Solutionbank C2
Edexcel Modular Mathematics for AS and A-Level

Differentiation
Exercise A, Question 1

Question:

Find the values of x for which f(x) is an increasing function, given that f(x) equals:

(a) $3x^2 + 8x + 2$

(b) $4x - 3x^2$

(c) $5 - 8x - 2x^2$

(d) $2x^3 - 15x^2 + 36x$

(e) $3 + 3x - 3x^2 + x^3$

(f) $5x^3 + 12x$

(g) $x^4 + 2x^2$

(h) $x^4 - 8x^3$

Solution:

(a) $f(x) = 3x^2 + 8x + 2$
$f'(x) = 6x + 8$
$f'(x) > 0 \quad \Rightarrow \quad 6x + 8 > 0$

So $x > \frac{-8}{6}$

i.e. $x > \frac{-4}{3}$

(b) $f(x) = 4x - 3x^2$
$f'(x) = 4 - 6x$
$f'(x) > 0 \quad \Rightarrow \quad 4 - 6x > 0$
So $4 > 6x$
i.e. $6x < 4$

$x < \frac{4}{6}$

$x < \frac{2}{3}$

(c) $f(x) = 5 - 8x - 2x^2$
$f'(x) = -8 - 4x$
$f'(x) > 0 \quad \Rightarrow \quad -8 - 4x > 0$
So $-8 > 4x$ (add $4x$ to both sides)
i.e. $4x < -8$
$x < -2$

(d) $f(x) = 2x^3 - 15x^2 + 36x$
$f'(x) = 6x^2 - 30x + 36$

$f'(x) > 0 \Rightarrow 6x^2 - 30x + 36 > 0$

So $6(x^2 - 5x + 6) > 0$
i.e. $6(x - 2)(x - 3) > 0$
By considering the 3 regions

	$x < 2$	$2 < x < 3$	$x > 3$
$6(x - 2)(x - 3)$	+ve	–ve	+ve

Then $x < 2$ or $x > 3$

(e) $f(x) = 3 + 3x - 3x^2 + x^3$
$f'(x) = 3 - 6x + 3x^2$
$f'(x) > 0 \Rightarrow 3 - 6x + 3x^2 > 0$
So $3(x^2 - 2x + 1) > 0$
i.e. $3(x - 1)^2 > 0$
So $x \in \mathbb{R}, x \neq 1$

(f) $f(x) = 5x^3 + 12x$
$f'(x) = 15x^2 + 12$
$f'(x) > 0 \Rightarrow 15x^2 + 12 > 0$
This is true for all real values of x.
So $x \in \mathbb{R}$

(g) $f(x) = x^4 + 2x^2$
$f'(x) = 4x^3 + 4x$
$f'(x) > 0 \Rightarrow 4x^3 + 4x > 0$
So $4x(x^2 + 1) > 0$
As $x^2 + 1 > 0$ for all x, $x > 0$

(h) $f(x) = x^4 - 8x^3$
$f'(x) = 4x^3 - 24x^2$
$f'(x) > 0 \Rightarrow 4x^3 - 24x^2 > 0$
So $4x^2(x - 6) > 0$
As $x^2 > 0$ for all x, $x - 6 > 0$
So $x > 6$

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Differentiation
Exercise A, Question 2

Question:

Find the values of x for which f(x) is a decreasing function, given that f(x) equals:

(a) $x^2 - 9x$

(b) $5x - x^2$

(c) $4 - 2x - x^2$

(d) $2x^3 - 3x^2 - 12x$

(e) $1 - 27x + x^3$

(f) $x + \frac{25}{x}$

(g) $x^{\frac{1}{2}} + 9x^{-\frac{1}{2}}$

(h) $x^2 (x + 3)$

Solution:

(a) $f (x) = x^2 - 9x$
$f ' (x) = 2x - 9$
$f ' (x) < 0 \quad \Rightarrow \quad 2x - 9 < 0$
So $2x < 9$
i.e. $x < 4.5$

(b) $f (x) = 5x - x^2$
$f ' (x) = 5 - 2x$
$f ' (x) < 0 \quad \Rightarrow \quad 5 - 2x < 0$
So $5 < 2x$
i.e. $2x > 5$
$x > 2.5$

(c) $f (x) = 4 - 2x - x^2$
$f ' (x) = - 2 - 2x$
$f ' (x) < 0 \quad \Rightarrow \quad - 2 - 2x < 0$
So $- 2 < 2x$
i.e. $2x > - 2$
$x > - 1$

(d) $f (x) = 2x^3 - 3x^2 - 12x$
$f ' (x) = 6x^2 - 6x - 12$
$f ' (x) < 0 \quad \Rightarrow \quad 6x^2 - 6x - 12 < 0$
So $6 (x^2 - x - 2) < 0$
i.e. $6 (x - 2) (x + 1) < 0$
By considering the 3 regions $x < - 1$, $- 1 < x < 2$, $x > 2$ determine
$- 1 < x < 2$

(e) $f(x) = 1 - 27x + x^3$
$f'(x) = -27 + 3x^2$
$f'(x) < 0 \Rightarrow -27 + 3x^2 < 0$
So $3x^2 < 27$
i.e. $x^2 < 9$
$-3 < x < 3$

(f) $f\left(x\right) = x + \frac{25}{x}$

$f'\left(x\right) = 1 - \frac{25}{x^2}$

$f'\left(x\right) < 0 \Rightarrow 1 - \frac{25}{x^2} < 0$

So $1 < \frac{25}{x^2}$
Multiply both sides by x^2:
$x^2 < 25$
$-5 < x < 5$

(g) $f\left(x\right) = x^{\frac{1}{2}} + 9x^{-\frac{1}{2}}$

$f'\left(x\right) = \frac{1}{2}x^{-\frac{1}{2}} - 9 \times \frac{1}{2}x^{-\frac{3}{2}}$

$f'\left(x\right) < 0 \Rightarrow \frac{1}{2}x^{-\frac{1}{2}} - \frac{9}{2}x^{-\frac{3}{2}} < 0$

So $\frac{x^{-\frac{3}{2}}}{2}\left(x - 9\right) < 0$

$x > 0$ or the function is not defined
So $0 < x < 9$

(h) $f(x) = x^3 + 3x^2$
$f'(x) = 3x^2 + 6x$
$f'(x) < 0 \Rightarrow 3x^2 + 6x < 0$
So $3x(x + 2) < 0$
Consider the regions $x < -2$, $-2 < x < 0$ and $x > 0$ to give
$-2 < x < 0$

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Differentiation
Exercise B, Question 1

Question:

Find the least value of each of the following functions:

(a) $f(x) = x^2 - 12x + 8$

(b) $f(x) = x^2 - 8x - 1$

(c) $f(x) = 5x^2 + 2x$

Solution:

(a) $f(x) = x^2 - 12x + 8$
$f'(x) = 2x - 12$
Put $f'(x) = 0$, then $2x - 12 = 0$, i.e. $x = 6$
$f(6) = 6^2 - 12 \times 6 + 8 = -28$
The least value of $f(x)$ is -28.

