General Certificate of Education
Advanced Level Examination
June 2015

Use of Mathematics (Pilot) USE3/PM

Mathematical Comprehension

Preliminary Material

Data Sheet

To be opened and issued to candidates between
Thursday 30 April 2015 and Thursday 14 May 2015

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INFORMATION

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Bacterial growth: friend or foe?

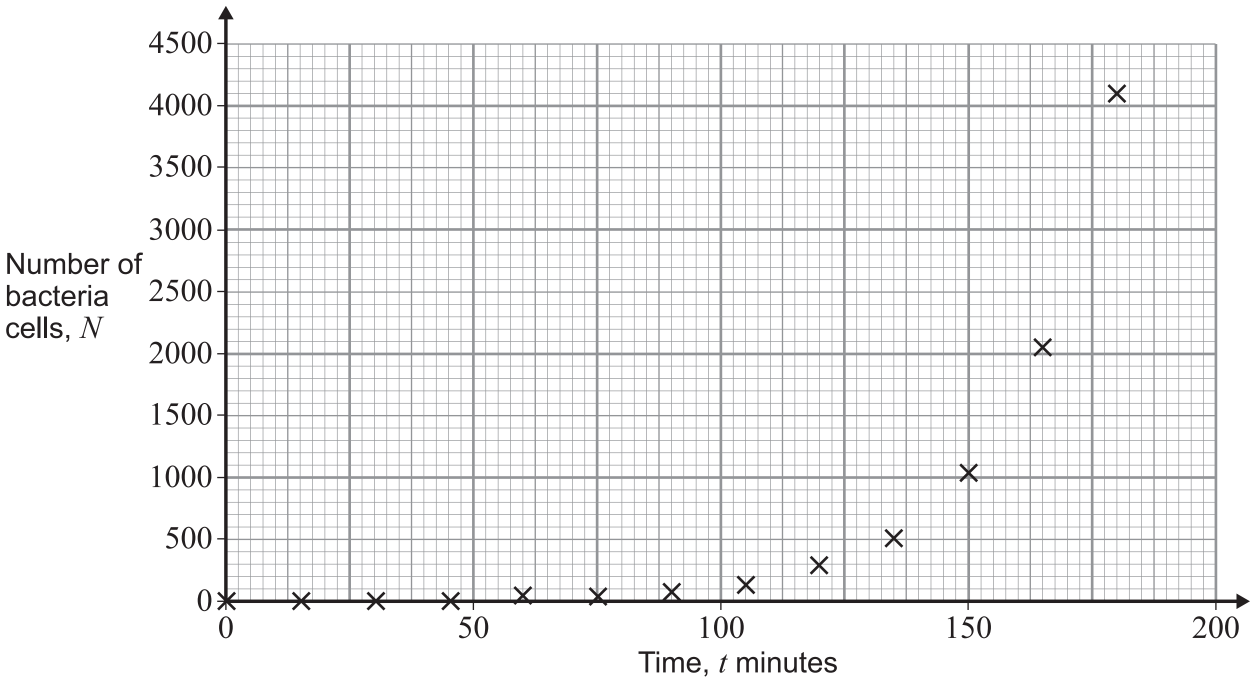
Bacteria are everywhere and, of course, many people think that they are dangerous. However, bacteria certainly should not be considered as always posing a problem, even when in the human body. For example, bacteria assist in the digestion of food and aid the body's defence system in fighting against illness and disease. However, some bacteria can be very dangerous when in the human body, giving rise to illnesses such as food poisoning.

Such illnesses can be difficult to bring under control because of the rapid growth of the bacteria, which can double in number in a relatively short time.

