}{9a^2}\left[\frac{9a^2x^2}{2} - 2ax^3 + \frac{x^4}{4}\right]_0^{3a}$$

$$= \frac{2m}{9a^2}\left[\frac{81a^4}{2} - 54a^4 + \frac{81a^4}{4}\right] = \frac{2m}{9a^2} \times \frac{27a^4}{4}$$

$$= \frac{3}{2}ma^2$$

ii About an axis through A parallel to BM

$$\delta I = (\delta m)x^2 = \left(y\delta x \times \frac{2m}{3a^2}\right)x^2$$

This axis is shown by a dotted line on the diagram above.

$$= \frac{2m}{9a^2}x^3\delta x$$

Hence

$$I = \frac{2m}{9a^2}\int_0^{3a} x^3\,\mathrm{d}x = \frac{2m}{9a^2}\left[\frac{x^4}{4}\right]_0^{3a}$$

$$= \frac{2m}{9a^2}\times\frac{81a^4}{4} = \frac{9}{2}ma^2$$

iii About AM

$$\delta I = \frac{4}{3}(\delta m)\left(\frac{y}{2}\right)^2 = \frac{4}{3}\left(y\delta x\times\frac{2m}{3a^2}\right)\left(\frac{y}{2}\right)^2$$

$$= \frac{2m}{9a^2}y^3\delta x = \frac{2m}{243a^2}x^3\delta x$$

Using the standard result, $I = \frac{4}{3}ml^2$, for a rod of length $2l$ about an axis through its end, with $m = \delta m$ and $2l = y$.

Using $y = \frac{1}{3}x$.

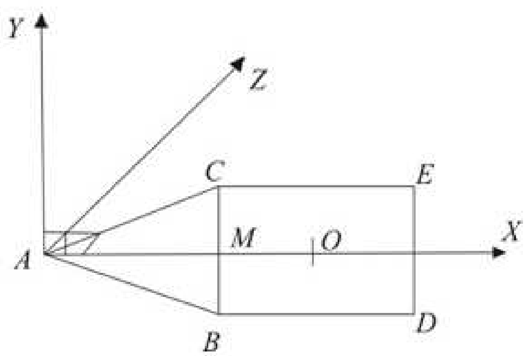
Hence

$$I = \frac{2m}{243a^2}\int_0^{3a} x^3\,\mathrm{d}x = \frac{2m}{243a^2}\left[\frac{x^4}{4}\right]_0^{3a}$$

$$= \frac{2m}{243a^2}\times\frac{81a^4}{4} = \frac{1}{6}ma^2$$

You can deduce this result, without integrating, from the answer to part **a i**. Replacing $3a$ by a, $I = \frac{3}{2}m\left(\frac{a}{3}\right)^2 = \frac{1}{6}ma^2$.

b For this part of the question, just consider triangle ABC in the diagram below. The full diagram is needed for part **c**.



The axis of rotation, AZ, is perpendicular to the plane of the lamina, that is, it is perpendicular to both AX and AY.

The moment of inertia of the two triangles about AX is given by

$$I_{AX} = 2 \times \frac{1}{6}ma^2 = \frac{1}{3}ma^2$$

Note that the triangle ABC has mass $2m$, so each right angled triangle has mass m.

That is twice the answer to part **a iii**.

The moment of inertia of the two triangles about AY is given by

$$I_{AY} = 2 \times \frac{9}{2}ma^2 = 9ma^2$$

That is twice the answer to part **a ii**.

By the perpendicular axes theorem, the moment of inertia of the two triangles about AZ is given by

$$I_{\text{triangles}} = I_{AX} + I_{AY} = \frac{1}{3}ma^2 + 9ma^2 = \frac{28}{3}ma^2$$

c The moment of inertia of the rectangle about an axis through its centre O perpendicular to the plane of the lamina is given by

The area of each triangle ACM and ABM is one quarter of the area of the rectangle $BDEC$. As the total mass is $6m$, the mass of each triangle is m and the mass of the rectangle is $4m$.

$$I_O = \frac{1}{3}(4m)\left(a^2 + \left(\frac{3a}{2}\right)^2\right) = \frac{13}{3}ma^2$$

Using the standard result that the moment of inertia of a rectangle, sides $2a$ and $2b$, about a perpendicular axis through its centre is $\frac{1}{3}m\left(a^2 + b^2\right)$ with $2b = 3a$. The mass of this rectangle is $4m$.

By the parallel axes theorem, the moment of inertia of the rectangle about AZ is given by

$$I_{\text{rectangle}} = I_O + 4mOA^2$$

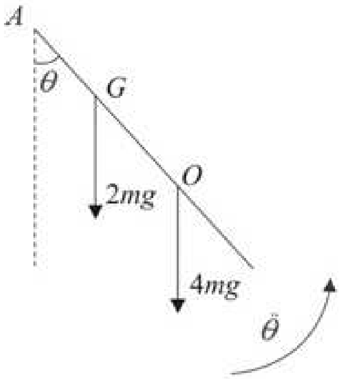
$$= \frac{13}{3}ma^2 + 4m\left(\frac{9a}{2}\right)^2 = \frac{256}{3}ma^2$$

The moment of inertia of the complete lamina about AZ is given by

$$I = I_{\text{triangles}} + I_{\text{rectangle}}$$

$$= \frac{28}{3}ma^2 + \frac{256}{3}ma^2 = \frac{284}{3}ma^2$$, as required

d



If G is the centre of mass of the triangle ABC, then, as AM is the median of the triangle,
$AG = \frac{2}{3}AM = \frac{2}{3} \times 3a = 2a$.

Equation of angular motion about AZ.

$$L = I\ddot{\theta}$$

$$-2mg \times AG\sin\theta - 4mg \times AO\sin\theta = \frac{284}{3}ma^2\ddot{\theta}$$

$$-2mg \times 2a\sin\theta - 4mg \times \frac{9a}{2}\sin\theta = \frac{284}{3}ma^2\ddot{\theta}$$

$$-22mga\sin\theta = \frac{284}{3}ma^2\ddot{\theta}$$

$$\ddot{\theta} = -\frac{33g}{142a}\sin\theta$$

For small θ, $\sin\theta \approx \theta$

Hence

$$\ddot{\theta} = -\frac{33g}{142a}\theta$$

Comparing with the standard equation for simple harmonic motion, $\ddot{\theta} = -\omega^2\theta$, the motion is approximately simply harmonic, with $\omega^2 = \frac{33g}{142a}$. The period of small oscillations is given by

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\left(\frac{142a}{33g}\right)}.$$