(b) $f(x) = x^2 - 8x - 1$
$f'(x) = 2x - 8$
Put $f'(x) = 0$, then $2x - 8 = 0$, i.e. $x = 4$
$f(4) = 4^2 - 8 \times 4 - 1 = -17$
The minimum value of $f(x)$ is -17.

(c) $f(x) = 5x^2 + 2x$
$f'(x) = 10x + 2$
Put $f'(x) = 0$, then $10x + 2 = 0$, i.e. $x = \frac{-2}{10}$ or $x = -\frac{1}{5}$

$$f\left(-\frac{1}{5}\right) = 5\left(-\frac{1}{5}\right)^2 + 2\left(-\frac{1}{5}\right) = \frac{5}{25} - \frac{2}{5} = -\frac{1}{5}$$

The least value of $f(x)$ is $-\frac{1}{5}$

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Differentiation
Exercise B, Question 2

Question:

Find the greatest value of each of the following functions:

(a) $f(x) = 10 - 5x^2$

(b) $f(x) = 3 + 2x - x^2$

(c) $f(x) = (6 + x)(1 - x)$

Solution:

(a) $f(x) = 10 - 5x^2$
$f'(x) = -10x$
Put $f'(x) = 0$, then $-10x = 0$, i.e. $x = 0$
$f(0) = 10 - 5 \times 0^2 = 10$
Maximum value of $f(x)$ is 10.

(b) $f(x) = 3 + 2x - x^2$
$f'(x) = 2 - 2x$
Put $f'(x) = 0$, then $2 - 2x = 0$, i.e. $x = 1$
$f(1) = 3 + 2 - 1 = 4$
The greatest value of $f(x)$ is 4.

(c) $f(x) = (6 + x)(1 - x) = 6 - 5x - x^2$
$f'(x) = -5 - 2x$
Put $f'(x) = 0$, then $-5 - 2x = 0$, i.e. $x = -2\frac{1}{2}$

$$f\left(-2\frac{1}{2}\right) = 3\frac{1}{2} \times 3\frac{1}{2} = 12\frac{1}{4}$$

The maximum value of $f(x)$ is $12\frac{1}{4}$.

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Differentiation
Exercise B, Question 3

Question:

Find the coordinates of the points where the gradient is zero on the curves with the given equations. Establish whether these points are maximum points, minimum points or points of inflexion, by considering the second derivative in each case.

(a) $y = 4x^2 + 6x$

(b) $y = 9 + x - x^2$

(c) $y = x^3 - x^2 - x + 1$

(d) $y = x (x^2 - 4x - 3)$

(e) $y = x + \frac{1}{x}$

(f) $y = x^2 + \frac{54}{x}$

(g) $y = x - 3 \sqrt{x}$

(h) $y = x^{\frac{1}{2}} \left(x - 6 \right)$

(i) $y = x^4 - 12x^2$

Solution:

(a) $y = 4x^2 + 6x$

$\frac{dy}{dx} = 8x + 6$

Put $\frac{dy}{dx} = 0$

Then $8x + 6 = 0$

$8x = - 6$

$x = - \frac{3}{4}$

When $x = - \frac{3}{4}$, $y = 4 \left(- \frac{3}{4} \right)^2 + 6 \left(- \frac{3}{4} \right) = \frac{9}{4} - \frac{9}{2} = - \frac{9}{4}$

So $\left(- \frac{3}{4} , - \frac{9}{4} \right)$ is a point of zero gradient

$\frac{d^2y}{dx^2} = 8 > 0$

So $\left(-\frac{3}{4}, -\frac{9}{4} \right)$ is a minimum point

(b) $y = 9 + x - x^2$

$\frac{dy}{dx} = 1 - 2x$

Put $\frac{dy}{dx} = 0$

Then $1 - 2x = 0$

$x = \frac{1}{2}$

When $x = \frac{1}{2}$, $y = 9 + \frac{1}{2} - \left(\frac{1}{2} \right)^2 = 9 \frac{1}{4}$

So $\left(\frac{1}{2}, 9 \frac{1}{4} \right)$ is a point with zero gradient

$\frac{d^2y}{dx^2} = -2 < 0$

So $\left(\frac{1}{2}, 9 \frac{1}{4} \right)$ is a maximum point

(c) $y = x^3 - x^2 - x + 1$

$\frac{dy}{dx} = 3x^2 - 2x - 1$

Put $\frac{dy}{dx} = 0$

Then $3x^2 - 2x - 1 = 0$

$(3x + 1) (x - 1) = 0$

$x = -\frac{1}{3}$ or $x = 1$

When $x = -\frac{1}{3}$, $y = \left(-\frac{1}{3} \right)^3 - \left(-\frac{1}{3} \right)^2 - \left(-\frac{1}{3} \right) + 1 = 1 \frac{5}{27}$

When $x = 1$, $y = 1^3 - 1^2 - 1 + 1 = 0$

So $\left(-\frac{1}{3}, 1 \frac{5}{27} \right)$ and (1, 0) are points of zero gradient

$\frac{d^2y}{dx^2} = 6x - 2$

When $x = -\frac{1}{3}$, $\frac{d^2y}{dx^2} = -4 < 0$

So $\left(-\frac{1}{3}, 1 \frac{5}{27} \right)$ is a maximum point

When $x = 1$, $\frac{d^2y}{dx^2} = 6 - 2 = 4 > 0$

So (1, 0) is a minimum point

(d) $y = x (x^2 - 4x - 3) = x^3 - 4x^2 - 3x$

$\frac{dy}{dx} = 3x^2 - 8x - 3$

Put $\frac{dy}{dx} = 0$

Then $3x^2 - 8x - 3 = 0$
$(3x + 1)(x - 3) = 0$
$x = -\frac{1}{3}$ or 3

When $x = -\frac{1}{3}$, $y = \left(-\frac{1}{3} \right)^3 - 4\left(-\frac{1}{3} \right)^2 - 3\left(-\frac{1}{3} \right) = \frac{14}{27}$

When $x = 3$, $y = 3^3 - 4 \times 3^2 - 3 \times 3 = -18$

So $\left(-\frac{1}{3}, -\frac{14}{27} \right)$ and $(3, -18)$ are points with zero gradient