Figure 1 Bacteria cells

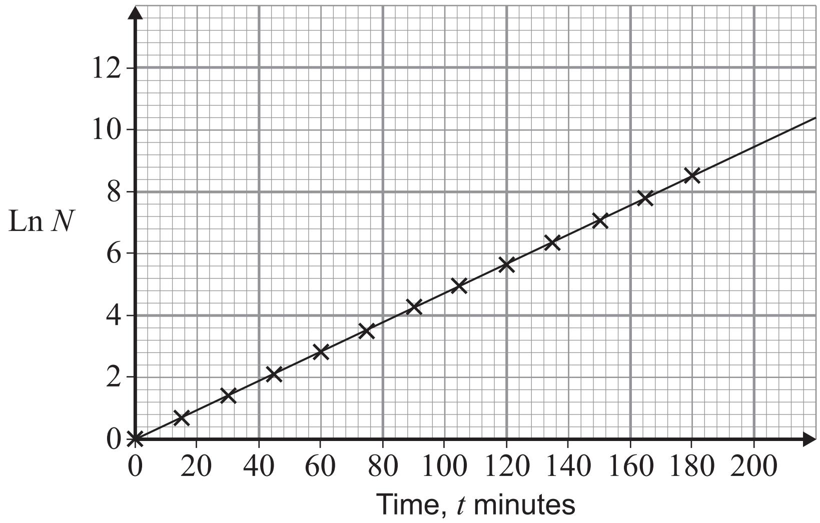
Consider a colony of bacteria which, in favourable conditions, can double in size every 15 minutes and starts with just one bacteria cell. A graph of the number of cells plotted against time is shown in **Figure 2**. The time taken for a colony of bacteria to double in size is known as the generation time, G, so in this example, the generation time is 15 minutes. As you can see from the graph, the number of bacteria cells increases very rapidly and in just a few hours there will be thousands of cells. Because of this rapid growth, logarithms may be useful when dealing with the mathematics of the growth.

Figure 2 Number of bacteria cells, N, plotted against time, t minutes, for a colony of bacteria with a generation time, G, of 15 minutes



A graph of the logarithm (base e) of the number of bacteria cells plotted against time for this colony is shown in **Figure 3**. As you can see, this is a straight line.

Figure 3 $\mathbf{Ln}\,N$, where N is the number of bacteria cells, plotted against time, t minutes, for a colony of bacteria with a generation time, G, of 15 minutes



To develop a general model of growth of bacteria, assume that there are N_0 bacteria at time $t = 0$, and after an interval of time, t, there are N bacteria.

The growth can be represented by a power relationship, $N = N_0 e^{kt}$, where k is a constant that is related to the specific type of bacteria and the conditions in which they are growing. If in the interval of time, t, there have been n generations, that is n doublings $N = N_0 \times 2^n = N_0 2^n$.

Since $G = \dfrac{t}{n}$,

$$N = N_0 2^n = N_0 2^{\frac{t}{G}}$$

Thus,

$$N_0 e^{kt} = N_0 2^{\frac{t}{G}}$$

Using logarithms, this can be rearranged to find an expression for k:

$$kt = \ln 2^{\frac{t}{G}}$$
$$= \frac{t}{G} \ln 2$$

so that, $k = \dfrac{\ln 2}{G}$

For any growth of this type you can check that the two expressions $N = N_0 2^{\frac{t}{G}}$ and $N = N_0 e^{\frac{\ln 2}{G} t}$ are equivalent using a graph plotter to plot both functions on the same axes. Such growth is known as exponential growth.

Of course, the growth rate, that is the rate of increase in the number of bacteria cells, is a very important factor in such growth.

An average growth rate between times t_1 and t_2 can be found using values for the number of bacteria N_1 and N_2 respectively:

$$\text{average growth rate} = \frac{N_2 - N_1}{t_2 - t_1}$$

Alternatively, the instantaneous growth rate, $\frac{\mathrm{d}N}{\mathrm{d}t}$, at time t, can be found:

using $N = N_0 \mathrm{e}^{\frac{\ln 2}{G}t}$ gives $\frac{\mathrm{d}N}{\mathrm{d}t} = N_0 \frac{\ln 2}{G} \mathrm{e}^{\frac{\ln 2}{G}t} = \frac{\ln 2}{G} N = kN$;

more easily, using $N = N_0 \mathrm{e}^{kt}$ gives $\frac{\mathrm{d}N}{\mathrm{d}t} = kN_0 \mathrm{e}^{kt} = kN$.

