General Certificate of Education
January 2005
Advanced Level Examination

MATHEMATICS (SPECIFICATION A) **MAP6**
Unit Pure 6

Tuesday 18 January 2005 Afternoon Session

In addition to this paper you will require:
- an 8-page answer book;
- the AQA booklet of formulae and statistical tables.

You may use a standard scientific calculator **only**.

Time allowed: 1 hour 20 minutes

Instructions
- Use blue or black ink or ball-point pen. Pencil should only be used for drawing.
- Write the information required on the front of your answer book. The *Examining Body* for this paper is AQA. The *Paper Reference* is MAP6.
- Answer **all** questions.
- All necessary working should be shown; otherwise marks for method may be lost.
- The **final** answer to questions requiring the use of tables or calculators should normally be given to three significant figures.
- Tie loosely any additional sheets you have used to the back of your answer book before handing it to the invigilator.

Information
- The maximum mark for this paper is 60.
- Mark allocations are shown in brackets.

Advice
- Unless stated otherwise, formulae may be quoted, without proof, from the booklet.

Answer **all** questions.

1 The matrix $\mathbf{M}$ is defined by

$$\mathbf{M} = \begin{bmatrix} 4 & 1 & 1 \\ 0 & 2 & -6 \\ 0 & -2 & 1 \end{bmatrix}.$$

(a) Show that $\lambda = 4$ is an eigenvalue and find the two other eigenvalues of $\mathbf{M}$. *(4 marks)*

(b) Verify that the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} -5 \\ 18 \\ 12 \end{bmatrix}$$

are eigenvectors corresponding to two of the eigenvalues. *(3 marks)*

(c) Find an eigenvector $\mathbf{v}_3$ corresponding to the third eigenvalue. *(3 marks)*

(d) Write down the image of the vector

$$\mathbf{r} = \alpha\mathbf{v}_1 + \beta\mathbf{v}_2 + \gamma\mathbf{v}_3$$

under the transformation represented by the matrix $\mathbf{M}$, giving your answer in the form

$$l\mathbf{v}_1 + m\mathbf{v}_2 + n\mathbf{v}_3 .$$ *(2 marks)*

2 (a) Expand and simplify

$$(\mathbf{a} + 3\mathbf{b}) \times (\mathbf{a} - 2\mathbf{b}) .$$ *(4 marks)*

(b) Show that, if the vectors $\mathbf{a}$ and $\mathbf{b}$ are perpendicular,

$$|(\mathbf{a} + 3\mathbf{b}) \times (\mathbf{a} - 2\mathbf{b})| = k\,|\,\mathbf{a}\,||\,\mathbf{b}\,| ,$$

where k is an integer. *(2 marks)*

3 The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{bmatrix} a & 5 & 4 \\ 4 & a & 0 \\ 3 & a & 1 \end{bmatrix},$$

where a is a constant.

(a) Express det $\mathbf{A}$ in terms of a. *(3 marks)*

(b) The non-singular matrix $\mathbf{B}$ is such that

$$\det(\mathbf{AB}) = \det \mathbf{B}.$$

Find the possible values of a. *(3 marks)*

4 The vectors $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$ are given by

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{u}_3 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}.$$

(a) (i) Find $(\mathbf{u}_1 \times \mathbf{u}_2) \cdot \mathbf{u}_3$. *(3 marks)*

(ii) Explain why it is **not** possible to express $\mathbf{u}_3$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$. *(2 marks)*

(b) The vector $\mathbf{u}_4$ is given by

$$\mathbf{u}_4 = \begin{bmatrix} 2 \\ 7 \\ 2 \end{bmatrix}.$$

(i) Find a vector $\mathbf{u}$ such that

$$\mathbf{u}_4 = \begin{bmatrix} 2 & 3 & 1 \\ 1 & -2 & 3 \\ 1 & 1 & 1 \end{bmatrix} \mathbf{u}.$$ *(6 marks)*

(ii) Hence, or otherwise, express $\mathbf{u}_4$ as a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$. *(2 marks)*

5 The points A, B and C have coordinates (3, 4, 1), (−2, 1, 4) and (1, 2, −1) respectively.

(a) Find:

(i) $\overrightarrow{AB} \times \overrightarrow{AC}$; *(3 marks)*

(ii) the exact value of the area of triangle ABC; *(2 marks)*

(iii) the Cartesian equation of the plane Π containing A, B and C. *(2 marks)*

(b) The line l passes through the point $D(0, -5, 0)$ and is perpendicular to Π. Find:

(i) the equation of l in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$; *(1 mark)*

(ii) the coordinates of the point of intersection of l with Π. *(3 marks)*

(c) Deduce the volume of the tetrahedron $ABCD$. *(3 marks)*
[The volume of a tetrahedron is $\frac{1}{3}$ area of base $\times$ height.]

6 (a) The transformation T_1 represented by the matrix

$$\mathbf{M}_1 = \begin{bmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}$$

is an anticlockwise rotation about the origin. Find the angle of rotation. *(2 marks)*

(b) The transformation T_2 represented by the matrix

$$\mathbf{M}_2 = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

is a reflection in the line $y = mx$. Find the value of m. *(3 marks)*

(c) (i) The matrix $\mathbf{M}_3$ is given by $\mathbf{M}_3 = \mathbf{M}_1\mathbf{M}_2$. Find $\mathbf{M}_3$. *(2 marks)*

(ii) The matrix $\mathbf{M}_3$ represents a single transformation T_3. Give a geometrical description of T_3. *(2 marks)*

END OF QUESTIONS