$\frac{d^2y}{dx^2} = 6x - 8$

When $x = -\frac{1}{3}$, $\frac{d^2y}{dx^2} = -10 < 0$

So $\left(-\frac{1}{3}, -\frac{14}{27} \right)$ is a maximum point

When $x = 3$, $\frac{d^2y}{dx^2} = +10 > 0$

So $(3, -18)$ is a minimum point

(e) $y = x + \frac{1}{x} = x + x^{-1}$

$\frac{dy}{dx} = 1 - x^{-2}$

Put $\frac{dy}{dx} = 0$

Then $1 - x^{-2} = 0$
$x^2 = 1$
$x = \pm 1$

When $x = 1$, $y = 1 + \frac{1}{1} = 2$

When $x = -1$, $y = -1 + \frac{1}{-1} = -2$

So (1, 2) and $(-1, -2)$ are points with zero gradient

$\frac{d^2y}{dx^2} = 2x^{-3}$

When $x = 1$, $\frac{d^2y}{dx^2} = 2 > 0$

So (1, 2) is a minimum point

When $x = -1$, $\frac{d^2y}{dx^2} = -2 < 0$

So $(-1, -2)$ is a maximum point

(f) $y = x^2 + \frac{54}{x} = x^2 + 54x^{-1}$

$\frac{dy}{dx} = 2x - 54x^{-2}$

Put $\frac{dy}{dx} = 0$

Then $2x - 54x^{-2} = 0$

$2x = \frac{54}{x^2}$

$x^3 = 27$
$x = 3$

When $x = 3$, $y = 3^2 + \frac{54}{3} = 27$

So (3, 27) is a point of zero gradient

$\frac{d^2y}{dx^2} = 2 + 108x^{-3}$

When $x = 3$, $\frac{d^2y}{dx^2} = 6 > 0$

So (3, 27) is a minimum point

(g) $y = x - 3\sqrt{x} = x - 3x^{\frac{1}{2}}$

$\frac{dy}{dx} = 1 - \frac{3}{2}x^{-\frac{1}{2}}$

Put $\frac{dy}{dx} = 0$

Then $1 - \frac{3}{2}x^{-\frac{1}{2}} = 0$

$1 = \frac{3}{2\sqrt{x}}$

$\sqrt{x} = \frac{3}{2}$

$x = \frac{9}{4}$

When $x = \frac{9}{4}$, $y = \frac{9}{4} - 3\sqrt{\frac{9}{4}} = \frac{-9}{4}$

So $\left(\frac{9}{4}, \frac{-9}{4}\right)$ is a point with zero gradient

$\frac{d^2y}{dx^2} = \frac{3}{4}x^{-\frac{3}{2}}$

When $x = \frac{9}{4}$, $\frac{d^2y}{dx^2} = \frac{3}{4} \times \left(\frac{9}{4}\right)^{-\frac{3}{2}} = \frac{3}{4} \times \left(\frac{2}{3}\right)^3 = \frac{2}{9} > 0$

So $\left(\frac{9}{4}, \frac{-9}{4}\right)$ is a minimum point

(h) $y = x^{\frac{1}{2}}\left(x - 6\right) = x^{\frac{3}{2}} - 6x^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}} - 3x^{-\frac{1}{2}}$

Put $\frac{dy}{dx} = 0$

Then $\frac{3}{2}x^{\frac{1}{2}} - 3x^{-\frac{1}{2}} = 0$

$\frac{3}{2}x^{\frac{1}{2}} = \frac{3}{x^{\frac{1}{2}}}$

Multiply both sides by $x^{\frac{1}{2}}$:

$\frac{3}{2}x = 3$

$x = 2$

When $x = 2$, $y = 2^{\frac{1}{2}}\left(-4\right) = -4\sqrt{2}$

So $\left(2, -4\sqrt{2}\right)$ is a point with zero gradient

$\frac{d^2y}{dx^2} = \frac{3}{4}x^{-\frac{1}{2}} + \frac{3}{2}x^{-\frac{3}{2}}$

When $x = 2$, $\frac{d^2y}{dx^2} = \frac{3}{4\sqrt{2}} + \frac{3}{4\sqrt{2}} > 0$

So $\left(2, -4\sqrt{2}\right)$ is a minimum point

(i) $y = x^4 - 12x^2$

$\frac{dy}{dx} = 4x^3 - 24x$

Put $\frac{dy}{dx} = 0$

Then $4x^3 - 24x = 0$

$4x\left(x^2 - 6\right) = 0$

$x = 0$ or $x = \pm\sqrt{6}$

When $x = 0$, $y = 0$

When $x = \pm\sqrt{6}$, $y = -36$

So $\left(0, 0\right)$, $\left(\sqrt{6}, -36\right)$ and $\left(-\sqrt{6}, -36\right)$ are points with zero gradient

$\frac{d^2y}{dx^2} = 12x^2 - 24$

When $x = 0$, $\frac{d^2y}{dx^2} = -24 < 0$

So (0, 0) is a maximum point

When $x^2 = 6$, $\frac{d^2y}{dx^2} = 48 > 0$

So $\left(\sqrt{6}, -36\right)$ and $\left(-\sqrt{6}, -36\right)$ are minimum points

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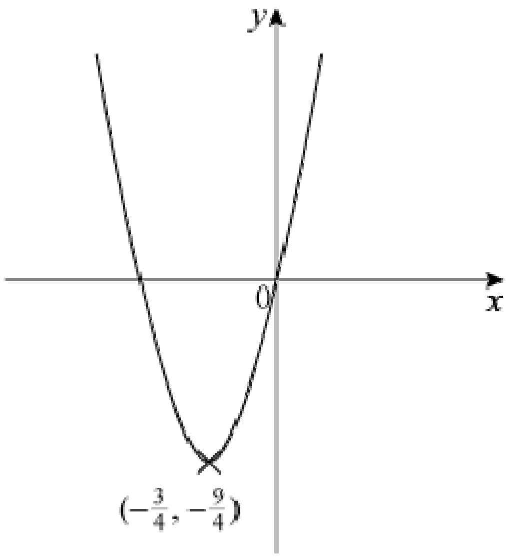
Differentiation
Exercise B, Question 4

Question:

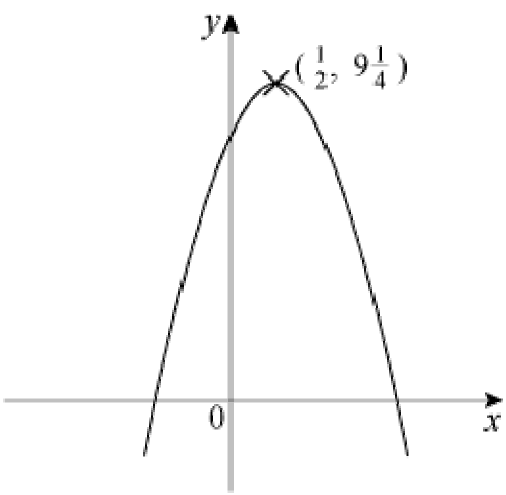
Sketch the curves with equations given in question 3 parts (a), (b), (c) and (d) labelling any stationary values.

Solution:

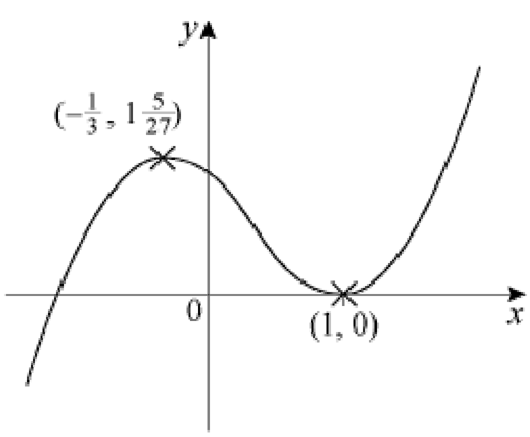
(a)


y
0
x
$(-\frac{3}{4}, -\frac{9}{4})$

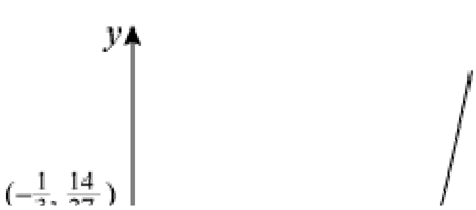
(b)


y
$(\frac{1}{2}, 9\frac{1}{4})$
0
x

(c)


y
$(-\frac{1}{3}, 1\frac{5}{27})$
0
(1, 0)
x

(d)


y

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Differentiation
Exercise B, Question 5

Question:

By considering the gradient on either side of the stationary point on the curve $y = x^3 - 3x^2 + 3x$, show that this point is a point of inflexion.
Sketch the curve $y = x^3 - 3x^2 + 3x$.