This demonstrates the distinctive property of the growth of a population that can be modelled using an exponential function, which is that its rate of growth, $\frac{\mathrm{d}N}{\mathrm{d}t}$, is proportional to the size of the population, N, that is

$$\frac{\mathrm{d}N}{\mathrm{d}t} = kN$$

When investigating exponential growth mathematically, we often start with this differential equation and solve it for the case where $N = N_0$ when $t = 0$, which gives

$$\ln N = kt + \ln N_0$$

and, therefore,

$$N = N_0 \mathrm{e}^{kt}$$

Further to this, $\frac{\mathrm{d}^2N}{\mathrm{d}t^2} = k^2 N$.

Table 1 gives the data from an experiment that monitored the growth of a colony of bacteria in a laboratory. These data give a measure of the population density, P, which is effectively a measure of the number of bacteria present, every 16 minutes. Thus G can be written in terms of P.

As $G = \frac{t}{n}$

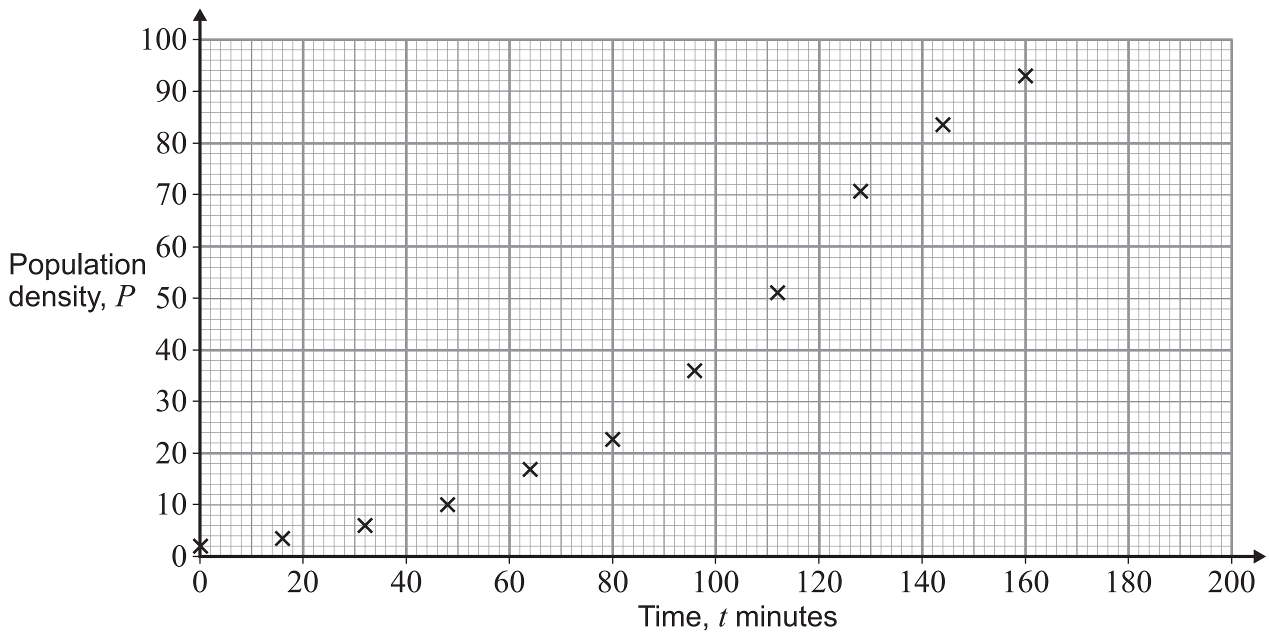
$$G = \frac{t \times \ln 2}{\ln \frac{P}{P_0}}$$

Table 1 Population density of a colony of bacteria measured every 16 minutes for 160 minutes

Time, t minutes	Population density, P
0	2.2
16	3.6
32	6.0
48	10.1
64	16.9
80	22.7
96	36.0
112	51.0
128	70.4
144	82.7
160	92.8