Solution:

$y = x^3 - 3x^2 + 3x$

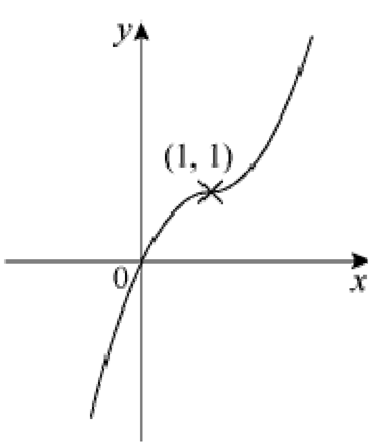
$\frac{dy}{dx} = 3x^2 - 6x + 3$

Put $\frac{dy}{dx} = 0$

Then $3x^2 - 6x + 3 = 0$
$3 (x^2 - 2x + 1) = 0$
$3 (x - 1) ^2 = 0$
$x = 1$
when $x = 1$, $y = 1$
So (1, 1) is a point with zero gradient.
Consider points near to (1, 1) and find the gradient at these points.

x	0.9	1	1.1
$\frac{dy}{dx}$	0.03	0	0.03
	+ve	0	+ve

The gradient on either side of (1, 1) is positive.
This is *not* a turning point—it is a point of inflexion.



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Differentiation
Exercise B, Question 6

Question:

Find the maximum value and hence the range of values for the function $f(x) = 27 - 2x^4$.

Solution:

$f(x) = 27 - 2x^4$
$f'(x) = -8x^3$
Put $f'(x) = 0$
Then $-8x^3 = 0$
So $x = 0$
$f(0) = 27$
So (0, 27) is a point of zero gradient
$f''(x) = -24x^2$
$f''(0) = 0$ —not conclusive
Find gradient on either side of (0, 27):

x	-0.1	0	0.1
$\frac{dy}{dx}$	$+0.008$	0	-0.008
	/	—	\

There is a maximum turning point at (0, 27).
So the maximum value of $f(x)$ is 27 and range of values is $f(x) \leq 27$.

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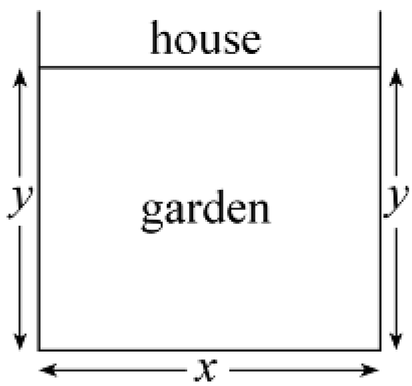
Differentiation
Exercise C, Question 1

Question:

A rectangular garden is fenced on three sides, and the house forms the fourth side of the rectangle.
Given that the total length of the fence is 80 m show that the area, A, of the garden is given by the formula $A = y \left(80 - 2y \right)$, where y is the distance from the house to the end of the garden.
Given that the area is a maximum for this length of fence, find the dimensions of the enclosed garden, and the area which is enclosed.

Solution:



Let the width of the garden be x m.
Then $x + 2y = 80$
So $x = 80 - 2y$ *
Area $A = xy$
So $A = y \left(80 - 2y \right)$

$A = 80y - 2y^2$

$\frac{dA}{dy} = 80 - 4y$

Put $\frac{dA}{dy} = 0$ for maximum area

Then $80 - 4y = 0$
So $y = 20$
Substitute in * to give $x = 40$.
So area $= 40 \text{ m} \times 20 \text{ m} = 800 \text{ m}^2$

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Differentiation
Exercise C, Question 2

Question:

A closed cylinder has total surface area equal to 600π. Show that the volume, V cm^3, of this cylinder is given by the formula $V = 300\pi r - \pi r^3$, where r cm is the radius of the cylinder.
Find the maximum volume of such a cylinder.

Solution:

Total surface area $= 2\pi \text{rh} + 2\pi r^2$
So $2\pi \text{rh} + 2\pi r^2 = 600\pi$
$\text{rh} = 300 - r^2$
Volume $= \pi r^2 h = \pi r (\text{rh}) = \pi r (300 - r^2)$
So $V = 300\pi r - \pi r^3$

For maximum volume $\frac{dV}{dr} = 0$

$\frac{dV}{dr} = 300\pi - 3\pi r^2$

Put $\frac{dV}{dr} = 0$

Then $300\pi - 3\pi r^2 = 0$
So $r^2 = 100$
$r = 10$
Substitute $r = 10$ into V to give
$V = 300\pi \times 10 - \pi \times 10^3 = 2000\pi$
Maximum volume $= 2000\pi$ cm^3

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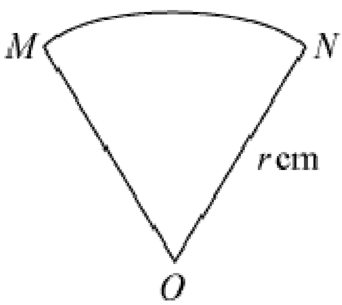
Differentiation
Exercise C, Question 3

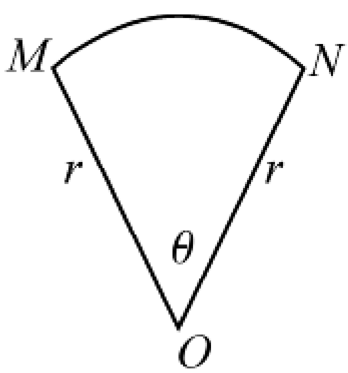
Question:

A sector of a circle has area 100 cm^2. Show that the perimeter of this sector is given by the formula $P = 2r + \frac{200}{r}, r > \sqrt{\frac{100}{\pi}}$.

Find the minimum value for the perimeter of such a sector.



Solution:



Let angle MON = θ radians.
Then perimeter $P = 2r + r\theta$ ①

and area $A = \frac{1}{2}r^2\theta$

But area is 100 cm^2 so

$\frac{1}{2}r^2\theta = 100$

$r\theta = \frac{200}{r}$

Substitute into ① to give

$P = 2r + \frac{200}{r}$ ②

Since area of circle > area of sector
$\pi r^2 > 100$

So $r > \sqrt{\frac{100}{\pi}}$

For the minimum perimeter $\frac{dP}{dr} = 0$

$\frac{dP}{dr} = 2 - \frac{200}{r^2}$

Put $\frac{dP}{dr} = 0$

Then $2 - \frac{200}{r^2} = 0$

So $r = 10$

Substitute into ② to give $P = 20 + 20 = 40$

Minimum perimeter $= 40$ cm

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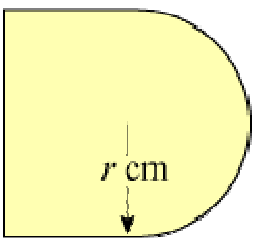
Differentiation
Exercise C, Question 4

Question:

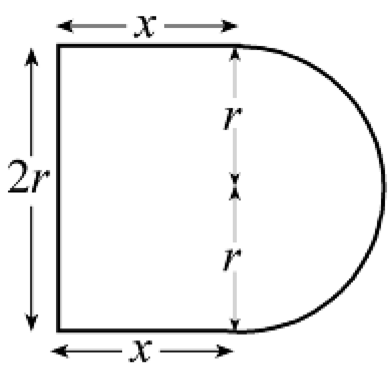
A shape consists of a rectangular base with a semicircular top, as shown. Given that the perimeter of the shape is 40 cm, show that its area, A cm^2, is given by the formula

$A = 40r - 2r^2 - \frac{\pi r^2}{2}$

where r cm is the radius of the semicircle. Find the maximum value for this area.



Solution:



Let the rectangle have dimensions $2r$ by x cm.
Then perimeter of figure is
$(2r + 2x + \pi r)$ cm
But perimeter is 40 cm so
$2r + 2x + \pi r = 40$

$x = \frac{40 - \pi r - 2r}{2}$ *

Area $= 2rx + \frac{1}{2}\pi r^2$ (rectangle + semicircle)

So $A = r \left(40 - \pi r - 2r \right) + \frac{1}{2}\pi r^2$ (substituting from *)

$\Rightarrow \quad A = 40r - 2r^2 - \frac{1}{2}\pi r^2$

To find maximum value, put $\frac{dA}{dr} = 0$:

$40 - 4r - \pi r = 0$

$r = \frac{40}{4 + \pi}$

Substitute into expression for A:

$$A = 40 \times \frac{40}{4+\pi} - 2\left(\frac{40}{4+\pi}\right)^2 - \frac{1}{2}\pi\left(\frac{40}{4+\pi}\right)^2$$

$$A = \frac{1600}{4+\pi} - \left(2 + \frac{1}{2}\pi\right)\left(\frac{40}{4+\pi}\right)^2$$

$$A = \frac{1600}{4+\pi} - \frac{4+\pi}{2} \times \frac{1600}{(4+\pi)^2}$$

$$A = \frac{1600}{4+\pi} - \frac{800}{4+\pi}$$

$$A = \frac{800}{4+\pi} \text{ cm}^2$$

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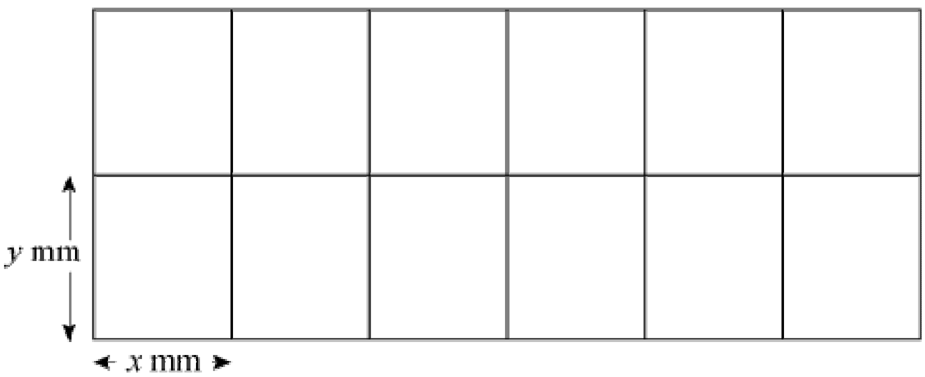
Differentiation
Exercise C, Question 5

Question:

The shape shown is a wire frame in the form of a large rectangle split by parallel lengths of wire into 12 smaller equal-sized rectangles.



Given that the total length of wire used to complete the whole frame is 1512 mm, show that the area of the whole shape is A mm^2, where $A = 1296x - \frac{108x^2}{7}$, where x mm is the width of one of the smaller rectangles.

Find the maximum area which can be enclosed in this way.

Solution:

Total length of wire is $(18x + 14y)$ mm
But length = 1512 mm so
$18x + 14y = 1512$
$y = \frac{1512 - 18x}{14}$ ①

Total area A mm^2 is given by
$A = 2y \times 6x$ ②
Substitute ① into ② to give

$A = 12x\left(\frac{1512 - 18x}{14}\right)$

$A = 1296x - \frac{108}{7}x^2$ *

For maximum area, put $\frac{dA}{dx} = 0$:

$\frac{dA}{dx} = 1296 - \frac{216}{7}x$

when $\frac{dA}{dx} = 0$, $x = \frac{7 \times 1296}{216} = 42$

Substitute $x = 42$ into * to give $A = 27216$
Maximum area $= 27216$ mm^2

(Check: $\frac{d^2A}{dx^2} = -\frac{216}{7} < 0 \therefore$ maximum)

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Differentiation
Exercise D, Question 1

Question:

Given that: $y = x^{\frac{3}{2}} + \frac{48}{x} \quad \left(x > 0 \right)$

(a) Find the value of x and the value of y when $\frac{dy}{dx} = 0$.

(b) Show that the value of y which you found in (a) is a minimum. **[E]**

Solution:

Given that $y = x^{\frac{3}{2}} + \frac{48}{x} \quad \left(x > 0 \right)$

(a) $\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}} - \frac{48}{x^2}$

Put $\frac{dy}{dx} = 0$:

$\frac{3}{2}x^{\frac{1}{2}} = \frac{48}{x^2}$

$x^{2\frac{1}{2}} = 32$

$x = 4$

Substitute $x = 4$ into $y = x^{\frac{3}{2}} + \frac{48}{x}$ to give

$y = 8 + 12 = 20$

So $x = 4$ and $y = 20$ when $\frac{dy}{dx} = 0$

(b) $\frac{d^2y}{dx^2} = \frac{3}{4}x^{-\frac{1}{2}} + \frac{96}{x^3}$

When $x = 4$, $\frac{d^2y}{dx^2} = \frac{3}{8} + \frac{96}{64} = \frac{15}{8} > 0 \therefore$ minimum

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Differentiation
Exercise D, Question 2

Question:

A curve has equation $y = x^3 - 5x^2 + 7x - 14$. Determine, by calculation, the coordinates of the stationary points of the curve C.

[E]

Solution:

$y = x^3 - 5x^2 + 7x - 14$

$\frac{dy}{dx} = 3x^2 - 10x + 7$

When $\frac{dy}{dx} = 0$

$3x^2 - 10x + 7 = 0$

$(3x - 7)(x - 1) = 0$

$x = \frac{7}{3}$ or $x = 1$

When $x = \frac{7}{3}$, $y = -12\frac{5}{27}$

When $x = 1$, $y = -11$

So $\left(\frac{7}{3}, -12\frac{5}{27}\right)$ and $(1, -11)$ are stationary points (where the gradient is zero)

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Differentiation
Exercise D, Question 3

Question:

The function f, defined for $x \in \mathbb{R}$, $x > 0$, is such that:

$$f'(x) = x^2 - 2 + \frac{1}{x^2}$$

(a) Find the value of $f''(x)$ at $x = 4$.