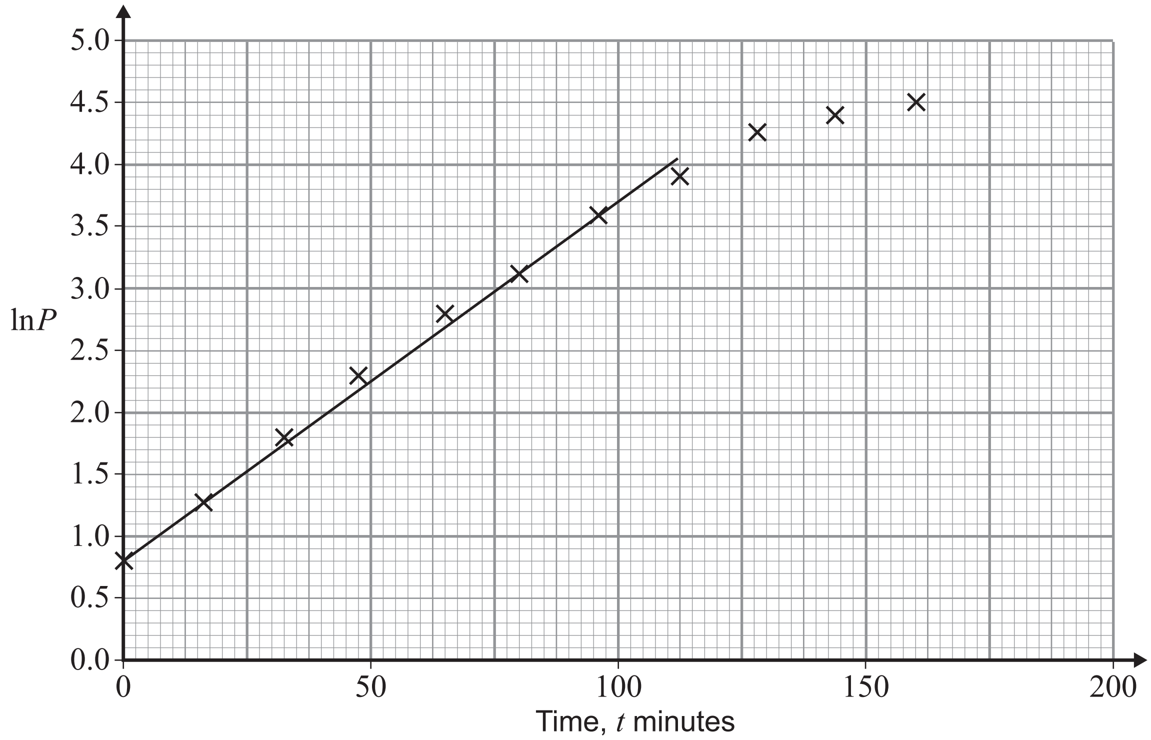
Figure 4 shows the data from **Table 1** plotted on a graph.

Figure 4 Graph showing the growth in population density, P, against time, t, for a colony of bacteria