(b) Given that $f(3) = 0$, find $f(x)$.

(c) Prove that f is an increasing function.

[E]

Solution:

$$f'(x) = x^2 - 2 + \frac{1}{x^2} \quad (x > 0)$$

(a) $f''(x) = 2x - \frac{2}{x^3}$

At $x = 4$, $f''(x) = 7\frac{31}{32}$

(b) $f(x) = \frac{x^3}{3} - 2x - \frac{1}{x} + c$

$f(3) = 0 \Rightarrow \frac{3^3}{3} - 2 \times 3 - \frac{1}{3} + c = 0$

$\Rightarrow c = -2\frac{2}{3}$

So $f(x) = \frac{x^3}{3} - 2x - \frac{1}{x} - 2\frac{2}{3}$

(c) For an increasing function, $f'(x) > 0$

$\Rightarrow x^2 - 2 + \frac{1}{x^2} > 0$

$\Rightarrow \left(x - \frac{1}{x}\right)^2 > 0$

This is true for all x, except $x = 1$ [where $f'(1) = 0$].
So the function is an increasing function.

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Differentiation
Exercise D, Question 4

Question:

A curve has equation $y = x^3 - 6x^2 + 9x$.
Find the coordinates of its maximum turning point.

[E]

Solution:

$y = x^3 - 6x^2 + 9x$

$\frac{dy}{dx} = 3x^2 - 12x + 9$

Put $\frac{dy}{dx} = 0$

Then $3x^2 - 12x + 9 = 0$
$3 (x^2 - 4x + 3) = 0$
$3 (x - 1) (x - 3) = 0$
$x = 1$ or $x = 3$

$\frac{d^2y}{dx^2} = 6x - 12$

When $x = 1$, $\frac{d^2y}{dx^2} = - 6 < 0 \therefore$ maximum point

When $x = 3$, $\frac{d^2y}{dx^2} = + 6 > 0 \therefore$ minimum point

So the maximum point is where $x = 1$.
Substitute $x = 1$ into $y = x^3 - 6x^2 + 9x$
Then $y = 1 - 6 + 9 = 4$
So (1, 4) is the maximum turning point.

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Differentiation
Exercise D, Question 5

Question:

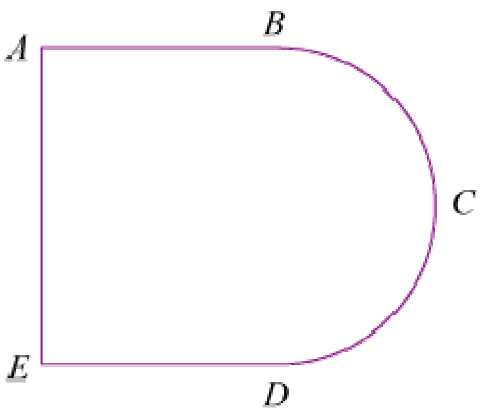
A wire is bent into the plane shape *ABCDEA* as shown. Shape *ABDE* is a rectangle and *BCD* is a semicircle with diameter *BD*. The area of the region enclosed by the wire is R m^2, $AE = x$ metres, $AB = ED = y$ metres. The total length of the wire is 2 m.

(a) Find an expression for y in terms of x.

(b) Prove that $R = \frac{x}{8}\left(8 - 4x - \pi x\right)$

Given that x can vary, using calculus and showing your working,

(c) find the maximum value of R. (You do not have to prove that the value you obtain is a maximum.)



[E]

Solution:

(a) The total length of wire is $\left(2y + x + \frac{\pi x}{2}\right)$ m

As total length is 2 m so

$2y + x\left(1 + \frac{\pi}{2}\right) = 2$

$y = 1 - \frac{1}{2}x\left(1 + \frac{\pi}{2}\right)$ ①

(b) Area $R = xy + \frac{1}{2}\pi\left(\frac{x}{2}\right)^2$

Substitute from ① to give

$R = x\left(1 - \frac{1}{2}x - \frac{\pi}{4}x\right) + \frac{\pi}{8}x^2$

$R = \frac{x}{8}\left(8 - 4x - 2\pi x + \pi x\right)$

$R = \frac{x}{8}\left(8 - 4x - \pi x\right)$ ②

(c) For maximum R, $\frac{dR}{dx} = 0$

$R = x - \frac{1}{2}x^2 - \frac{\pi}{8}x^2$

So $\frac{dR}{dx} = 1 - x - \frac{\pi}{4}x$

Put $\frac{dR}{dx} = 0$ to obtain $x = \frac{1}{1 + \frac{\pi}{4}}$

So $x = \frac{4}{4 + \pi}$

Substitute into ② to give

$R = \frac{1}{2\left(4 + \pi\right)}\left(8 - \frac{16}{4 + \pi} - \frac{4\pi}{4 + \pi}\right)$

$R = \frac{1}{2\left(4 + \pi\right)} \times \frac{32 + 8\pi - 16 - 4\pi}{4 + \pi}$

$R = \frac{1}{2\left(4 + \pi\right)} \times \frac{16 + 4\pi}{4 + \pi}$

$R = \frac{4\left(4 + \pi\right)}{2\left(4 + \pi\right)^2}$

$R = \frac{2}{4 + \pi}$

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Differentiation
Exercise D, Question 6

Question:

The fixed point A has coordinates $(8, -6, 5)$ and the variable point P has coordinates $(t, t, 2t)$.

(a) Show that $AP^2 = 6t^2 - 24t + 125$.

(b) Hence find the value of t for which the distance AP is least.

(c) Determine this least distance.

[E]

Solution:

(a) From Pythagoras
$AP^2 = (8 - t)^2 + (-6 - t)^2 + (5 - 2t)^2$
$AP^2 = 64 - 16t + t^2 + 36 + 12t + t^2 + 25 - 20t + 4t^2$
$AP^2 = 6t^2 - 24t + 125$ *

(b) AP is least when AP^2 is least.
$\frac{d(AP^2)}{dt} = 12t - 24$

Put $\frac{d(AP^2)}{dt} = 0$, then $t = 2$

(c) Substitute $t = 2$ into * to obtain
$AP^2 = 24 - 48 + 125 = 101$
So AP = $\sqrt{101}$

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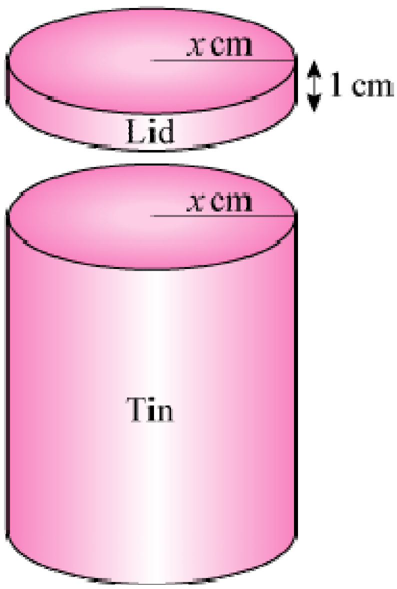
Differentiation
Exercise D, Question 7

Question:



A cylindrical biscuit tin has a close-fitting lid which overlaps the tin by 1 cm, as shown. The radii of the tin and the lid are both x cm. The tin and the lid are made from a thin sheet of metal of area $80\pi \text{cm}^2$ and there is no wastage. The volume of the tin is V cm^3.

(a) Show that $V = \pi (40x - x^2 - x^3)$.
Given that x can vary:

(b) Use differentiation to find the positive value of x for which V is stationary.

(c) Prove that this value of x gives a maximum value of V.

(d) Find this maximum value of V.

(e) Determine the percentage of the sheet metal used in the lid when V is a maximum.

[E]

Solution:

(a) Let the height of the tin be h cm.
The area of the curved surface of the tin $= 2\pi xh$ cm^2
The area of the base of the tin $= \pi x^2$ cm^2
The area of the curved surface of the lid $= 2\pi x$ cm^2
The area of the top of the lid $= \pi x^2$ cm^2
Total area of sheet metal is 80π cm^2
So $2\pi x^2 + 2\pi x + 2\pi xh = 80\pi$
Rearrange to give

$$h = \frac{40 - x - x^2}{x}$$

The volume, V, of the tin is given by
$V = \pi x^2 h$

So $V = \frac{\pi x^2 (40 - x - x^2)}{x} = \pi \left(40x - x^2 - x^3 \right)$

(b) $\frac{dV}{dx} = \pi \left(40 - 2x - 3x^2 \right)$

When V is stationary $\frac{dV}{dx} = 0$

So $40 - 2x - 3x^2 = 0$

$\Rightarrow \quad (10 - 3x) (4 + x) = 0$

$\Rightarrow \quad x = \frac{10}{3}$ or -4

But x is positive so $x = \frac{10}{3}$ is the required value.

(c) $\frac{d^2V}{dx^2} = \pi \left(-2 - 6x \right)$

When $x = \frac{10}{3}$, $\frac{d^2V}{dx^2} = \pi \left(-2 - 20 \right) < 0$

So V has a maximum value.

(d) Substitute $x = \frac{10}{3}$ into the expression given in part (a):

$V = \frac{2300\pi}{27}$

(e) The metal used in the lid $= 2\pi x + \pi x^2$ with $x = \frac{10}{3}$

i.e. $A_{\text{lid}} = \frac{160\pi}{9}$

Total area $= 80\pi$

So percentage used in the lid $= \left(\frac{160\pi}{9} \div 80\pi \right) \times 100 = 22\frac{2}{9}\%$.

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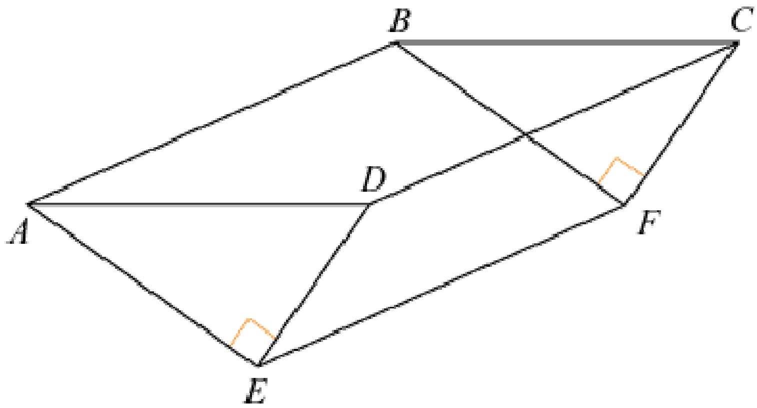
Differentiation
Exercise D, Question 8

Question:



The diagram shows an open tank for storing water, *ABCDEF*. The sides *ABFE* and *CDEF* are rectangles. The triangular ends *ADE* and *BCF* are isosceles, and $\angle$ AED = $\angle$ BFC = 90 ° . The ends *ADE* and *BCF* are vertical and *EF* is horizontal.

Given that AD = x metres:

(a) show that the area of triangle *ADE* is $\frac{1}{4}x^2$ m^2.

Given also that the capacity of the container is 4000 m^3 and that the total area of the two triangular and two rectangular sides of the container is S m^2:

(b) Show that $S = \frac{x^2}{2} + \frac{16000\sqrt{2}}{x}$.

Given that x can vary:

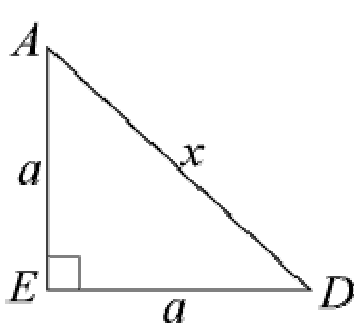
(c) Use calculus to find the minimum value of S.

(d) Justify that the value of S you have found is a minimum.

[E]

Solution:

(a) Let the equal sides of $\triangle ADE$ be a metres.



Then $a^2 + a^2 = x^2$ (Pythagoras' Theorem)
So $2a^2 = x^2$

$$\Rightarrow \quad a^2 = \frac{x^2}{2}$$

Area of $\triangle ADE = \frac{1}{2}\text{base} \times \text{height} = \frac{1}{2} a \times a = \frac{x^2}{4}$

(b) Area of two triangular sides is $2 \times \frac{x^2}{4} = \frac{x^2}{2}$

Let the length AB = CD = y metres

Area of two rectangular sides is $2 \times \text{ay} = 2\text{ay} = 2\sqrt{\frac{x^2}{2}}\, y$

Then $S = \frac{x^2}{2} + 2\sqrt{\frac{x^2}{2}}\, y \quad *$

But capacity of storage tank is $\frac{1}{4}x^2 \times y$ so

$\frac{1}{4}x^2 y = 4000$

$y = \frac{16000}{x^2}$

Substitute this into equation * to give

$S = \frac{x^2}{2} + \frac{16000\sqrt{2}}{x}$

(c) $\frac{\mathrm{d}S}{\mathrm{d}x} = x - \frac{16000\sqrt{2}}{x^2}$

Put $\frac{\mathrm{d}S}{\mathrm{d}x} = 0$

Then $x - \frac{16000\sqrt{2}}{x^2} = 0$

$x = \frac{16000\sqrt{2}}{x^2}$

$x^3 = 16000\sqrt{2}$
$x = 20\sqrt{2}$ or 28.28
Substitute into expression for S to give
$S = 400 + 800 = 1200$

(d) $\frac{\mathrm{d}^2S}{\mathrm{d}x^2} = 1 + \frac{32000\sqrt{2}}{x^3}$

When $x = 20\sqrt{2}$, $\frac{\mathrm{d}^2S}{\mathrm{d}x^2} = 3 > 0 \therefore$ minimum value

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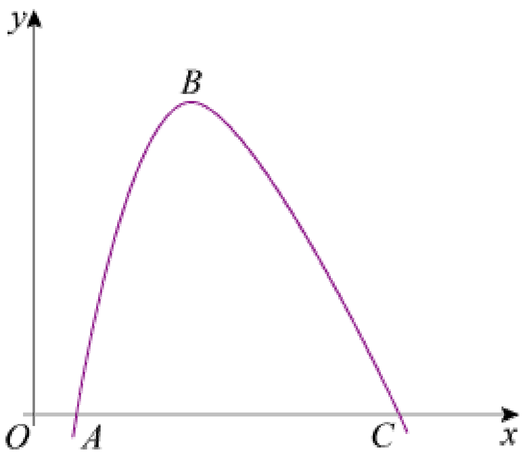
Differentiation
Exercise D, Question 9

Question:



The diagram shows part of the curve with equation $y = f(x)$, where:

$f(x) \equiv 200 - \frac{250}{x} - x^2, x > 0$

The curve cuts the x-axis at the points A and C.
The point B is the maximum point of the curve.