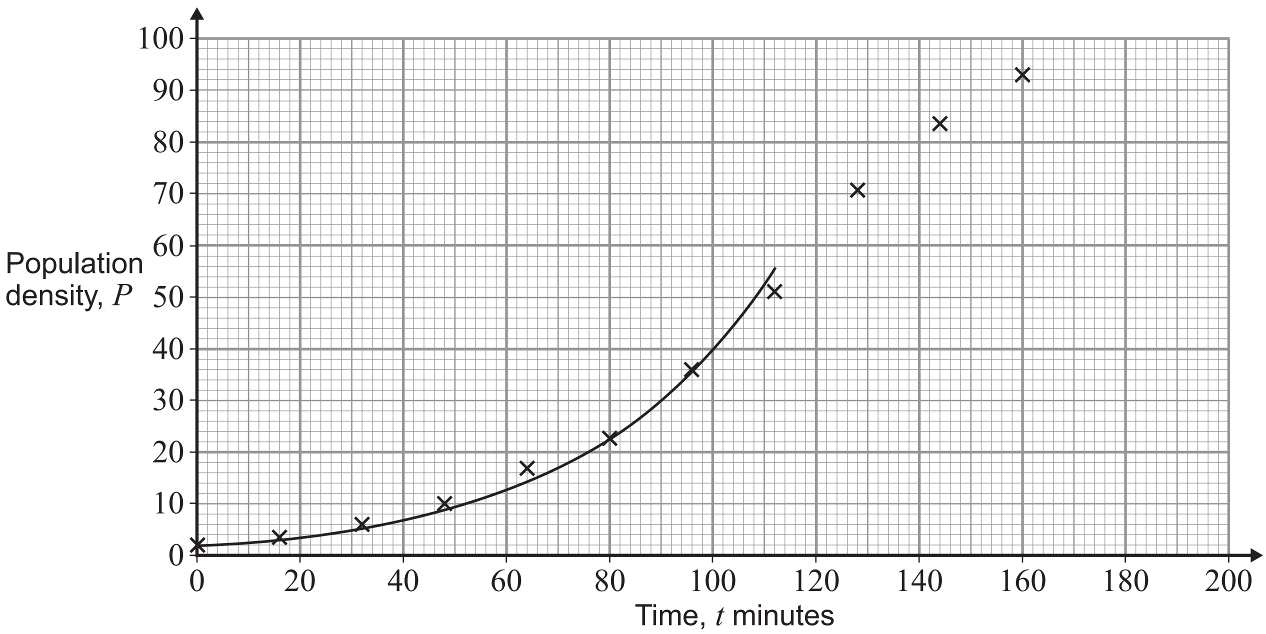
Inspection of this graph suggests that the first eight data points appear to exhibit exponential growth but, after this, the rate of growth appears to slow down, perhaps due to lack of nutrients. This slowdown is confirmed by a graph that plots $\ln P$ against time, t, as shown in **Figure 5**.

Figure 5 Graph showing $\ln P$ plotted against time



This graph also shows a straight line which is a good fit to the first eight points. The equation of this line can be found to be $\ln P = 0.788 + 0.029t$, which suggests that the exponential function $P = 2.2e^{0.029t}$ can be used effectively as a model for the early stages of growth. This is confirmed by the graph of **Figure 6**, which shows the data plotted against time together with a graph of this function. As you can see, this is a close fit for $t < 100$.

Figure 6 Graph plotting population density data for a colony of bacteria together with the function $P = 2.2e^{0.029t}$



As you can see from the mathematics so far, graphs can help us to see how we might proceed in attempting to fit functions to model biological phenomena such as the growth of bacteria. For the case examined here, it seems that we have a reasonable model for the first 90 minutes or so of growth.

Another useful technique that is relatively straightforward is to look at the growth rate using the data. This is done in **Table 2**, where the growth in population density, δP, at time t_n, when the population density is P_n, is found by calculating $P_{n+1} - P_n$.

Additionally, a relative growth rate, $\dfrac{\delta P}{P}$, is calculated and given in **Table 2**.

Table 2 The growth rate and relative growth rate for the growth in population density of a colony of bacteria

Time, t minutes	Population density, P	Growth rate, δP	Relative growth rate, $\dfrac{\delta P}{P}$
0	2.2	1.4	0.64
16	3.6	2.4	0.67
32	6.0	4.1	0.68
48	10.1	6.8	0.67
64	16.9	5.8	0.34
80	22.7	13.3	0.59
96	36.0	15.0	0.42
112	51.0	19.4	0.38
128	70.4	12.3	0.17
144	82.7	10.1	0.12
160	92.8		

As you can see, this suggests that the relative growth rate is only relatively constant for the first four data points and we might therefore expect to find an exponential function that is a good fit to the data for these few points. To some extent, this is confirmed by the graph in **Figure 6**. However, as we found earlier, the final few data points do not fit the exponential function. This is confirmed by the relative growth rates calculated here.

This article provides some ideas about how mathematics might start to be used to make sense of growth that might be described as exponential, that is where the rate of growth of a population is proportional to the size of the population. The techniques explored here have applications in many areas across the sciences and beyond, often allowing us to make sense of growth.

END OF DATA SHEET

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