(a) Find $f'(x)$.

(b) Use your answer to part (a) to calculate the coordinates of B.

[E]

Solution:

(a) $f(x) = 200 - \frac{250}{x} - x^2$

$f'(x) = \frac{250}{x^2} - 2x$

(b) At the maximum point, B, $f'(x) = 0$. So

$\frac{250}{x^2} - 2x = 0$

$\frac{250}{x^2} = 2x$

$250 = 2x^3$
$x^3 = 125$
$x = 5$ at point B
As $y = f(x)$, $y = f(5)$ at point B. So $y = 125$.
The coordinates of B are $(5, 125)$.

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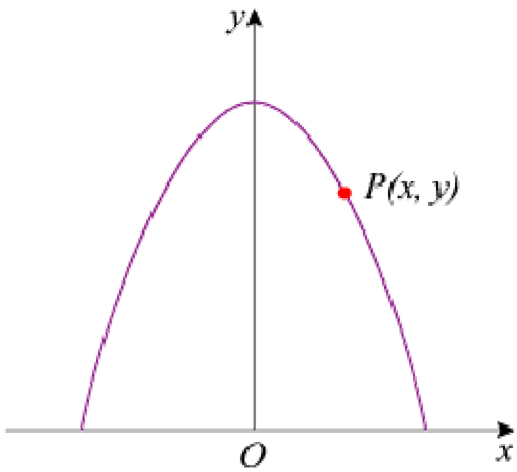
Differentiation
Exercise D, Question 10

Question:



The diagram shows the part of the curve with equation $y = 5 - \frac{1}{2}x^2$ for which $y \geq 0$.

The point $P(x, y)$ lies on the curve and O is the origin.

(a) Show that $OP^2 = \frac{1}{4}x^4 - 4x^2 + 25$.

Taking $f\left(x\right) \equiv \frac{1}{4}x^4 - 4x^2 + 25$:

(b) Find the values of x for which $f'\left(x\right) = 0$.

(c) Hence, or otherwise, find the minimum distance from O to the curve, showing that your answer is a minimum.

[E]

Solution:

(a) P has coordinates $\left(x , 5 - \frac{1}{2}x^2\right)$. So

$OP^2 = \left(x - 0\right)^2 + \left(5 - \frac{1}{2}x^2 - 0\right)^2 = x^2 + 25 - 5x^2 + \frac{1}{4}x^4 = \frac{1}{4}x^4 - 4x^2 + 25$

(b) Given $f\left(x\right) = \frac{1}{4}x^4 - 4x^2 + 25$

$f'\left(x\right) = x^3 - 8x$
When $f'\left(x\right) = 0$,
$x^3 - 8x = 0$
$x\left(x^2 - 8\right) = 0$
$x = 0$ or $x^2 = 8$

$x = 0$ or $x = \pm 2\sqrt{2}$

(c) Substitute $x^2 = 8$ into $f(x)$:

$OP^2 = \frac{1}{4} \times 8^2 - 4 \times 8 + 25 = 9$

So $OP = 3$ when $x = \pm 2\sqrt{2}$
$f''(x) = 3x^2 - 8 = 16 > 0$ when $x^2 = 8 \Rightarrow$ minimum value for OP^2 and hence OP.
So minimum distance from O to the curve is 3.

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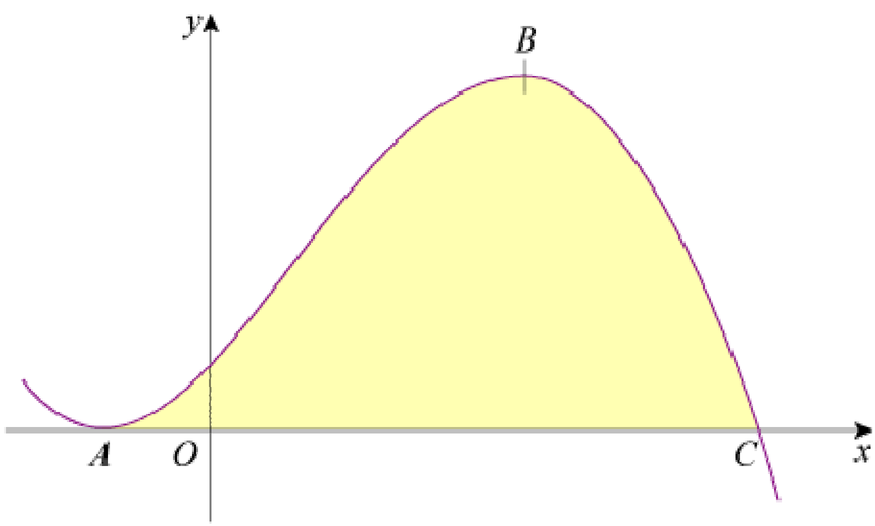
Differentiation
Exercise D, Question 11

Question:



The diagram shows part of the curve with equation $y = 3 + 5x + x^2 - x^3$. The curve touches the x-axis at A and crosses the x-axis at C. The points A and B are stationary points on the curve.

(a) Show that C has coordinates (3, 0).

(b) Using calculus and showing all your working, find the coordinates of A and B.

Solution:

(a) $y = 3 + 5x + x^2 - x^3$
Let $y = 0$, then
$3 + 5x + x^2 - x^3 = 0$
$\left(3 - x \right) \left(1 + 2x + x^2 \right) = 0$
$\left(3 - x \right) \left(1 + x \right)^2 = 0$
$x = 3$ or $x = -1$ when $y = 0$
The curve touches the x-axis at $x = -1$ $\left(A \right)$ and cuts the axis at $x = 3$ $\left(C \right)$.
C has coordinates $\left(3 , 0 \right)$

(b) $\frac{dy}{dx} = 5 + 2x - 3x^2$

Put $\frac{dy}{dx} = 0$, then

$5 + 2x - 3x^2 = 0$
$\left(5 - 3x \right) \left(1 + x \right) = 0$
$x = \frac{5}{3}$ or $x = -1$

When $x = \frac{5}{3}$, $y = 3 + 5 \left(\frac{5}{3} \right) + \left(\frac{5}{3} \right)^2 - \left(\frac{5}{3} \right)^3 = 9 \frac{13}{27}$

So $\left(\frac{5}{3}, 9 \frac{13}{27} \right)$ is the point B.

When $x = -1$, $y = 0$

So $(-1, 0)$ is the point A.